Algorithms and Data Structures

Modelling with Flows

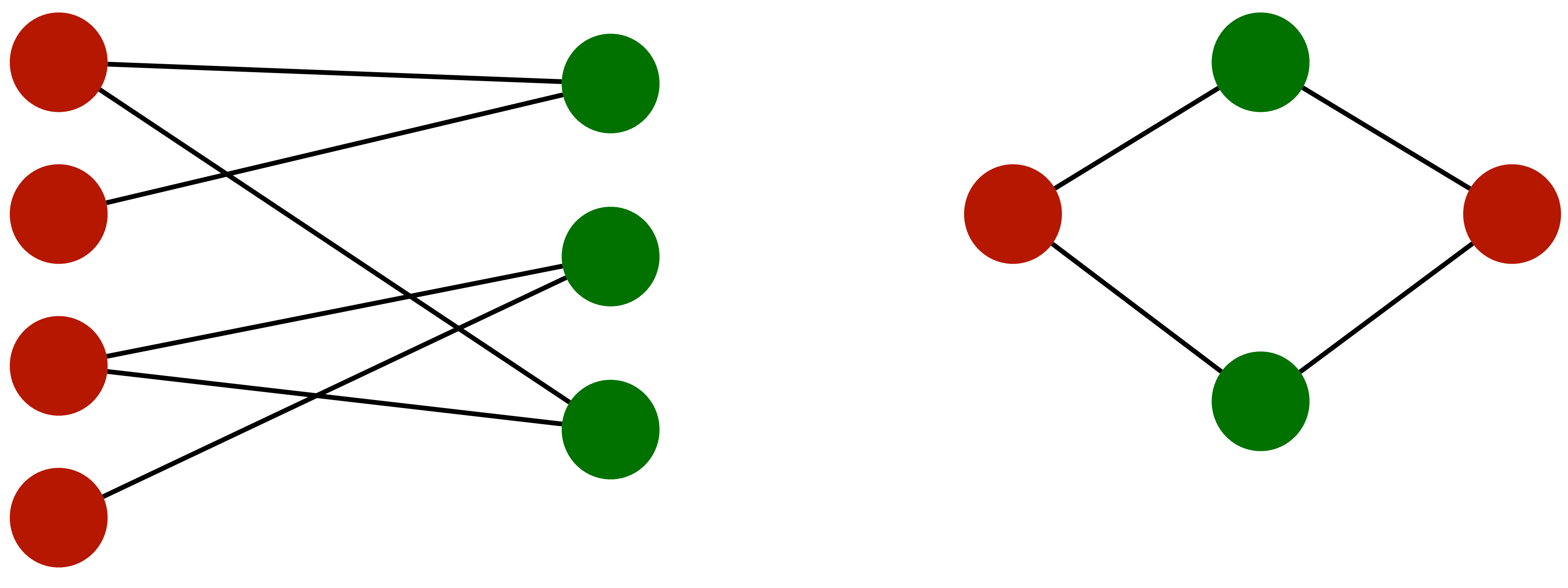
Bipartite Matching

Maximum Bipartite Matching or Maximum matching on a bipartite graph G.

Bipartite graphs

A graph G=(V,E) is bipartite *if any only if* it can be partitioned into sets A and B such that each edge has one endpoint in A and one endpoint in B.

Often, we write G=(L,R,E).



Bipartite Matching

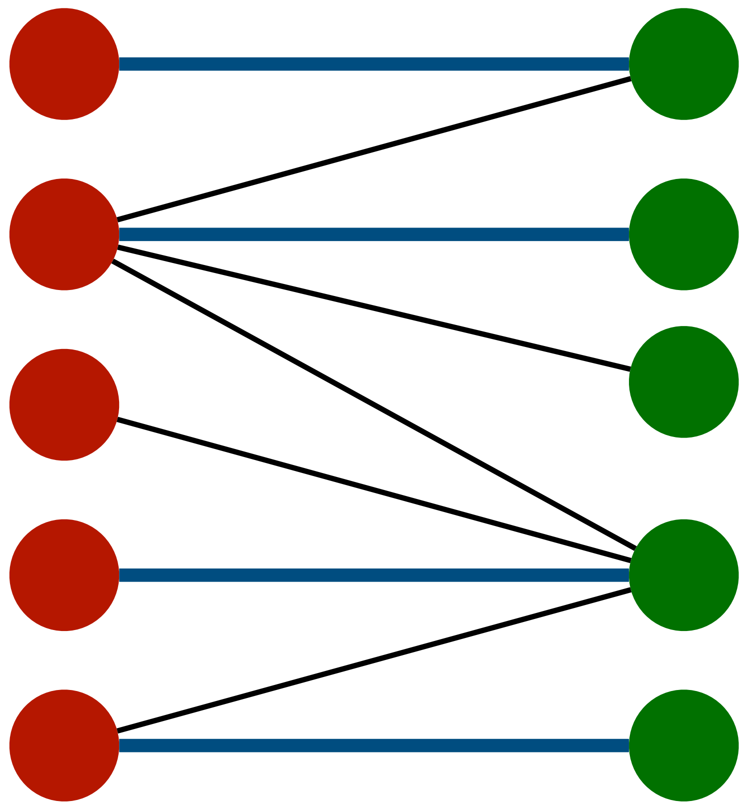
Maximum Bipartite Matching or Maximum matching on a bipartite graph G.

Matching: A subset M of the edges E such that each node v of V appears in at most one edge e in E.

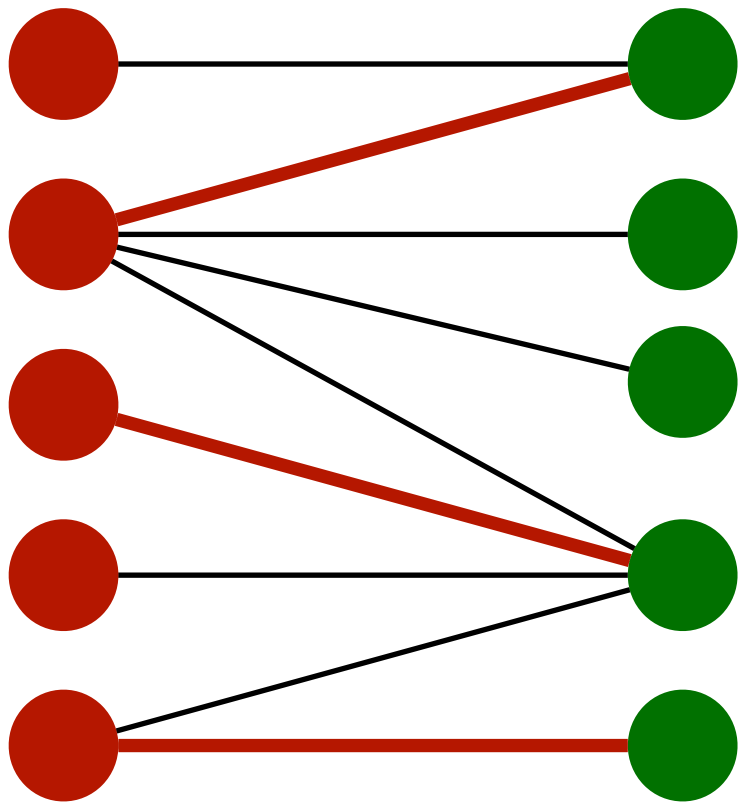
Maximum matching: A matching with maximum cardinality. (i.e., |M| is maximised).

Example

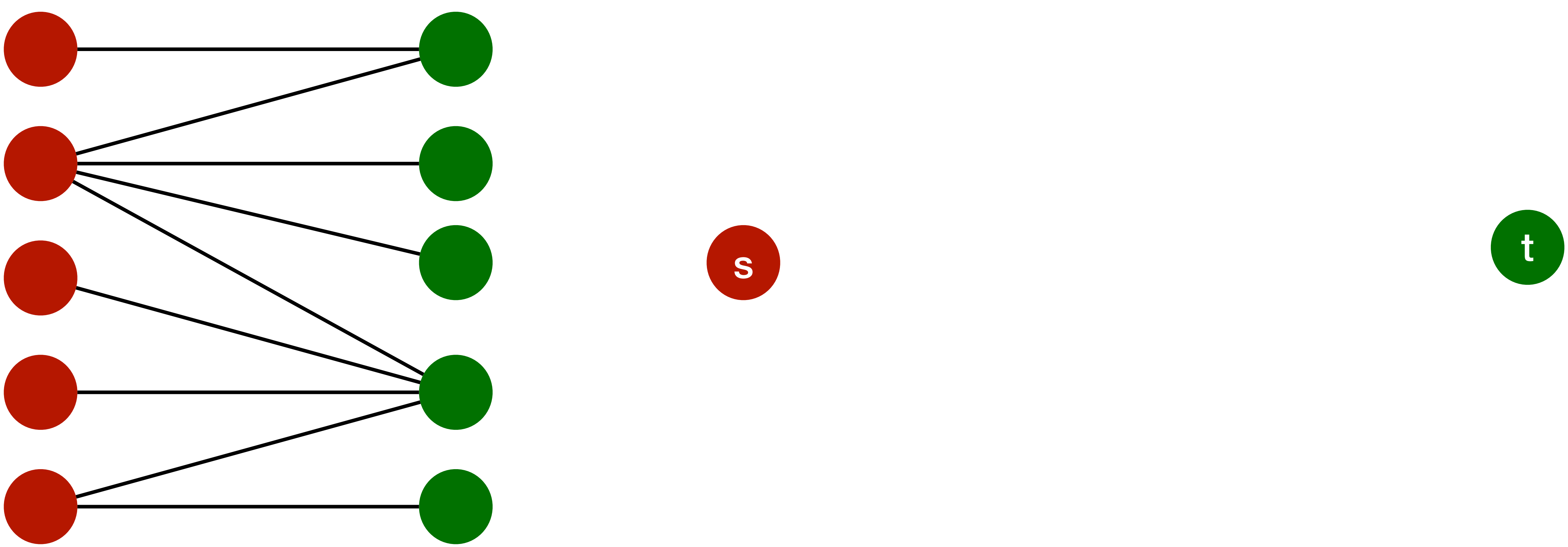
A maximum matching



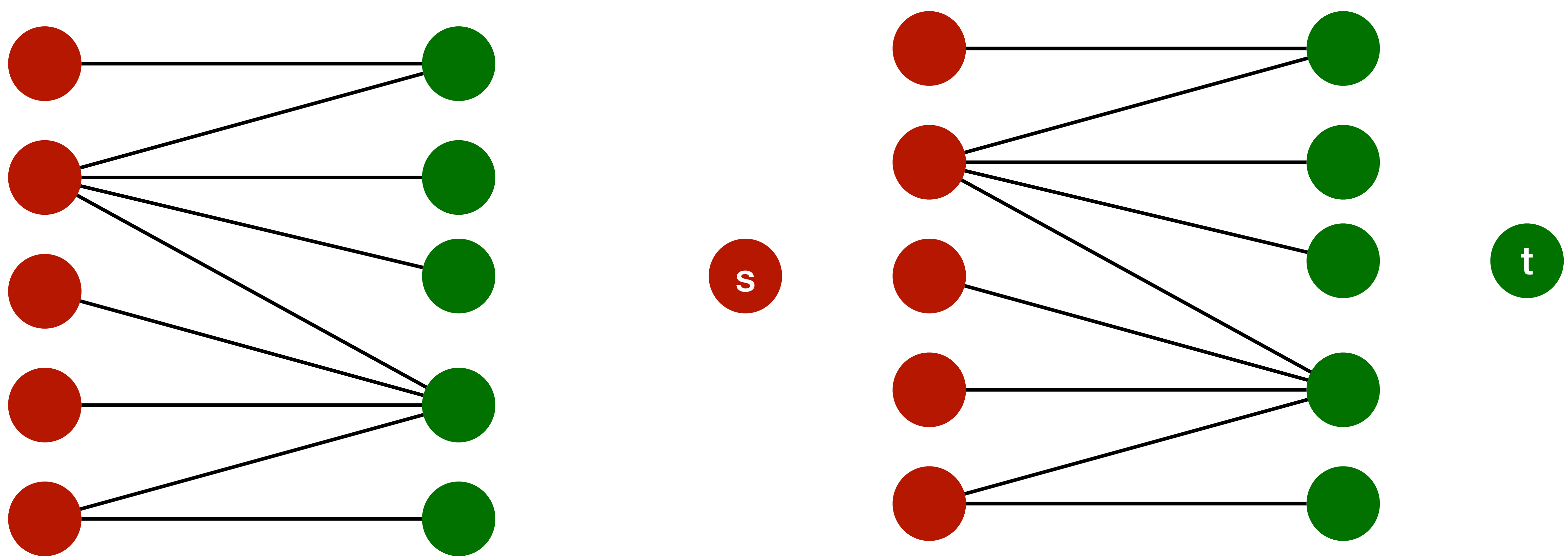
A maximal matching



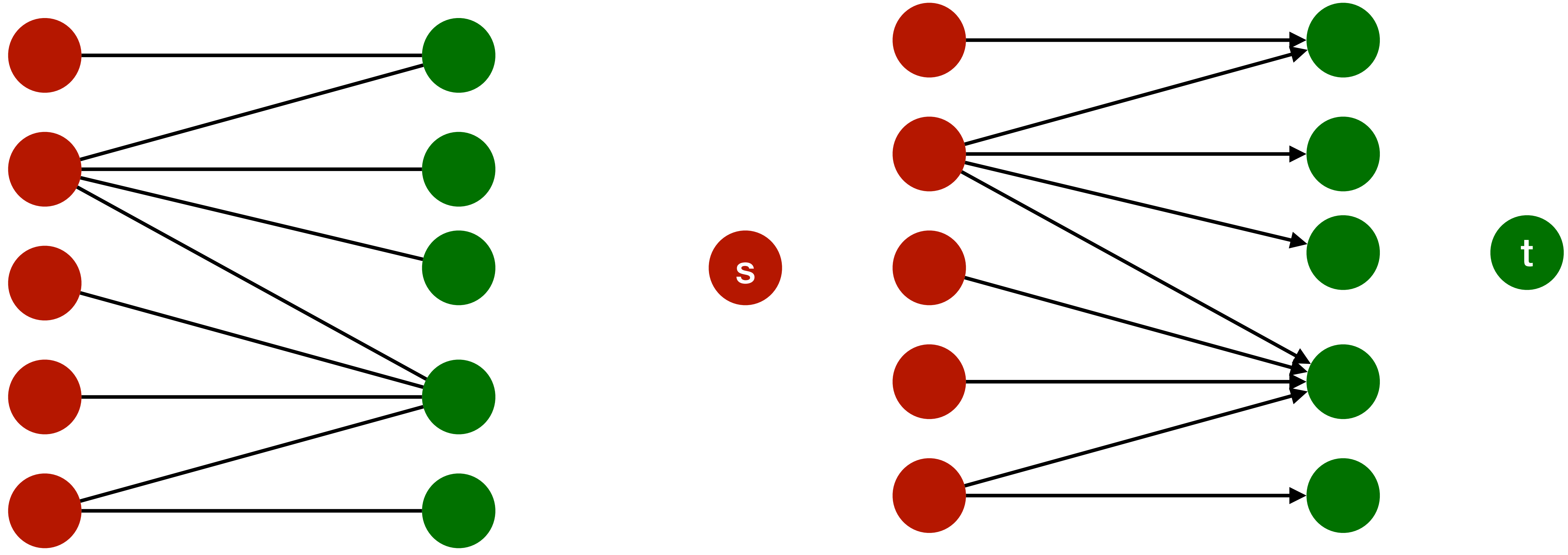
From matchings to flows



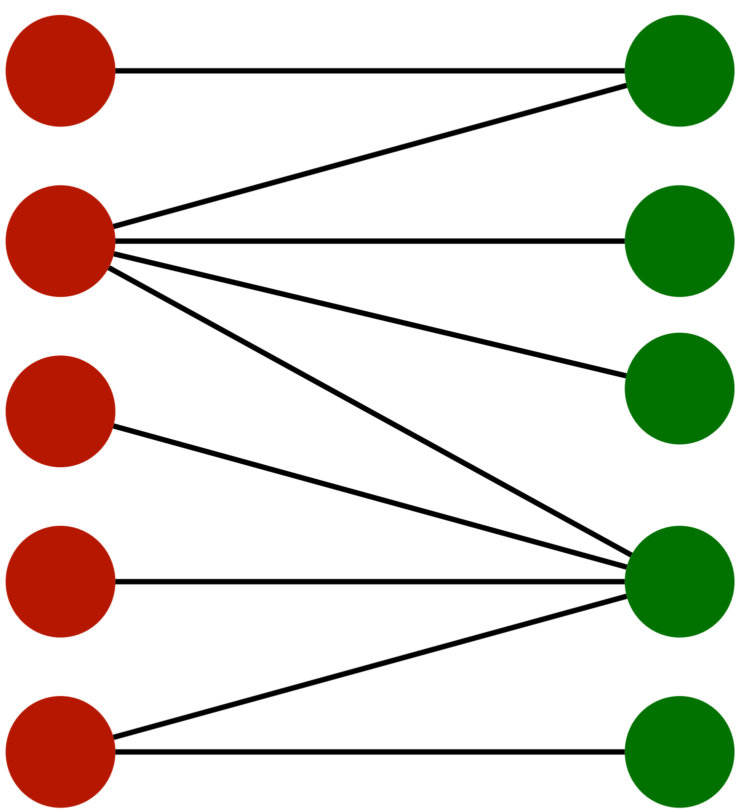
From matchings to flows



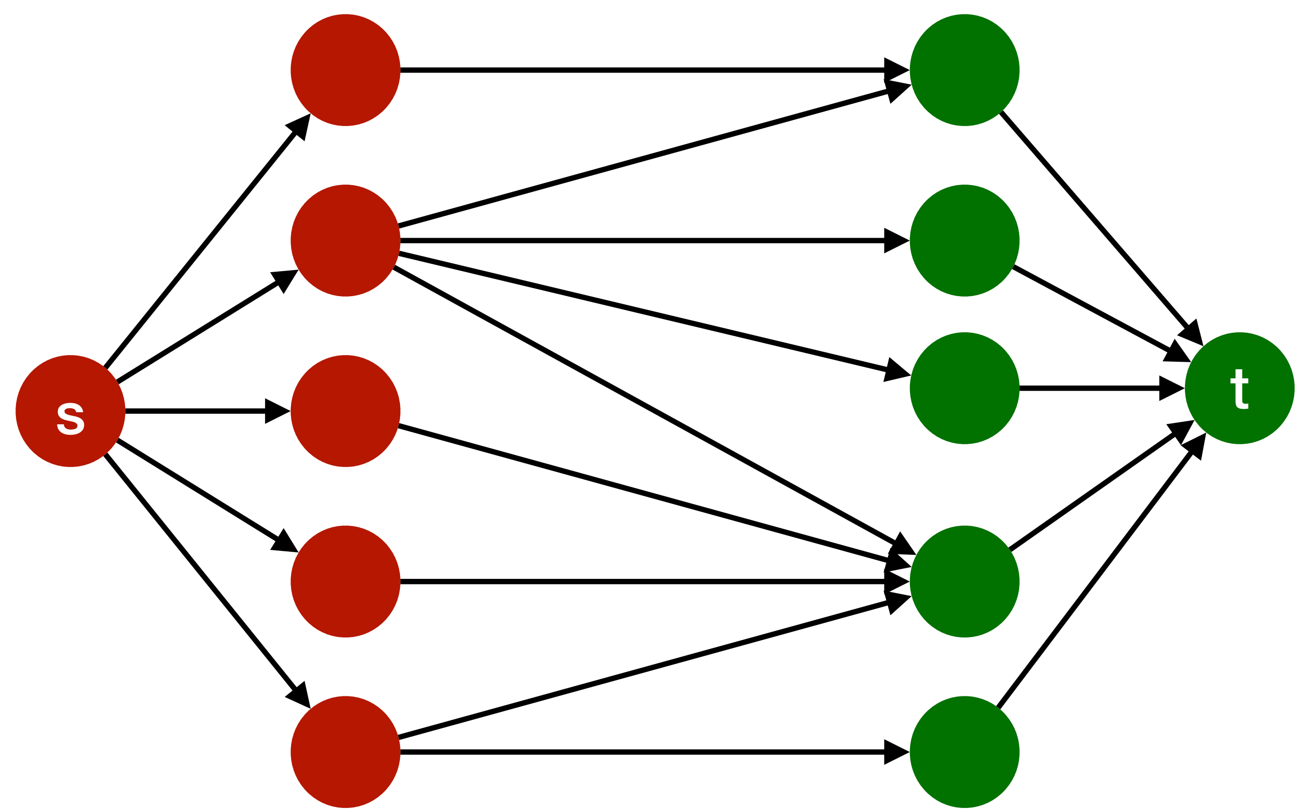
From matchings to flows



From matchings to flows



All capacities are set to 1.



From matchings to flows

Claim: Assume that there is a matching M of size k on G. Then there is a flow f of value k in G^f.

Consider the matching

$M = \{(u_1, v_1), (u_2, v_2), \ldots , (u_k, v_k)\}$

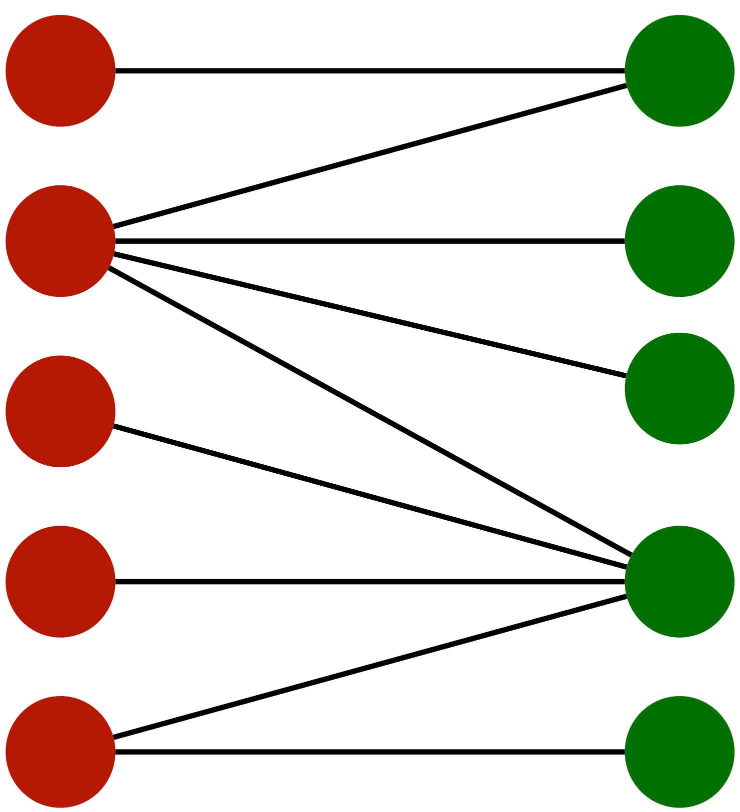
Consider the flow such that

$f(s, u_i) = f(u_i, v_i) = f(v_i, t) = 1$ for all $i = 1, \ldots , k$

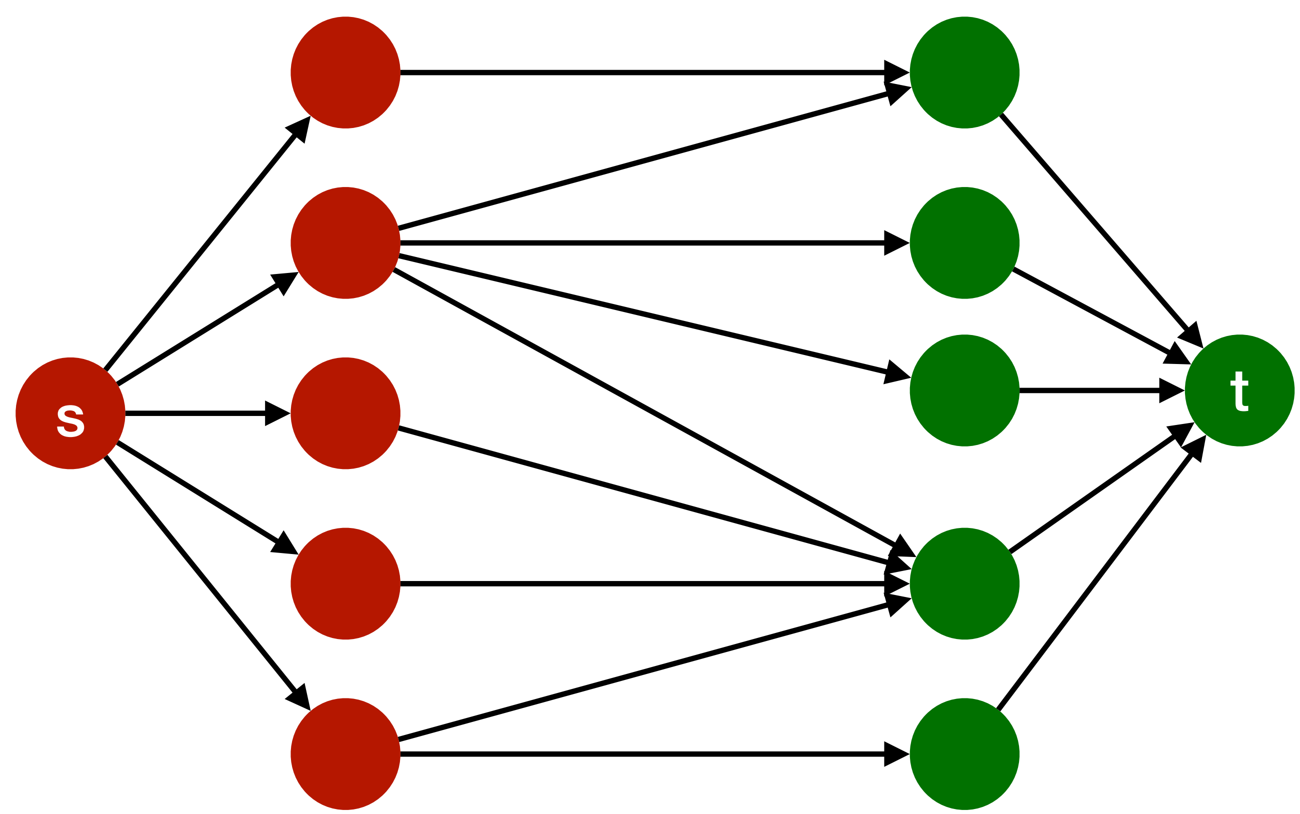
$f(e) = 0$, otherwise

This is a feasible flow and obviously has value k.

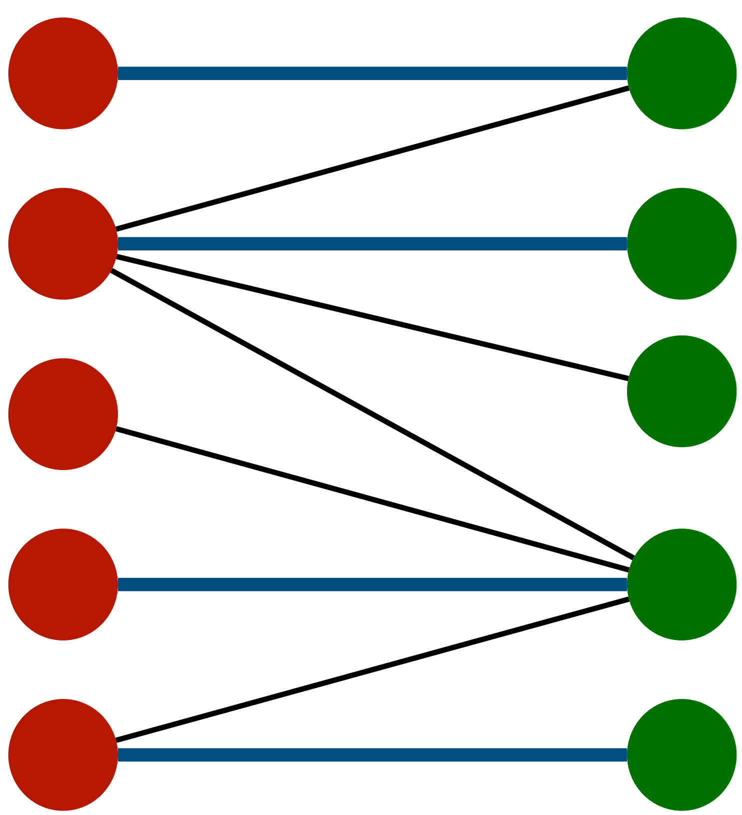
From matchings to flows



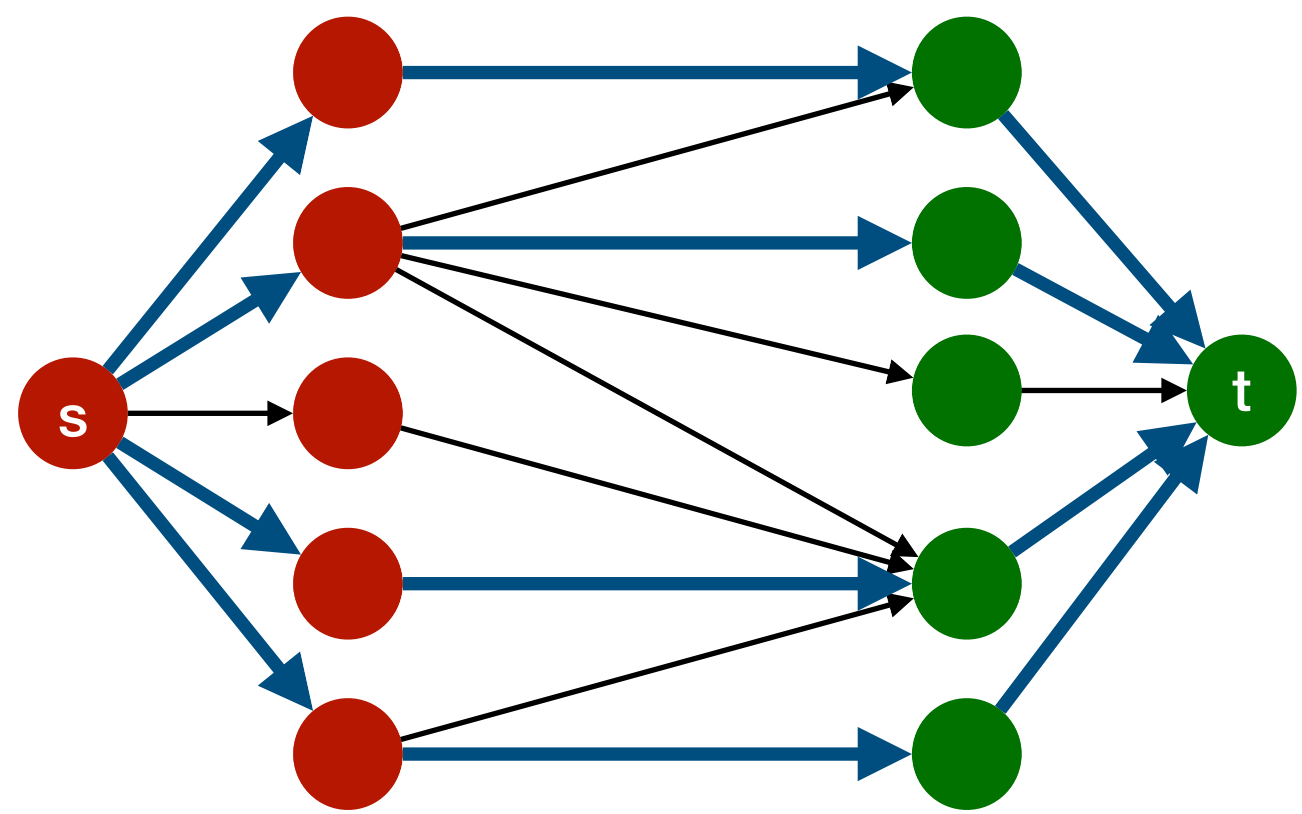
All capacities are set to 1.



From matchings to flows



All capacities are set to 1.



From flows to matchings

Claim: Assume that there is a a flow f of value k in G^f. Then there is a matching M of size k on G.

For an edge e, $f(e)$ is either *0* or *1*. (why?)

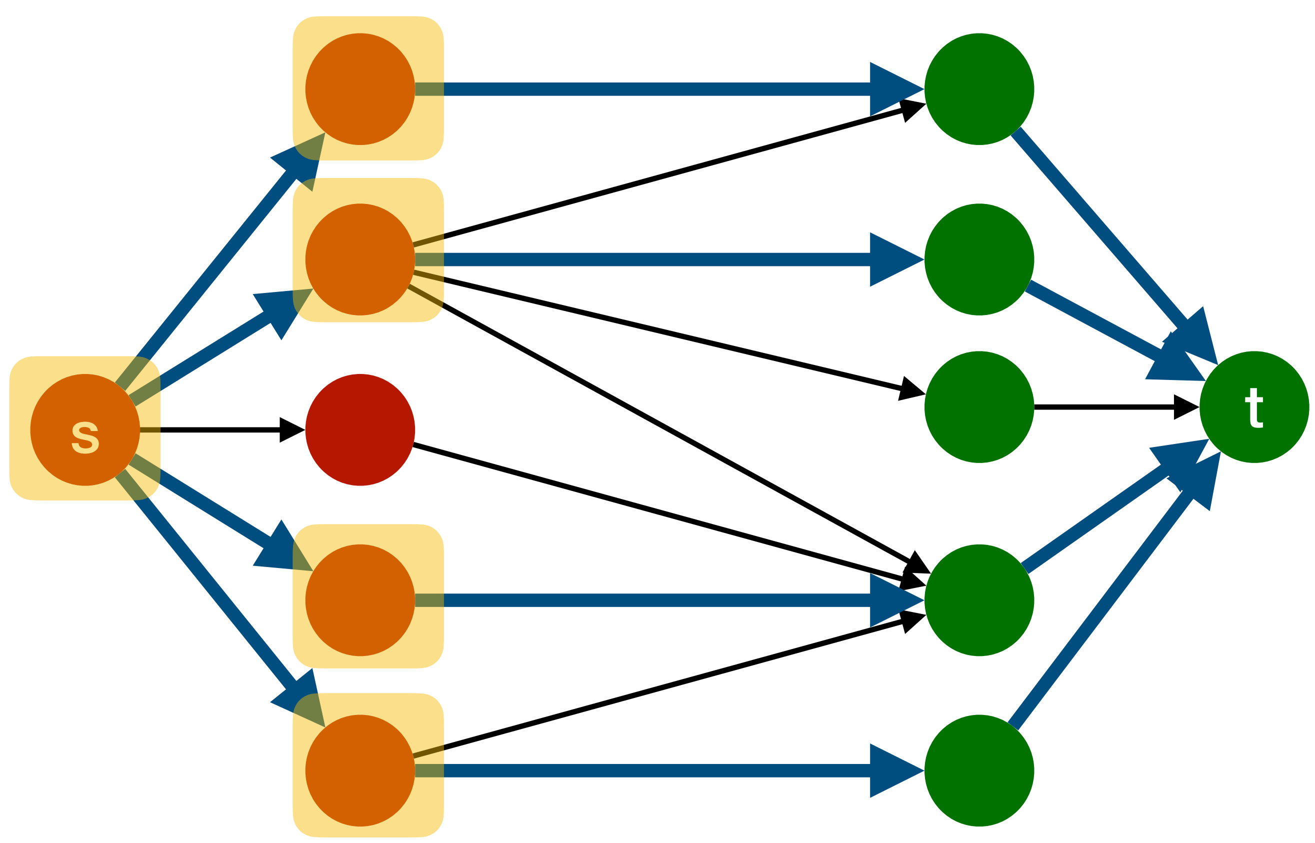
Consider the set M' of edges (of the middle level) with $f(e) = 1$.

From flows to matchings

Consider the set M’ of edges with f(e) = *1*.

Claim: |M’| = k.

All capacities are set to 1.

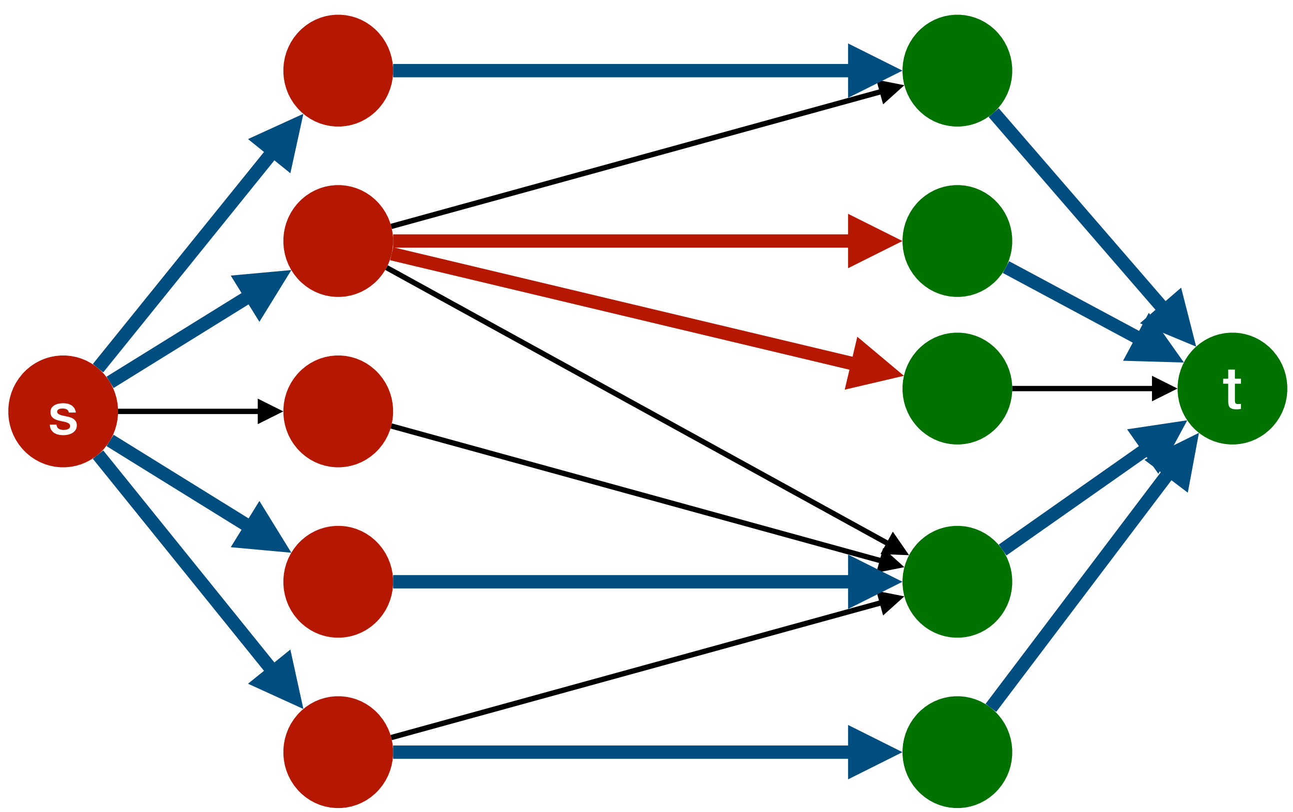


From flows to matchings

Consider the set M’ of edges with f(e) = *1*.

Claim: M’ is a matching.

All capacities are set to 1.



Maximum Flow and Maximum matching

The size of the maximum matching M in G is equal to the value of the maximum flow f in G^f.

The edges of M are the edges that carry flow from A to B in G^f.

What was the crucial part, that allows us to establish this?

The integrality theorem.

Running time

What is the running time of the algorithm?

By Edmonds - Karp, we get **O(**nm^2**).**

Running time

What is the running time of the algorithm?

By Ford-Fulkerson, we get **O($m$$F$).**

How large is F here?

It is at most $\max\{|L|, |R|\}$.

Running time **O(nm).**

Polynomial Time Reduction

We are given a problem A that we want to solve.

We can *reduce* solving problem A to solving some other problem B.

This is a transformation of an instance of problem A to an instance of problem B.

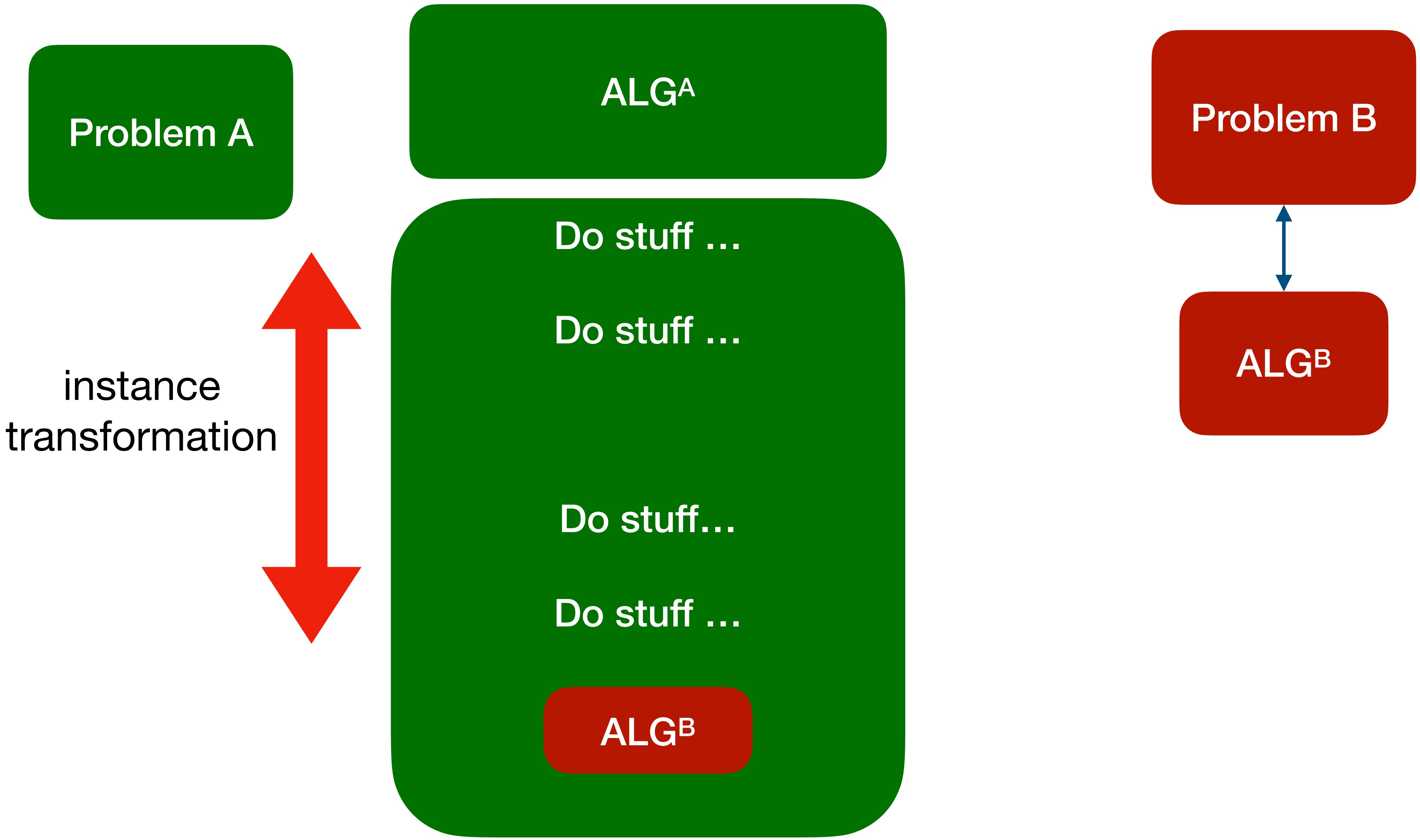
Assume that we had an algorithm ALG^B for solving problem B.

We can construct an algorithm ALG^A for solving problem A, which

- does the transformation,
- uses the algorithm ALG^B,
- transforms the solution back to a solution to problem A.

If ALG^A is a polynomial time algorithm, then this is a *polynomial time reduction*.

Pictorially



Polynomial Time Reduction

We are given **the maximum bipartite matching problem**

We can *reduce* solving **the MBP problem** to solving **the maximum flow problem**

This is a transformation of an instance of **the MBP problem** to an instance **the MF problem**

We have an algorithm ALG^B for solving **the MF problem**

We can construct an algorithm ALG^A for solving **the MBP problem** which

- does the transformation,
- uses the algorithm ALG^B,
- transforms the solution back to a solution to **the MBP problem**

If ALG^A is a polynomial time algorithm, then this is a *polynomial time reduction*.

Baseball Elimination

In the baseball league, there are 4 teams with the following number of wins:

New York	92
Baltimore	91
Toronto	91
Boston	90

Assume Boston wins all remaining games.

Baltimore and Toronto must win one game each.

New York must lose all remaining games.

Baltimore or Toronto must win one more game (BLT vs TOR).

There are five games left in the season.

NY vs BLT, NY vs TOR, BLT vs TOR, BLT vs BOS, TOR vs BOS

Question: Can Boston finish (possibly tied for) first?

The answer is no.

Baseball Elimination

In the baseball league, there are 4 teams with the following number of wins:

New York	90
Baltimore	88
Toronto	87
Boston	79

These are the games left in the season:

NY vs BLT

NY vs TOR ***6 games***

BLT vs TOR

BOS vs ANY ***4 games (12 games total)***

Question: Can Boston finish (possibly tied for) first?

Baseball Elimination

Generally:

We have a set S of teams.

For each team x in S, the current number of wins is w_x.

For teams x and y in S, they still have to play g_{xy} games against each other.

We are given a designated team z.

Can z win the tournament (possibly in a tie?)

From baseball to flows

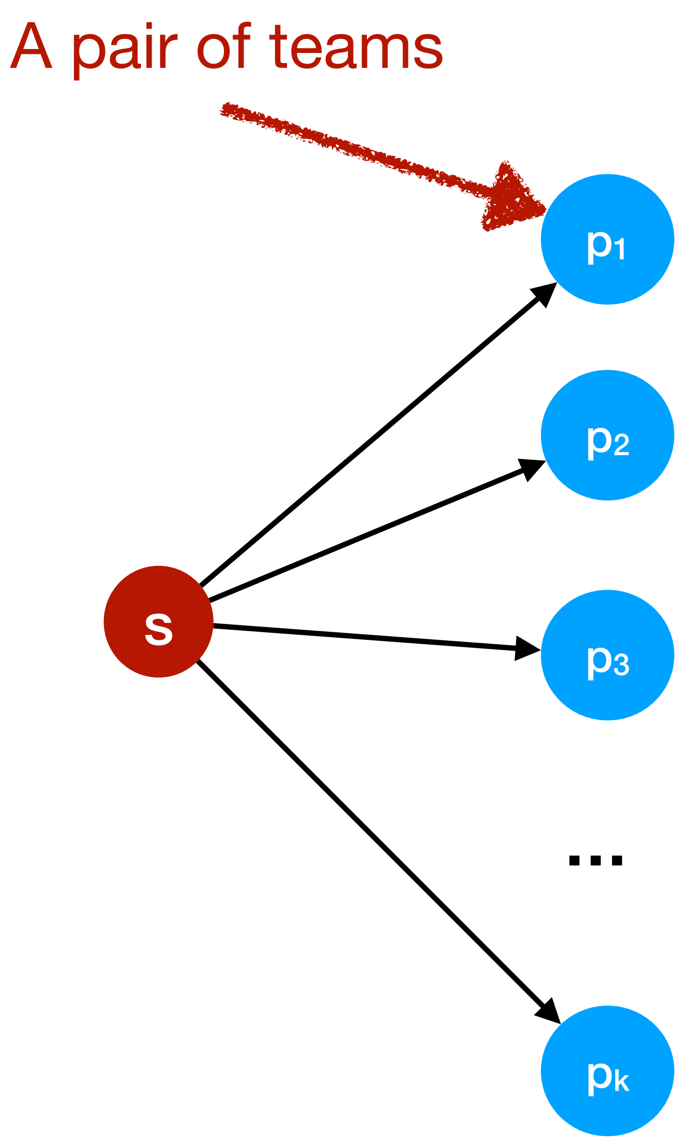
Observation: If there is a way for z to be first, there is a way for z to be first *when winning all remaining games*.

Suppose that in the end, team z has m wins.

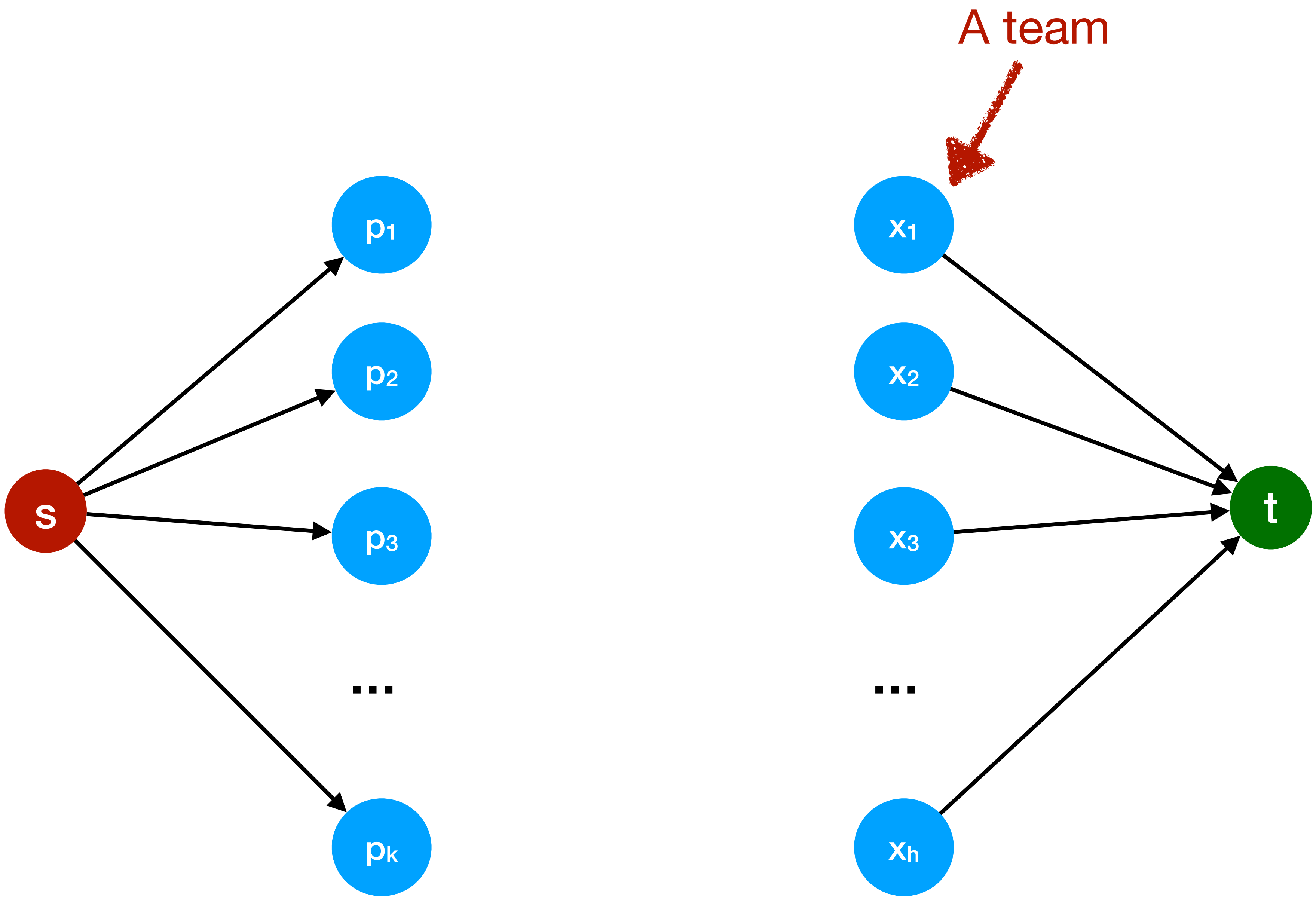
What are we looking for?

Is there an allocation of all the remaining g^* games (between the other teams) such that no team ends up with more than m wins?

From baseball to flows

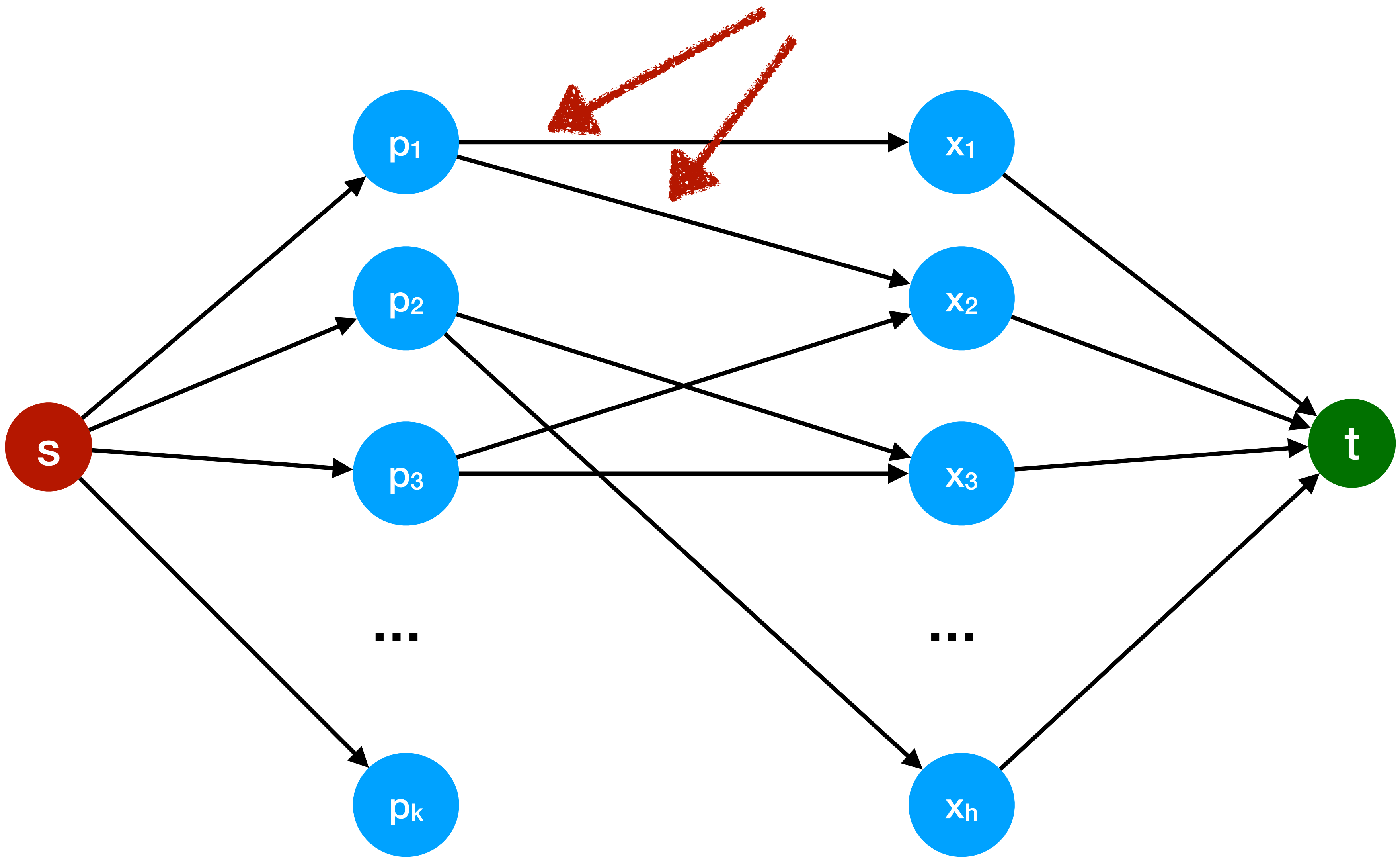


From baseball to flows

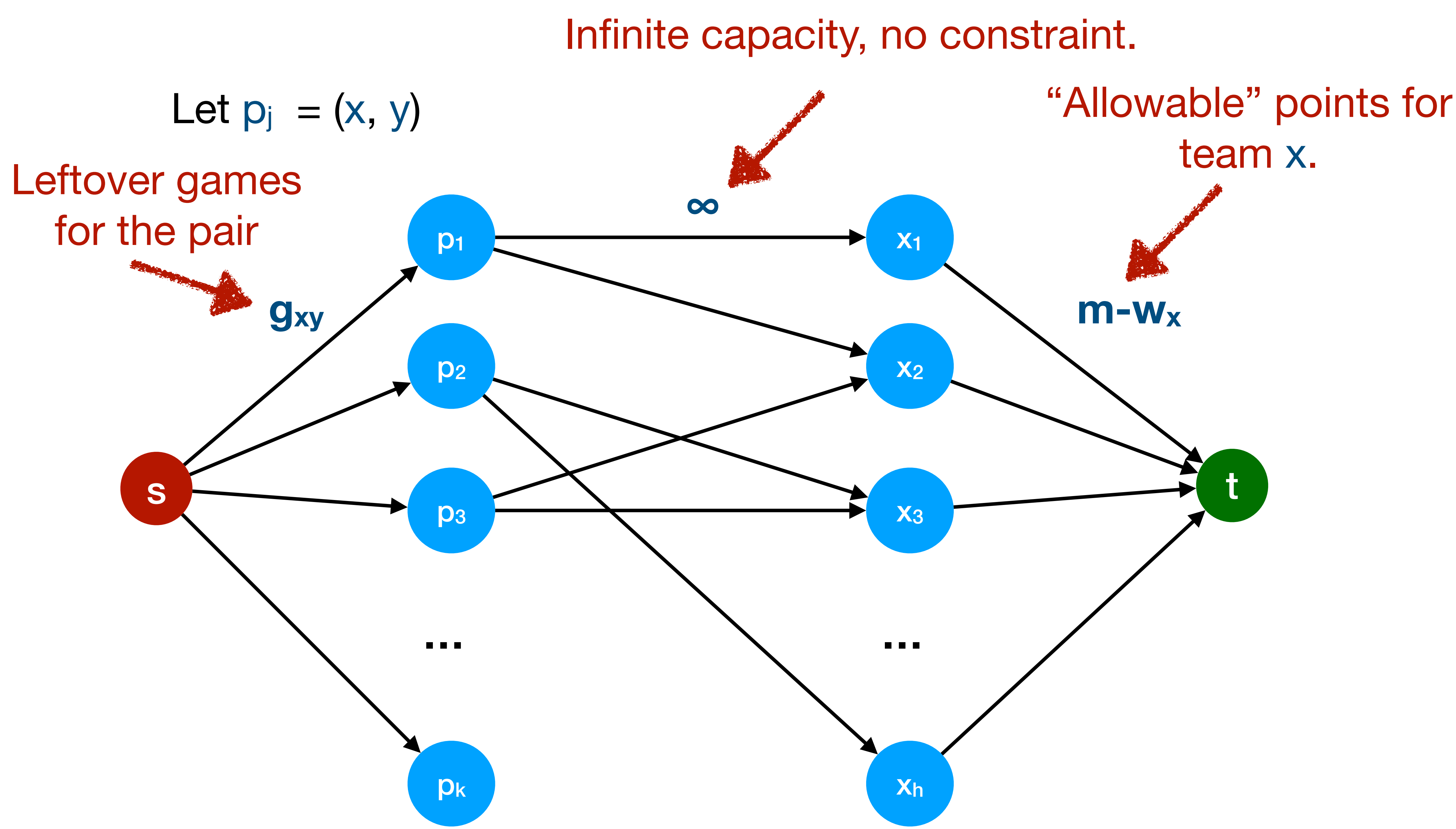


From baseball to flows

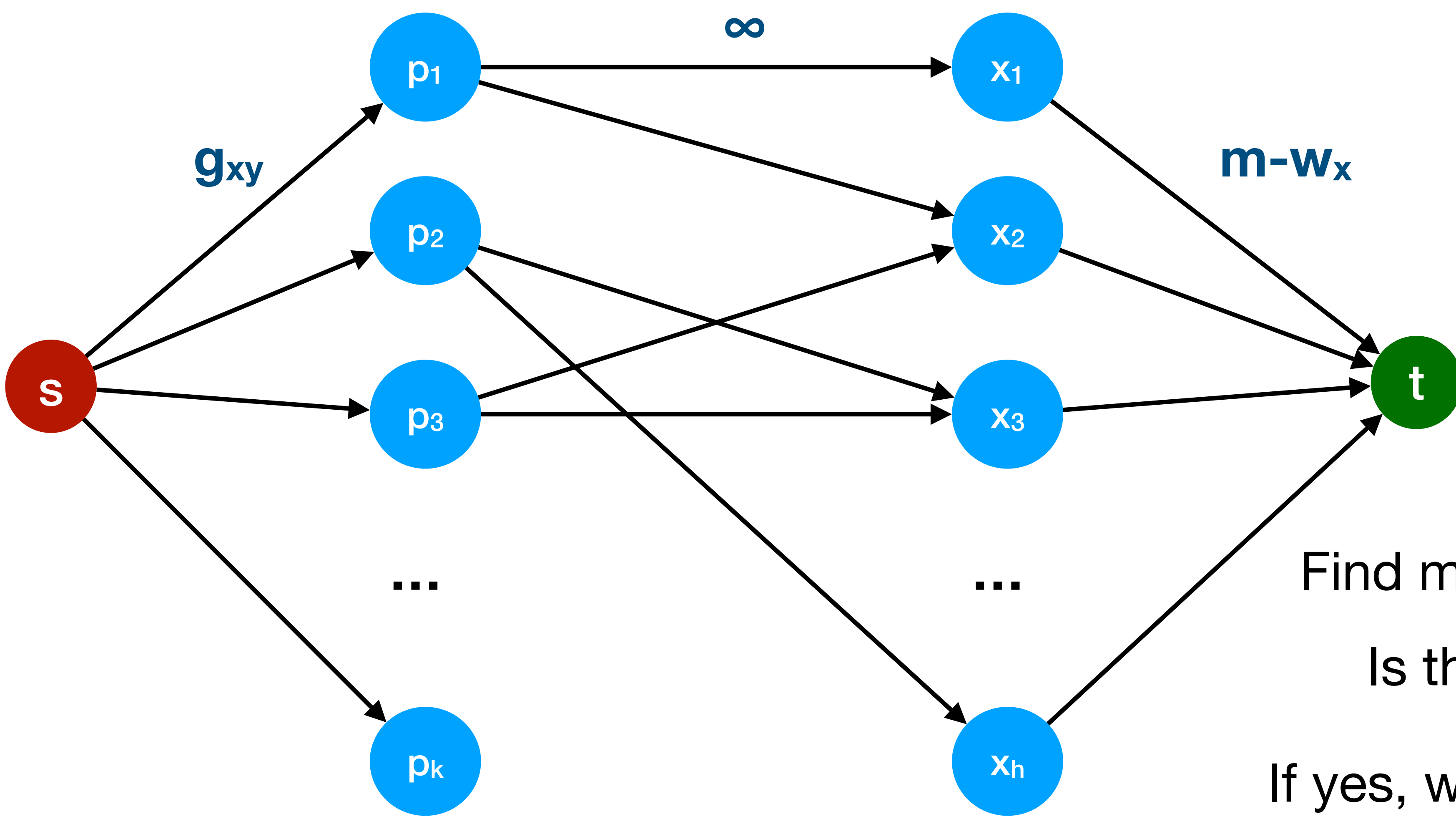
Two edges if teams in p_j still have games to play between them.



From baseball to flows



From baseball to flows



Find max flow in the network

Is the value at least g^*?

If yes, winning is still possible

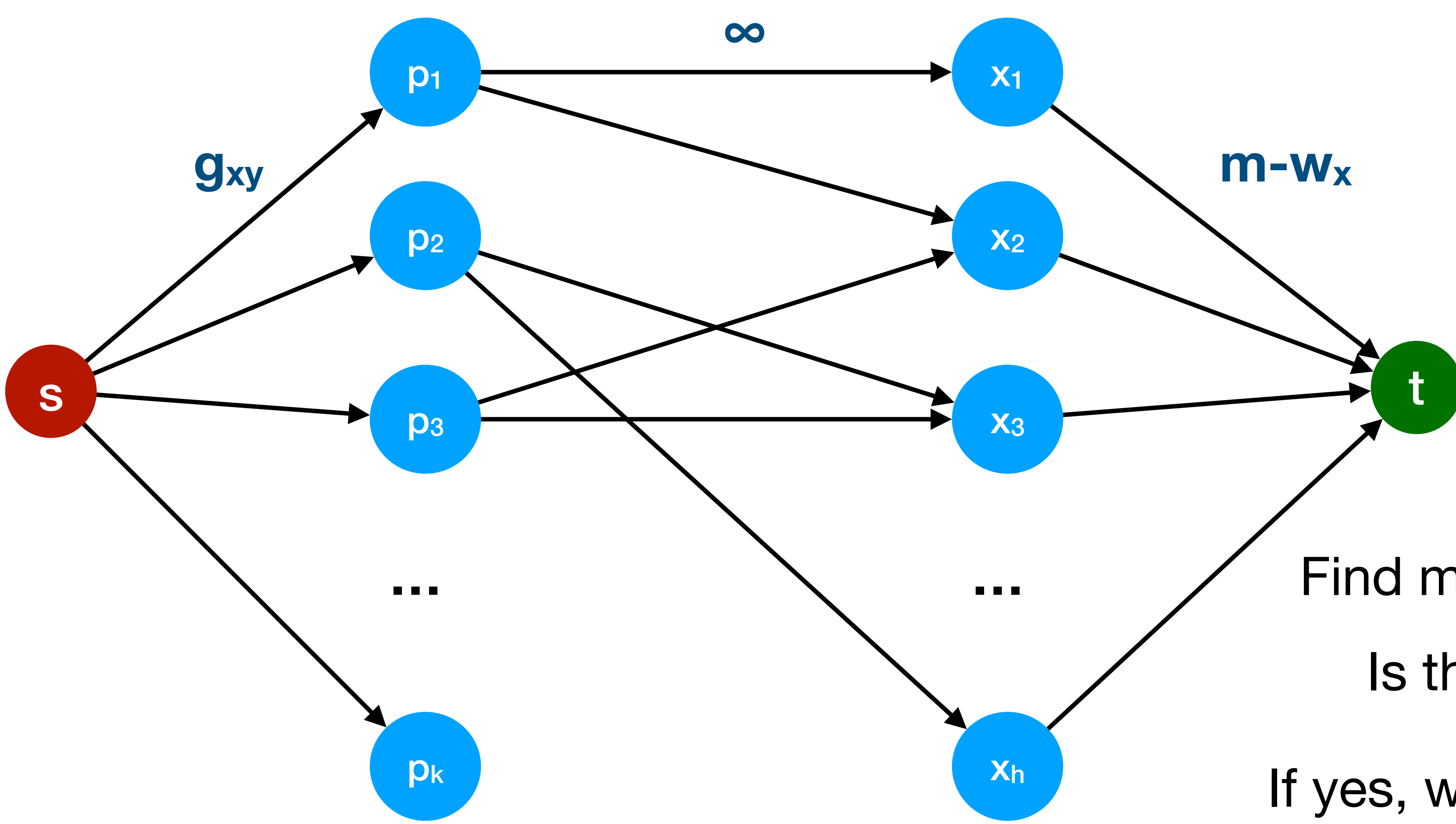
If no, winning is not possible

Why does this work?

Assume that the algorithm says yes.

The value of the flow is equal to the number of remaining games. (why?)

From baseball to flows



Find max flow in the network

Is the value at least g^*?

If yes, winning is still possible

If no, winning is not possible

Why does this work?

Assume that the algorithm says yes.

The value of the flow is equal to the number of remaining games. (why?)

The following hold:

A pair (x, y) will play exactly g_{xy} games.

A team x will win at most $m-w_x$ games.

Team z can win.

Why does this work?

Assume that the algorithm says no.

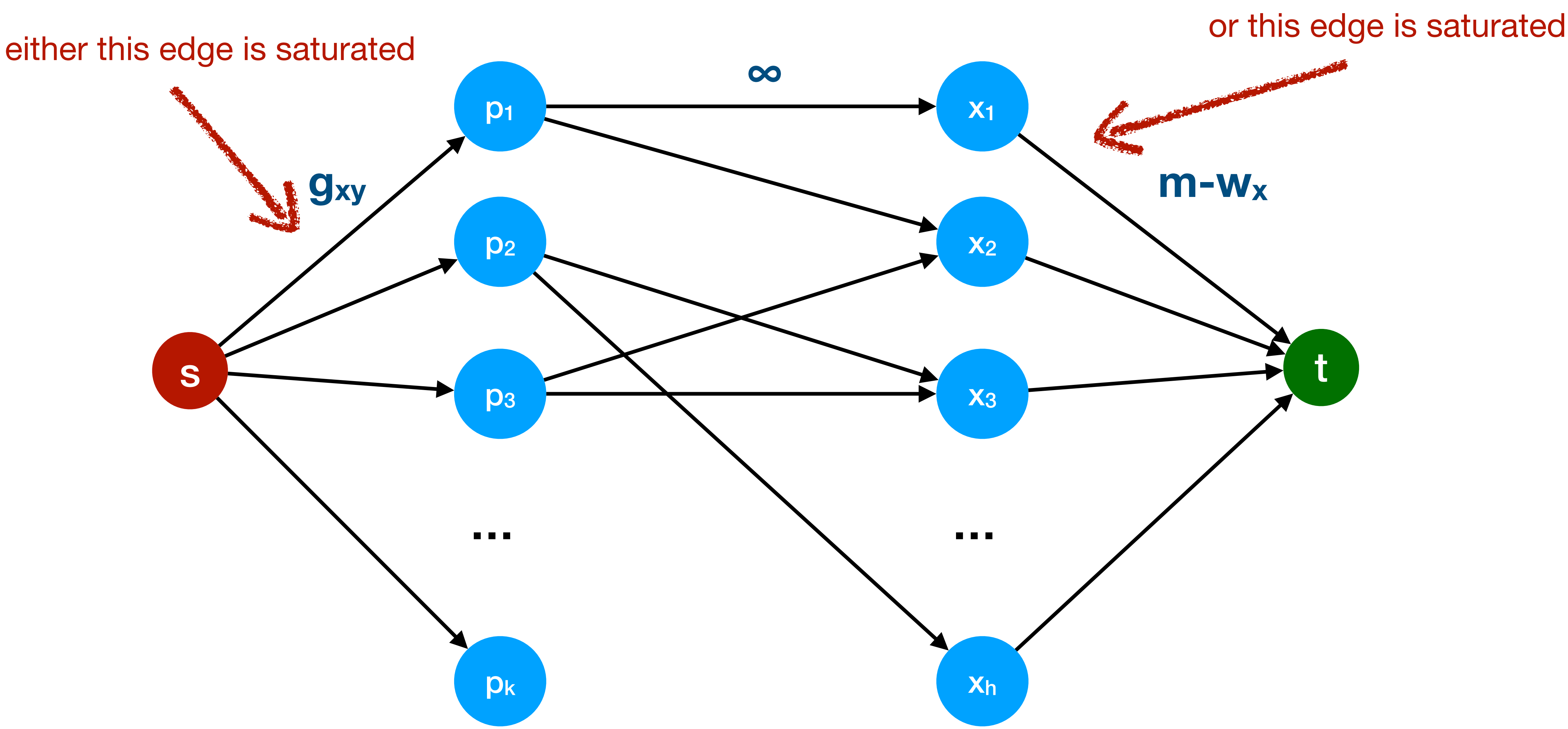
The maximum flow has value < g^*.

It is not possible to play all the remaining games without giving some team x more than $m - w_x$ points.

Team z cannot win.

Another way to think about it

Let's look at the final residual graph. Why do we have no augmenting paths?



Either all the games have been played, or some team cannot win any more games.

Example

In the baseball league, there are 4 teams with the following number of wins:`

New York	90
Baltimore	88
Toronto	87
Boston	79

There are five games left in the season.

NY vs BLT

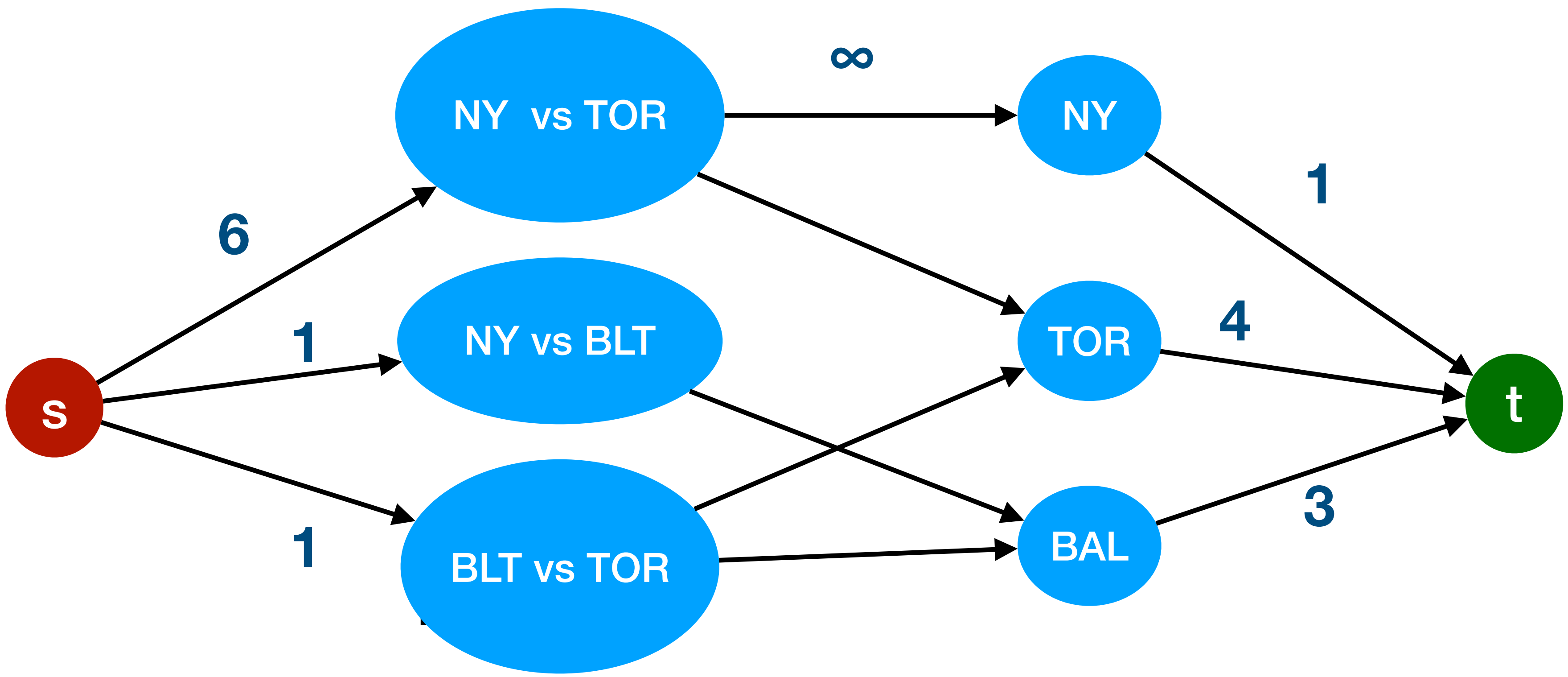
NY vs TOR ***6 games***

BLT vs TOR

BOS vs ANY ***4 games (12 games total)***

Question: Can Boston finish (possibly tied for) first?

Example



m = 91

Open pit mining

We extract blocks of earth from the surface, trying to find gold.

Each block z that we mine has

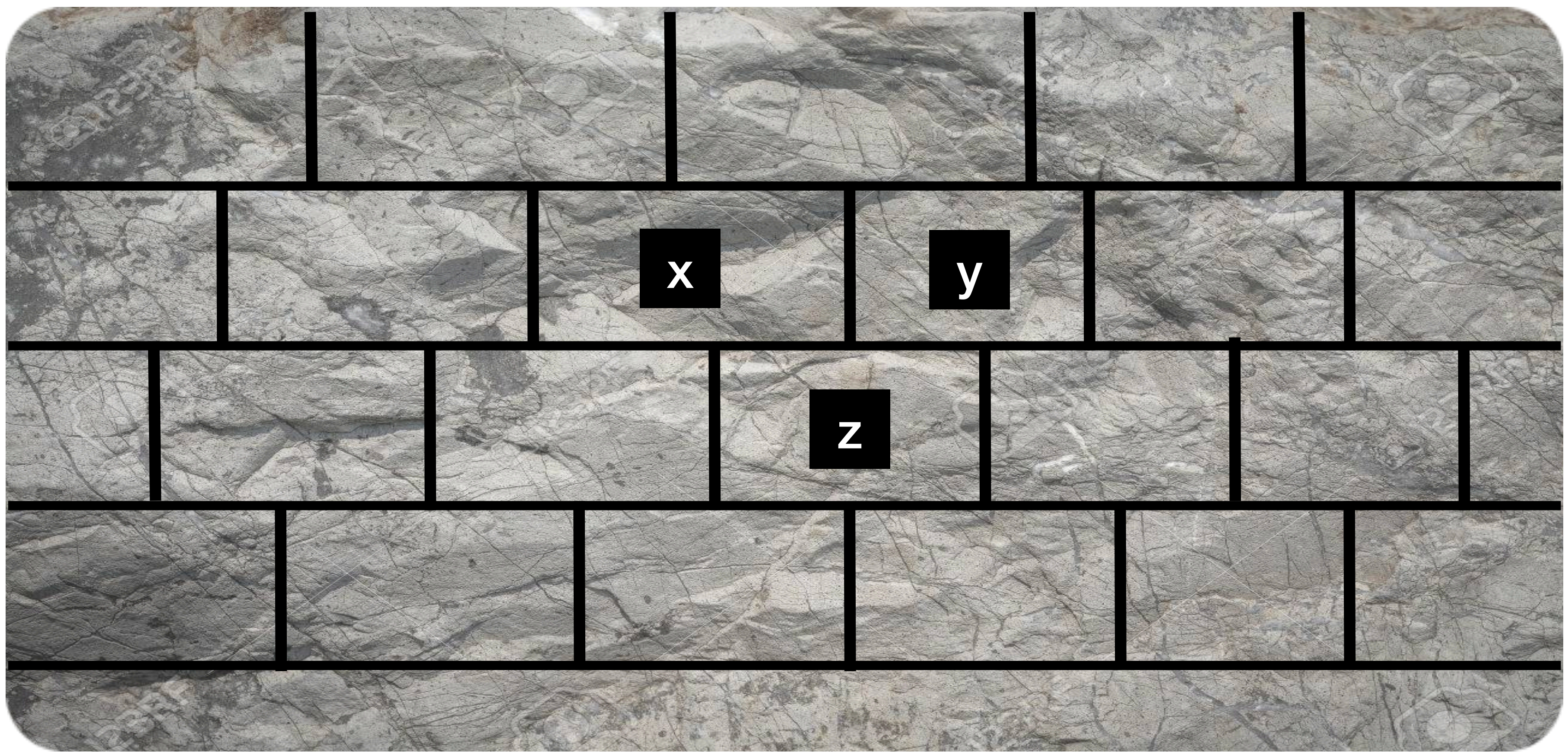
a value p_z

a mining cost c_z

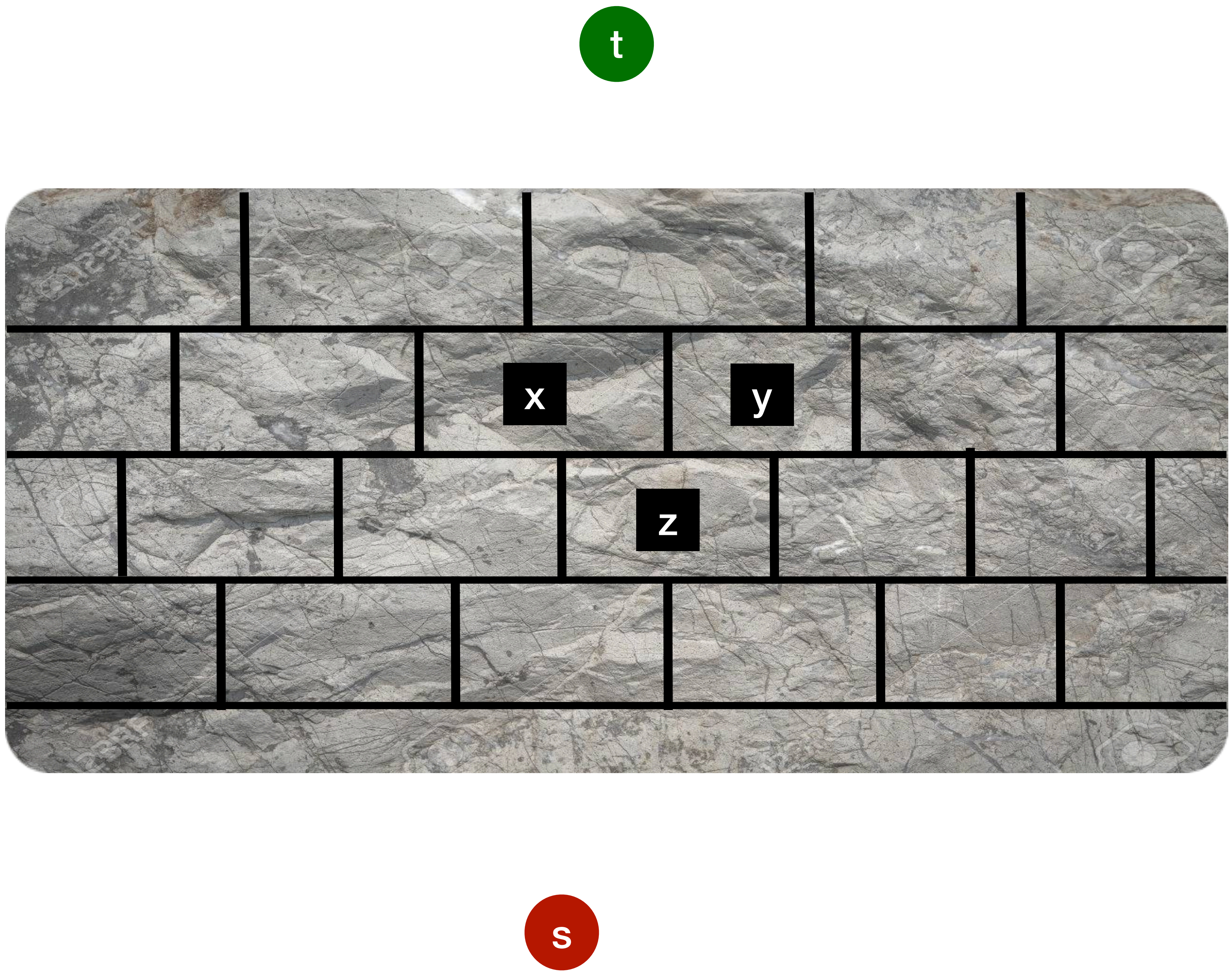
Constraint: We can not mine a block z unless we mine the two blocks x and y on top of it.

We want to earn as much money as possible.

Open pit mining

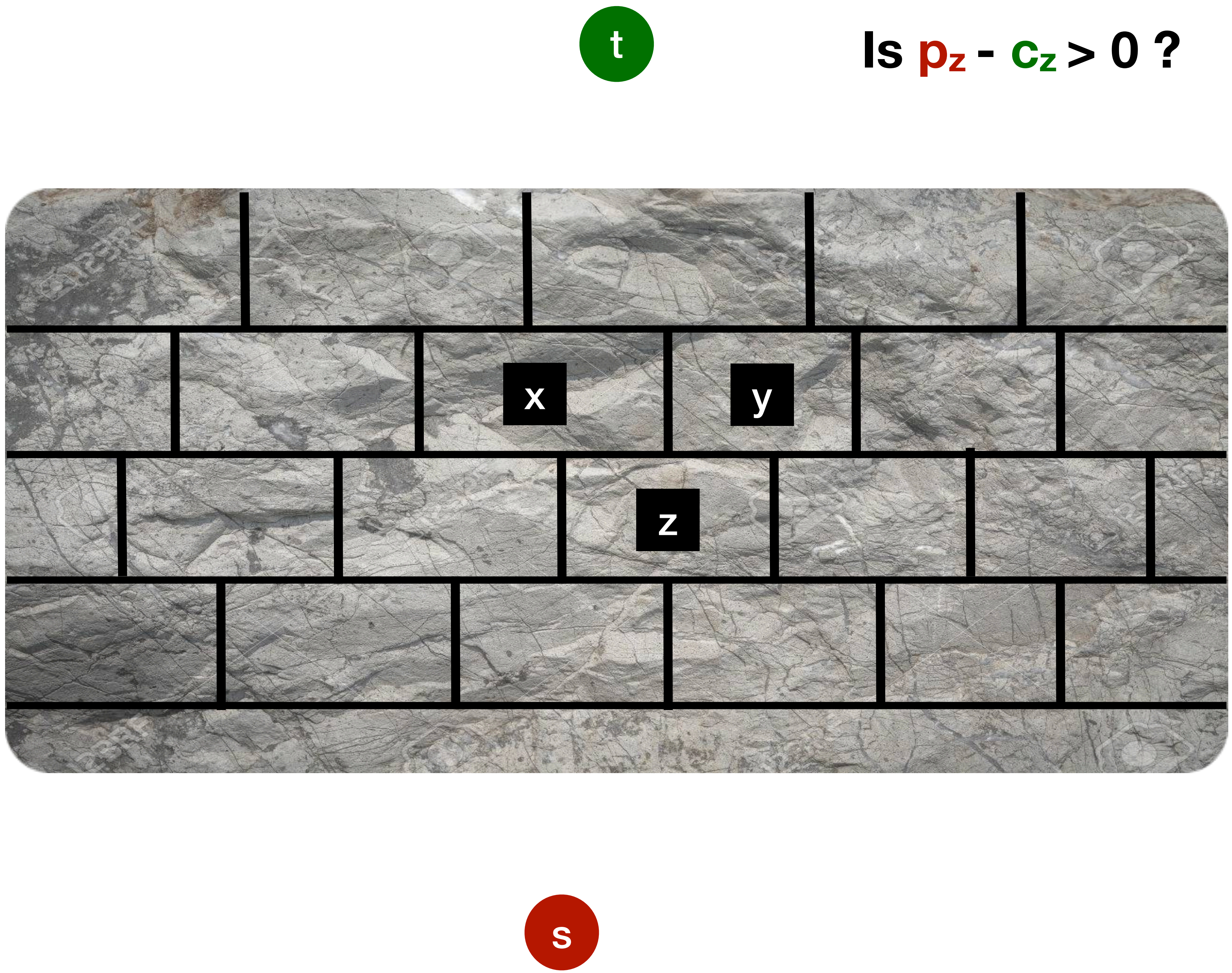


From pits to flows

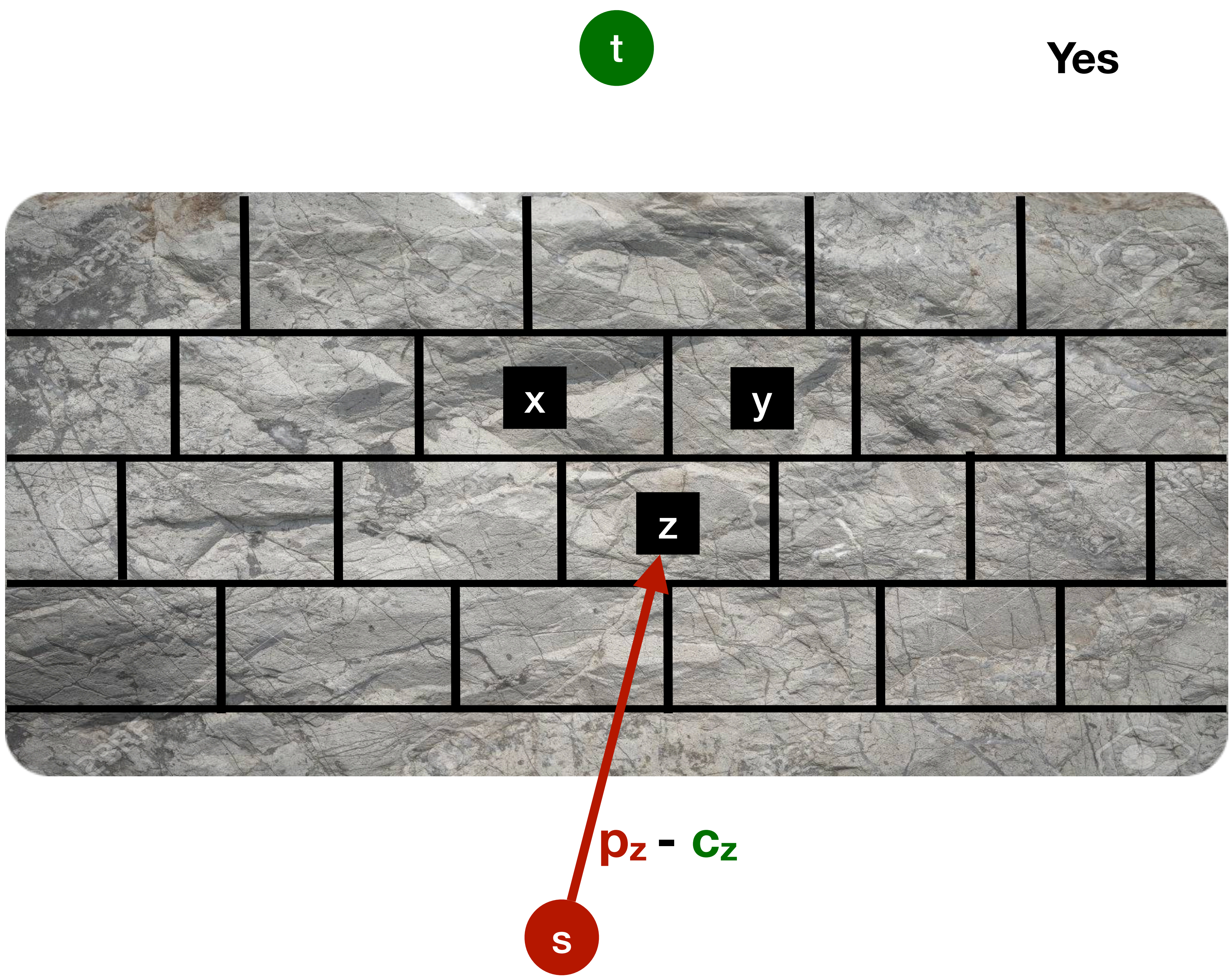


From pits to flows

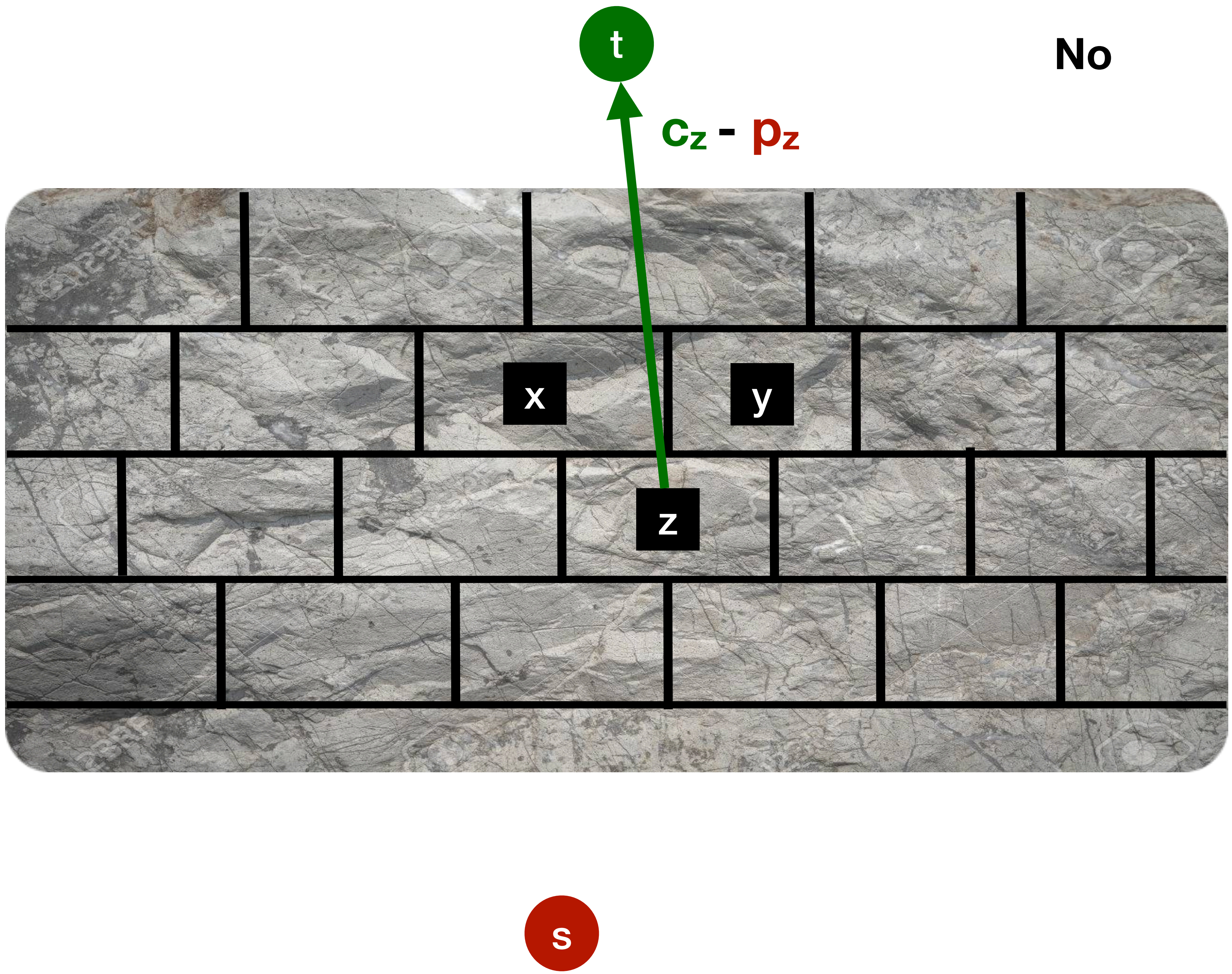
Is $p_z - c_z > 0$?



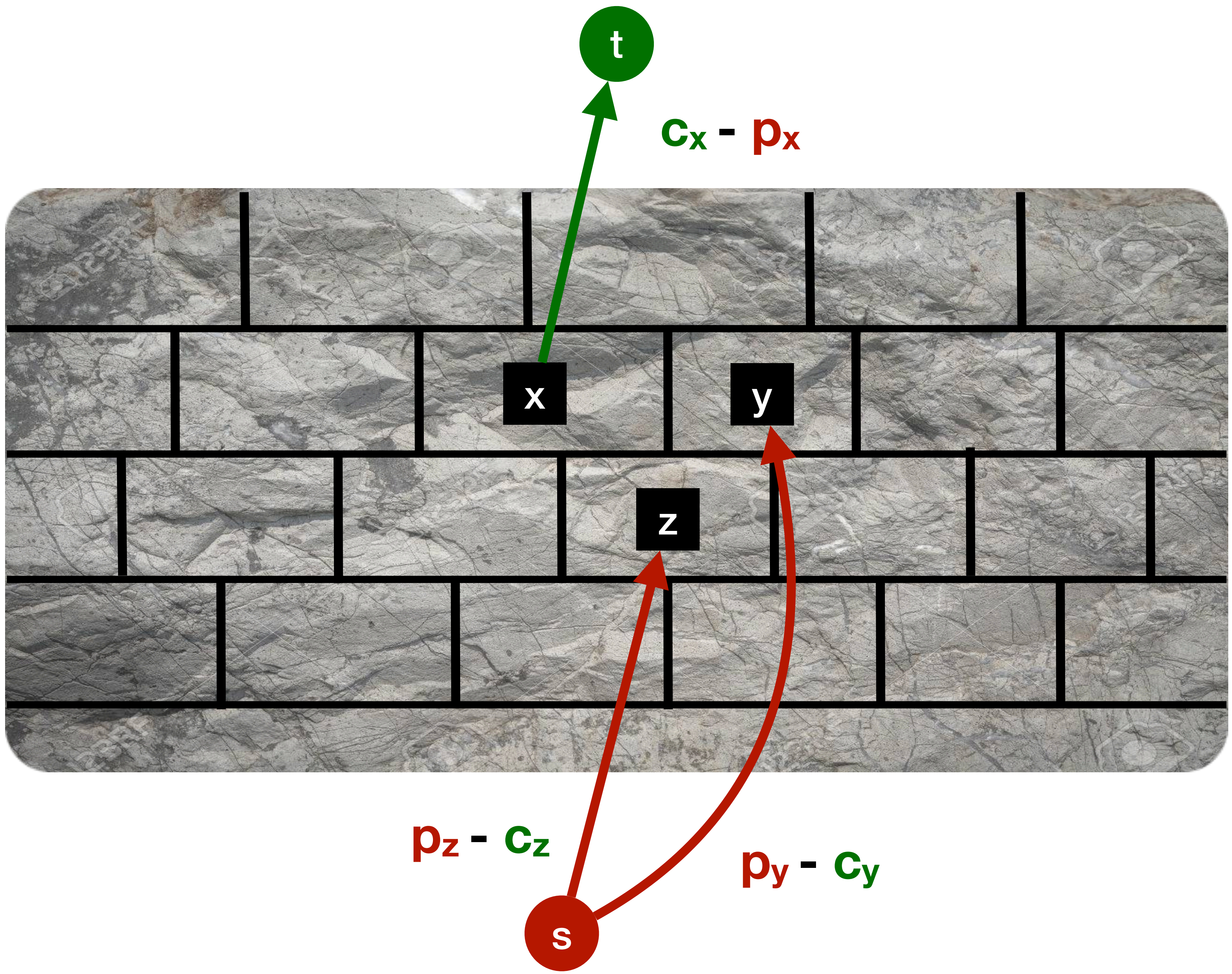
From pits to flows



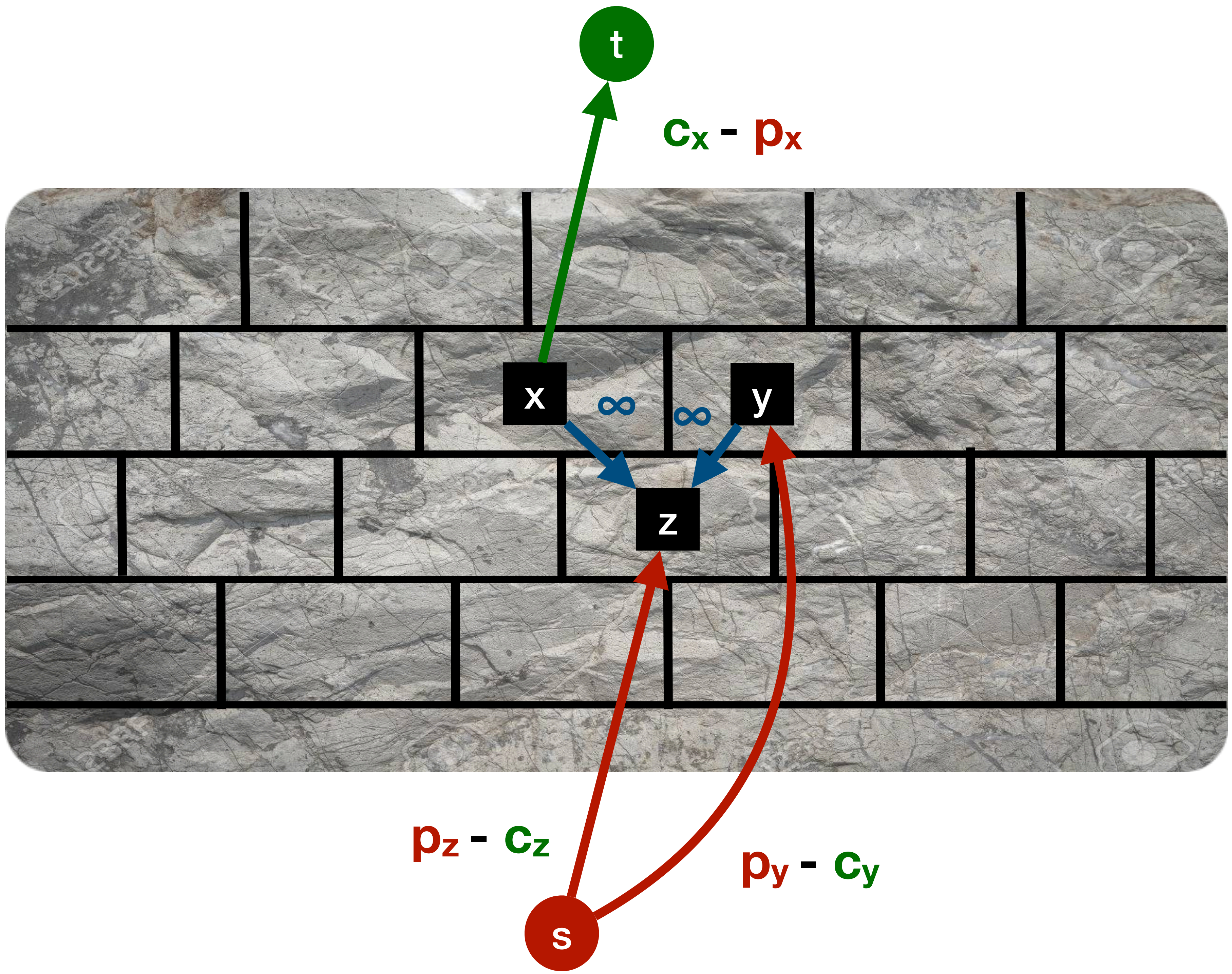
From pits to flows



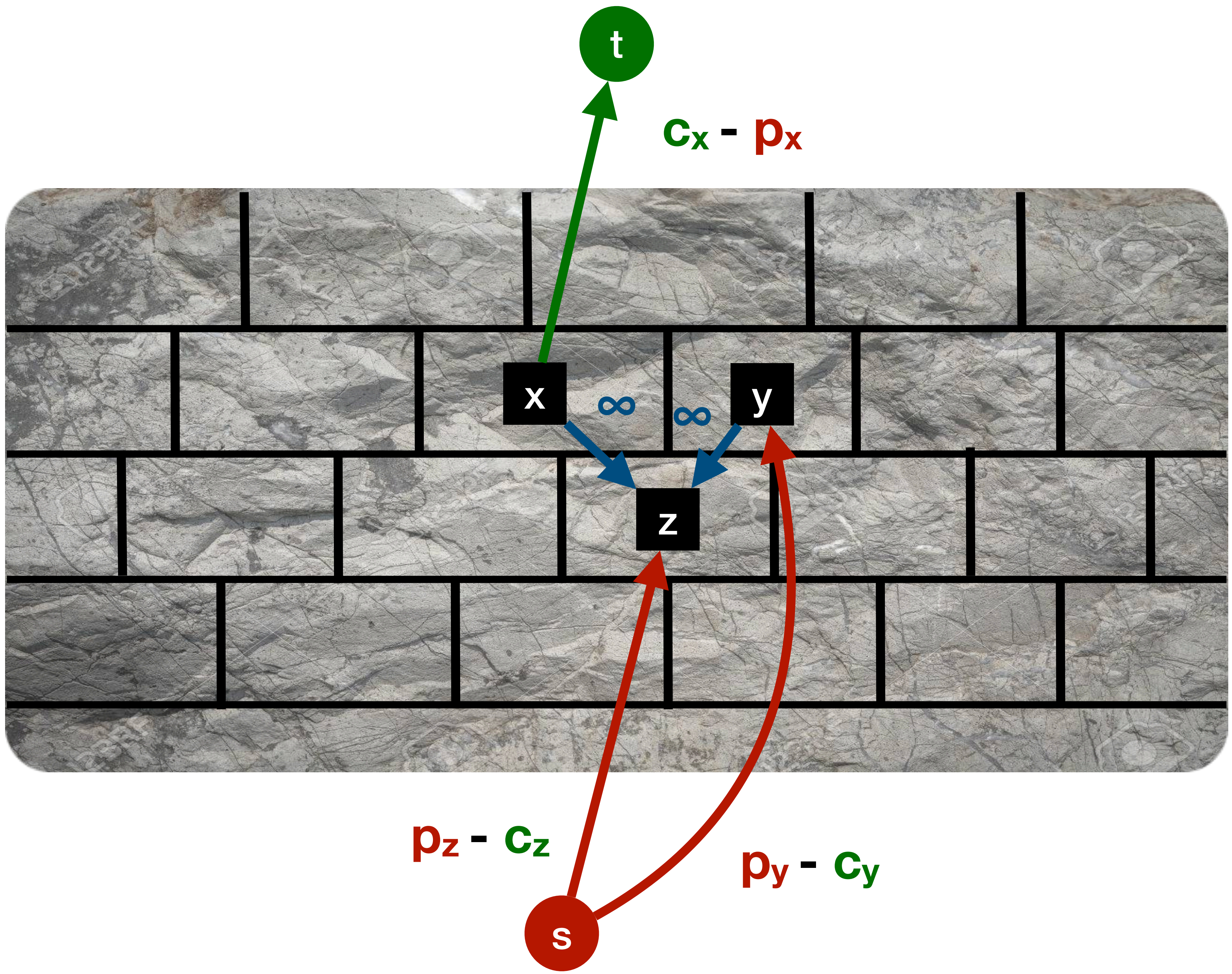
From pits to flows



From pits to flows



From pits to cuts



From pits to cuts

Consider an (S, T) cut C.

We will mine S - {s}.

If C is minimum, we cannot have nodes that are connected with an *infinite capacity* edge in different sides of the cut.

If S contains z, it must contain x and y (needed to mine z).

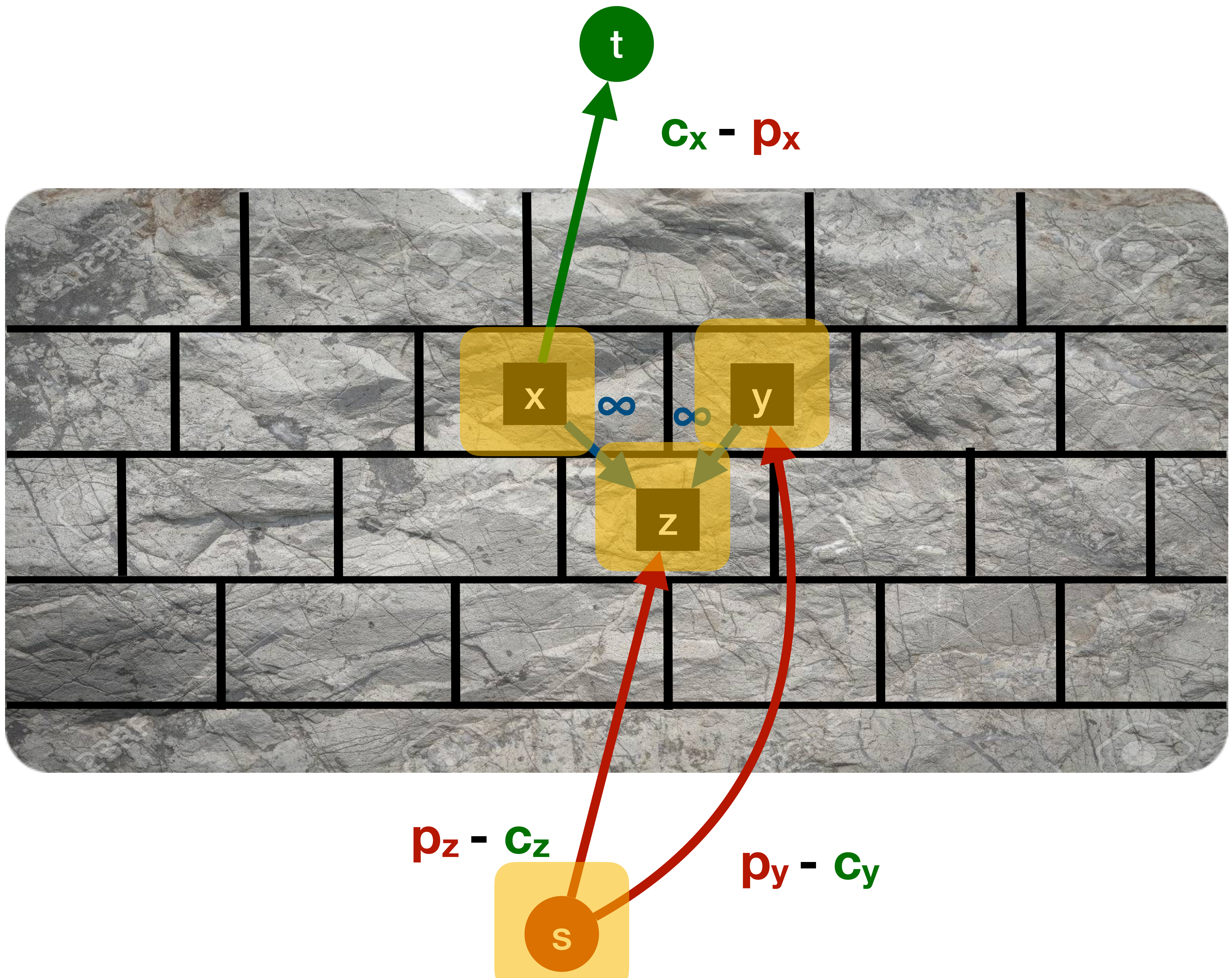
Feasibility guaranteed by the above fact.

Optimality?

Optimality of our mining set

$$c(S,T) = \sum_{z \in T \,:\, p_z - c_z > 0} (p_z - c_z) + \sum_{z \in S \,:\, p_z - c_z < 0} (c_z - p_z)$$

From pits to cuts

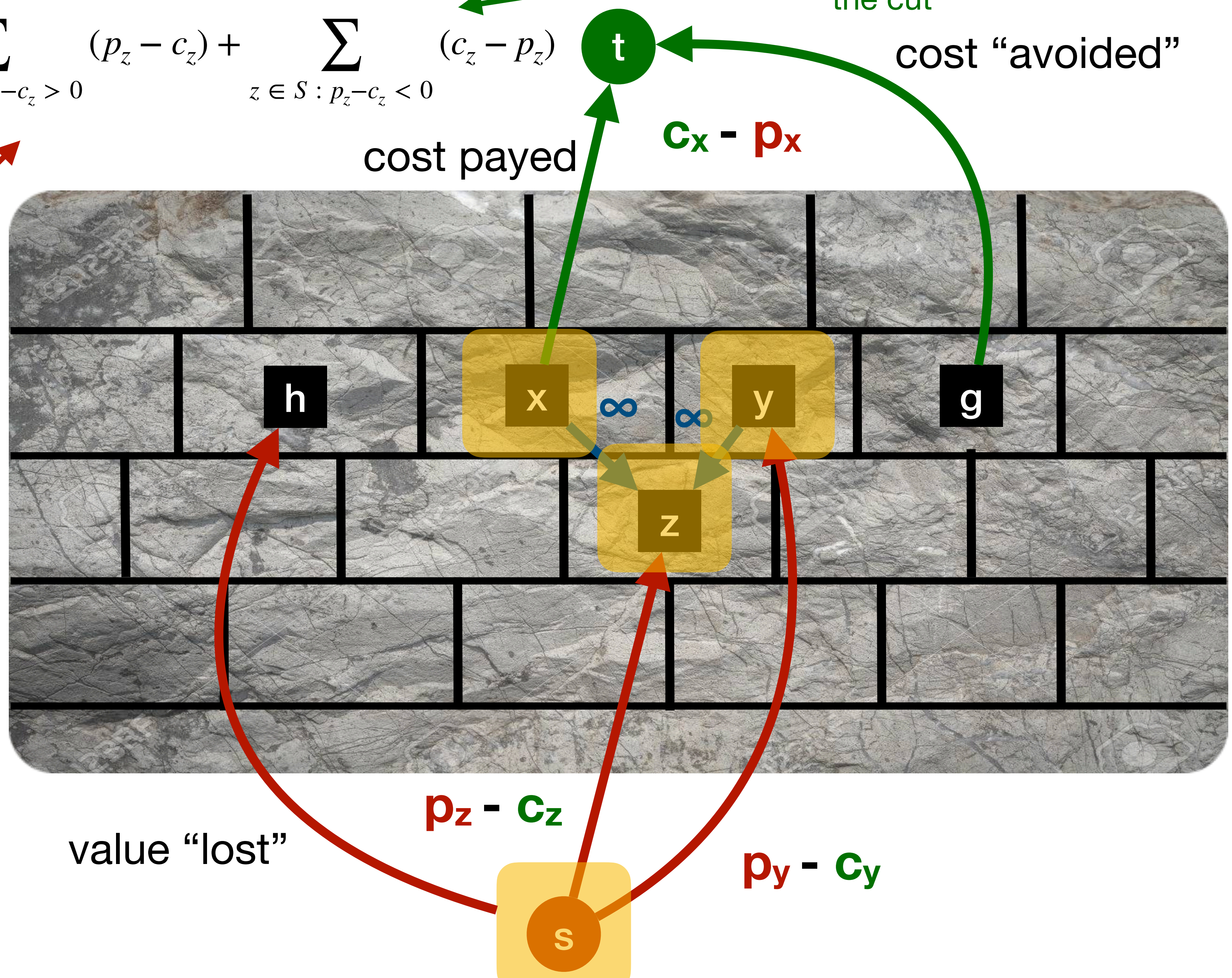


From pits to cuts

$$c(S,T) = \sum_{z \in T\,:\,p_z - c_z > 0} (p_z - c_z) + \sum_{z \in S\,:\,p_z - c_z < 0} (c_z - p_z)$$

Sum of capacities
of green edges crossing
the cut

Sum of capacities
of red edges crossing
the cut.



Optimality of our mining set.

$$c(S,T) = \sum_{z \in T \,:\, p_z - c_z > 0} (p_z - c_z) + \sum_{z \in S \,:\, p_z - c_z < 0} (c_z - p_z)$$

$$c(S,T) = \sum_{z \in T \,:\, p_z - c_z > 0} (p_z - c_z) - \sum_{z \in S \,:\, p_z - c_z < 0} (p_z - c_z)$$

Add and subtract this: $$c(S,T) = \sum_{z \in S \,:\, p_z - c_z > 0} (p_z - c_z)$$

$$c(S,T) = \underbrace{\boxed{\sum_{z \in V \,:\, p_z - c_z > 0} (p_z - c_z)}}_{\text{constant}} - \underbrace{\boxed{\sum_{z \in S} (p_z - c_z)}}_{\text{Mining profit}}$$

Open-pit mining - Summarising

Construct the flow network.

Run Ford-Fulkerson to find a maximum flow.

Find a minimum cut using the final residual graph.

Mine the blocks in the S part of the cut.