Algorithms and Data Structures

Introduction to Linear Programming

A problem

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The available processing time on machine A is *40 hours* and on machine B it is *35 hours*.

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The available processing time on machine A is *40 hours* and on machine B it is *35 hours*.

The demand for X in the week is *75 units* and for Y it is *95 units*.

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Goal: Maximise the combined sum of units of X and Y in stock at the end of the week.

A linear program

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Maximise $(x + 30 - 75) + (y + 90 - 95)$

A linear program

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$$50x + 24y \leq 2400 \qquad x \geq 75 - 30$$

$$30x + 33y \leq 2100 \qquad y \geq 95 - 90$$

A linear program

Maximise $x + y - 50$

subject to

$$50x + 24y \leq 2400$$
$$30x + 33y \leq 2100$$
$$x \geq 45$$
$$y \geq 5$$

Linear programming (LP)

$$\text{maximise} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1\in}^{n} \alpha_{ij} x_j \leq b_i, \quad i = 1, ..., m$$

$$x_j \geq 0, \quad j = 1, ..., n$$

Linear programming (in matrix form)

$$\begin{aligned} \text{maximise} \quad & c^{\mathrm{T}} x \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \end{aligned}$$

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Geometric Interpretation

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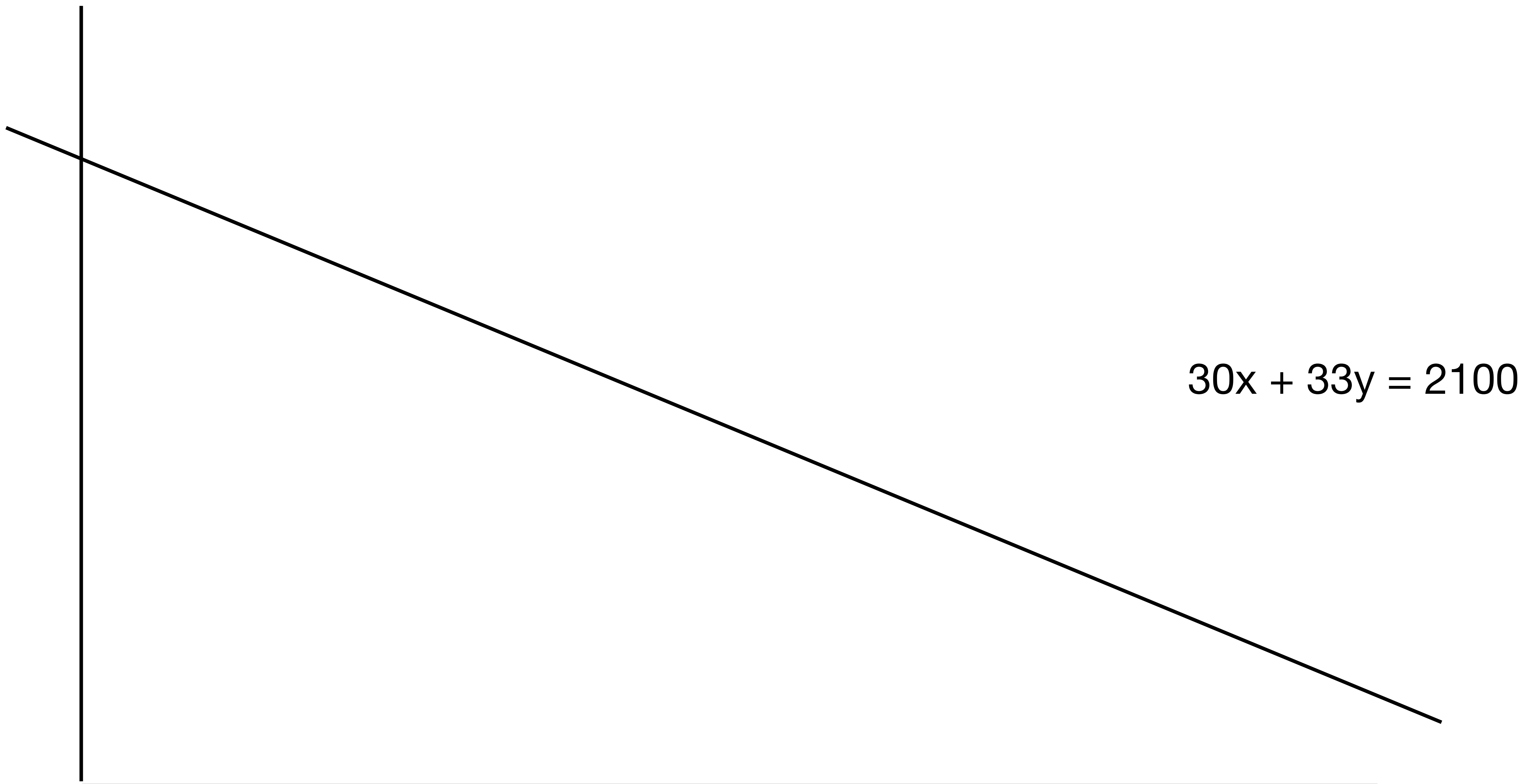

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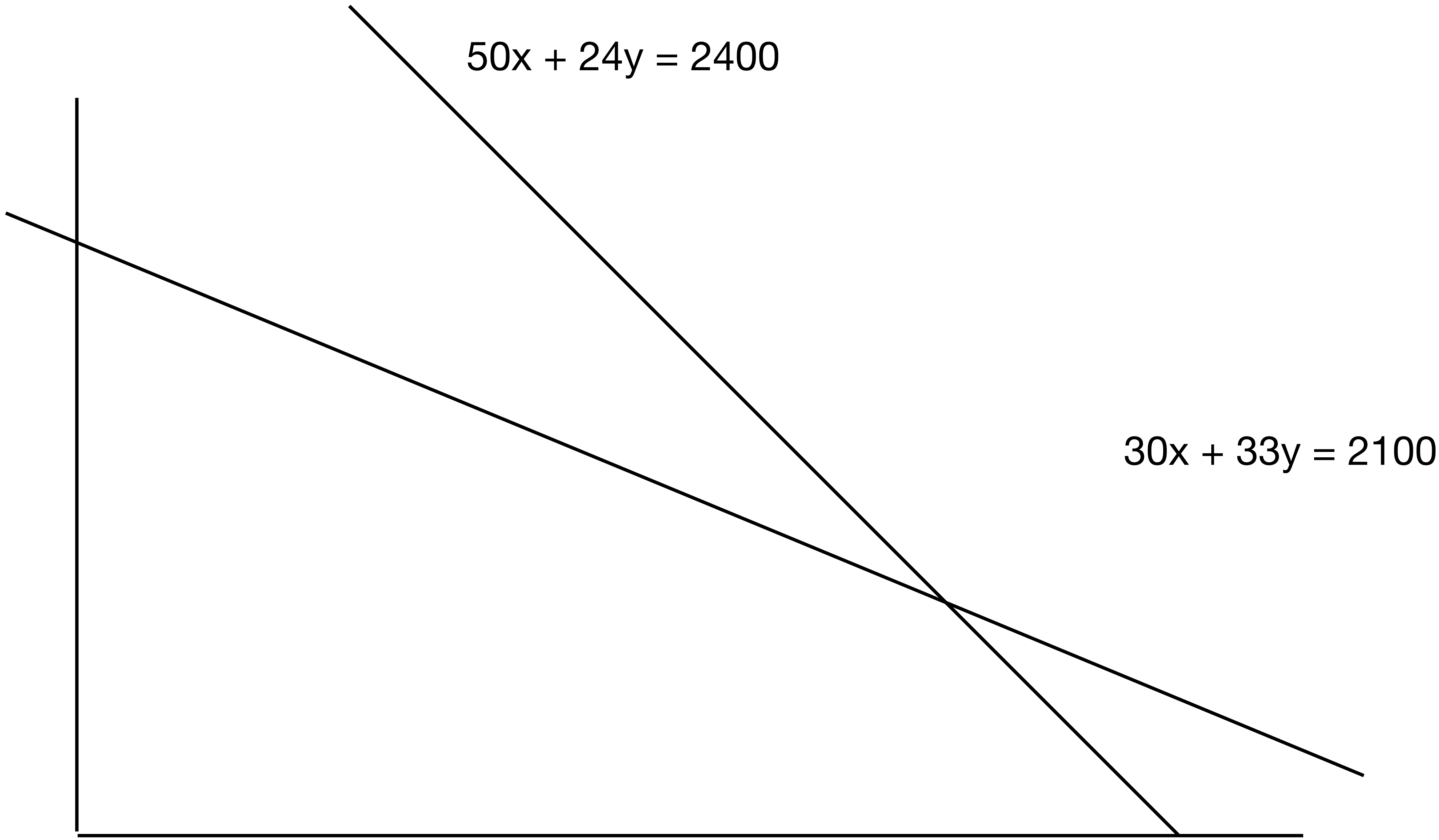

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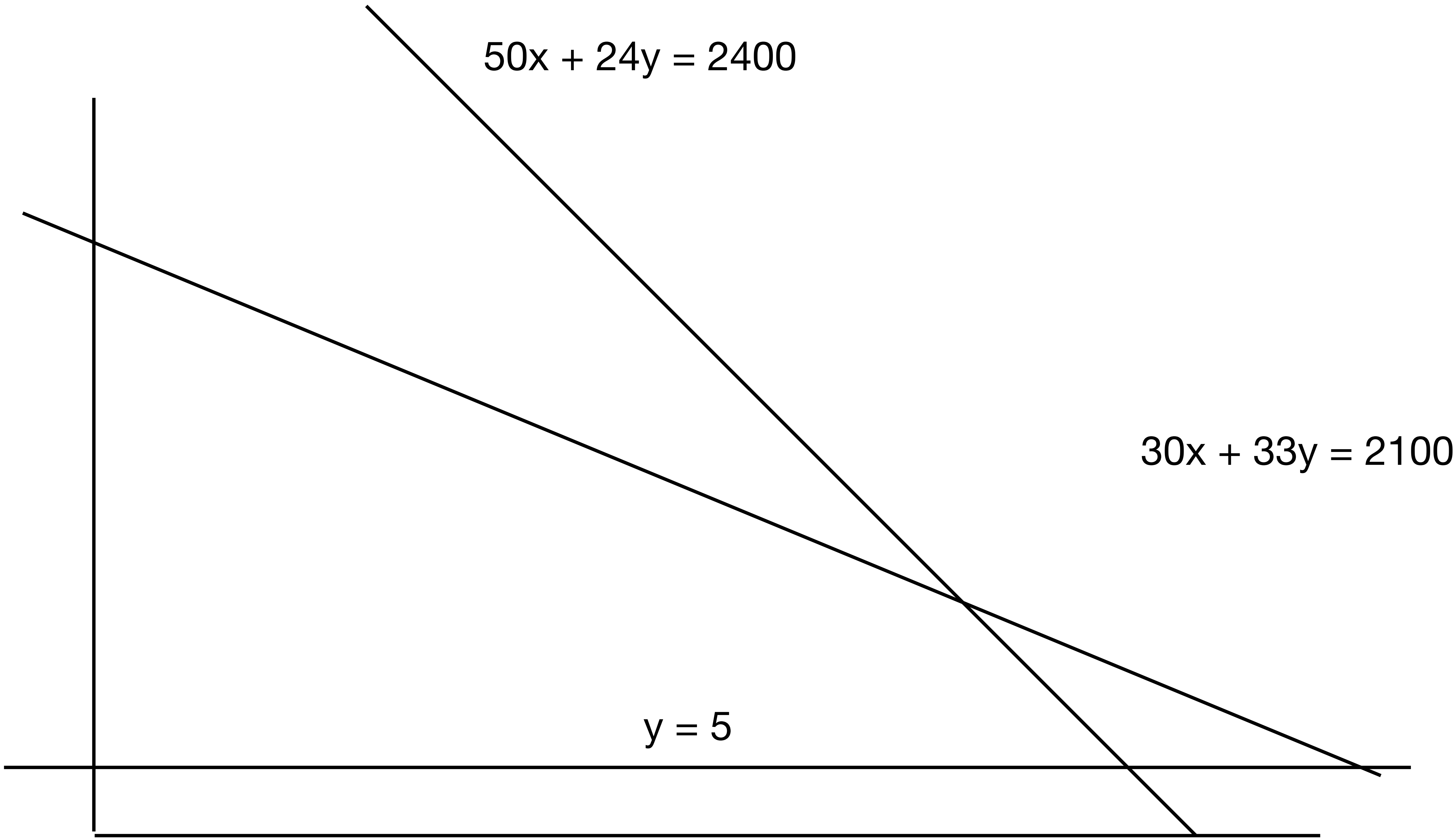

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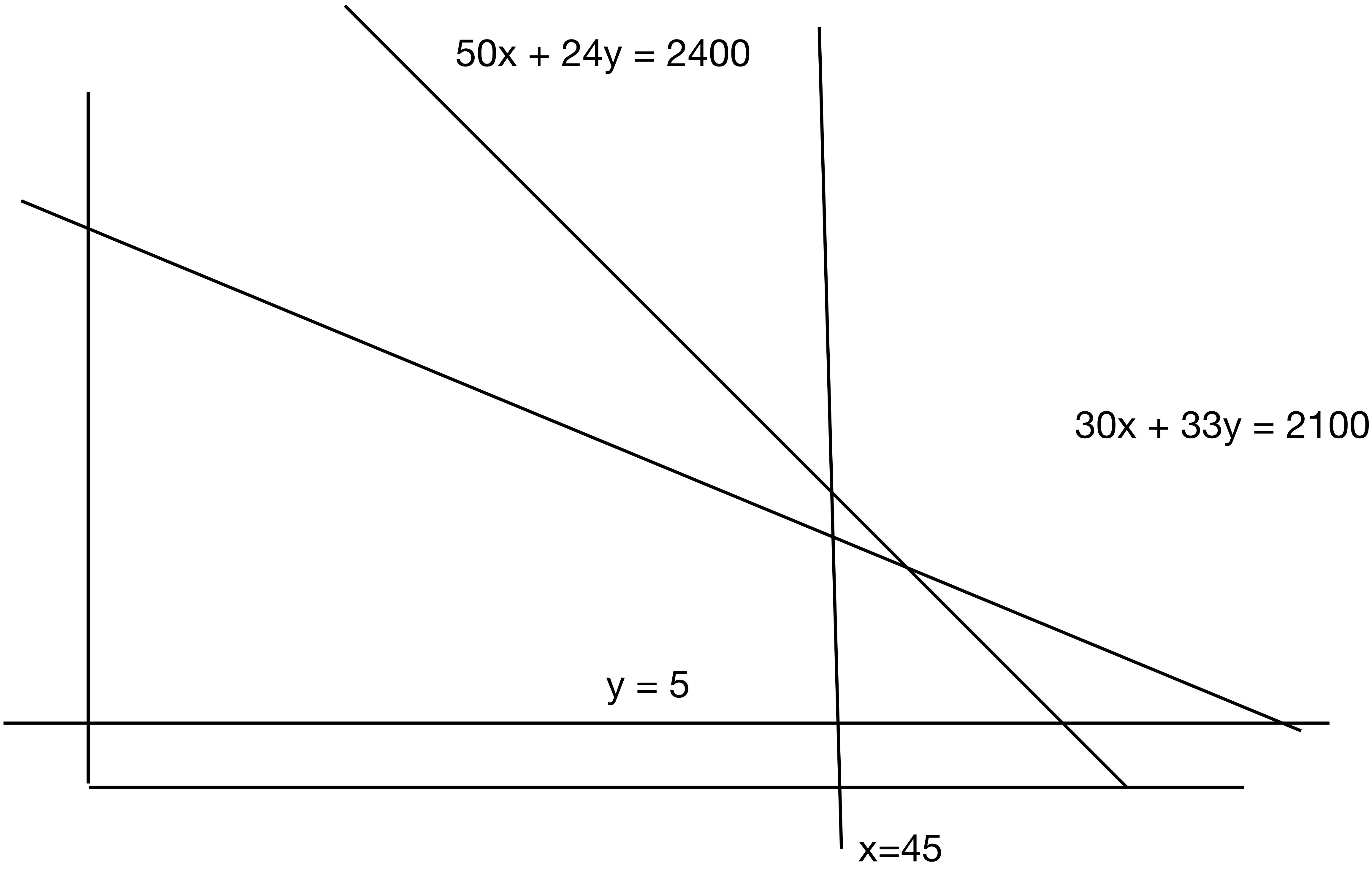

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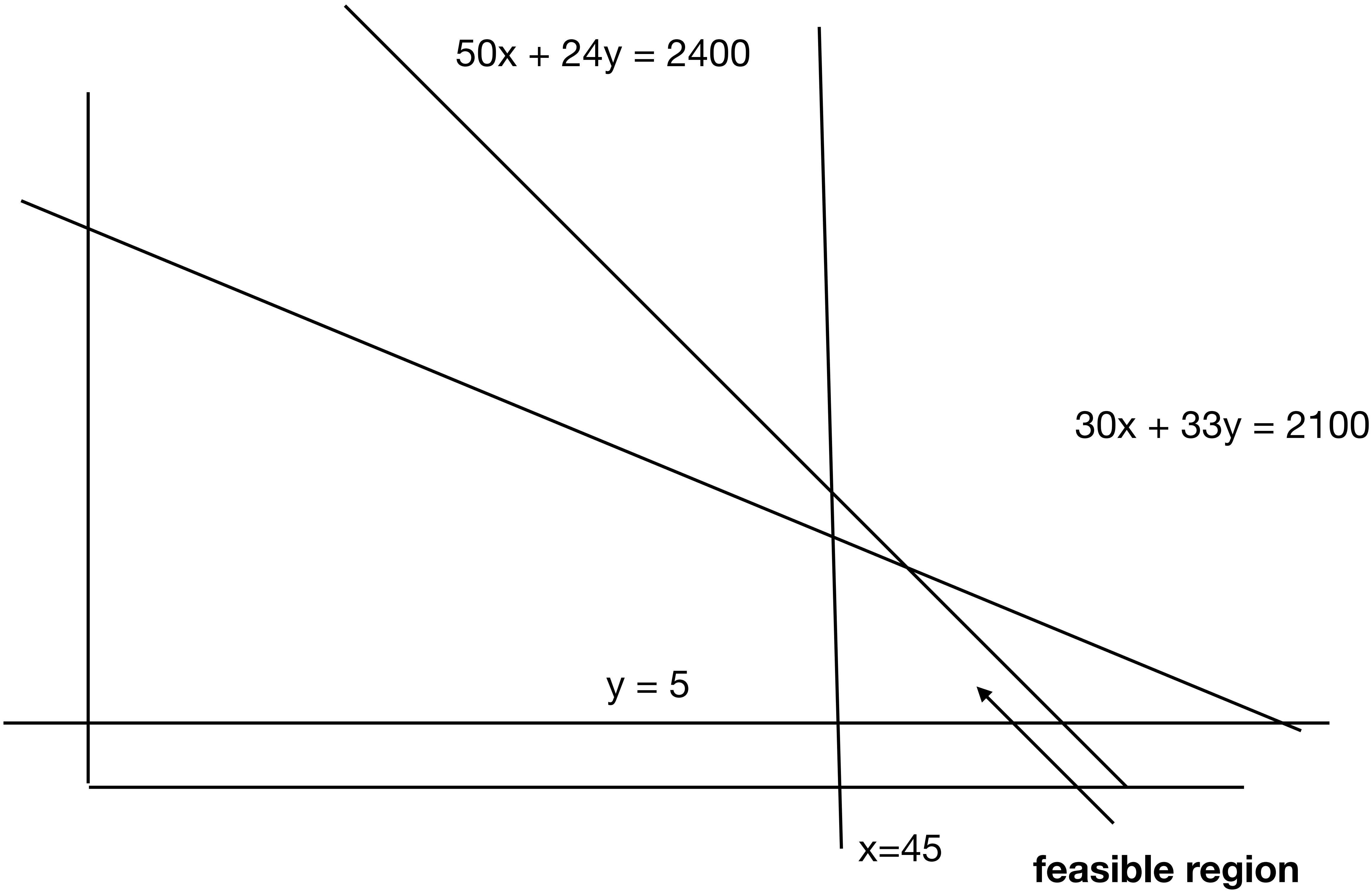

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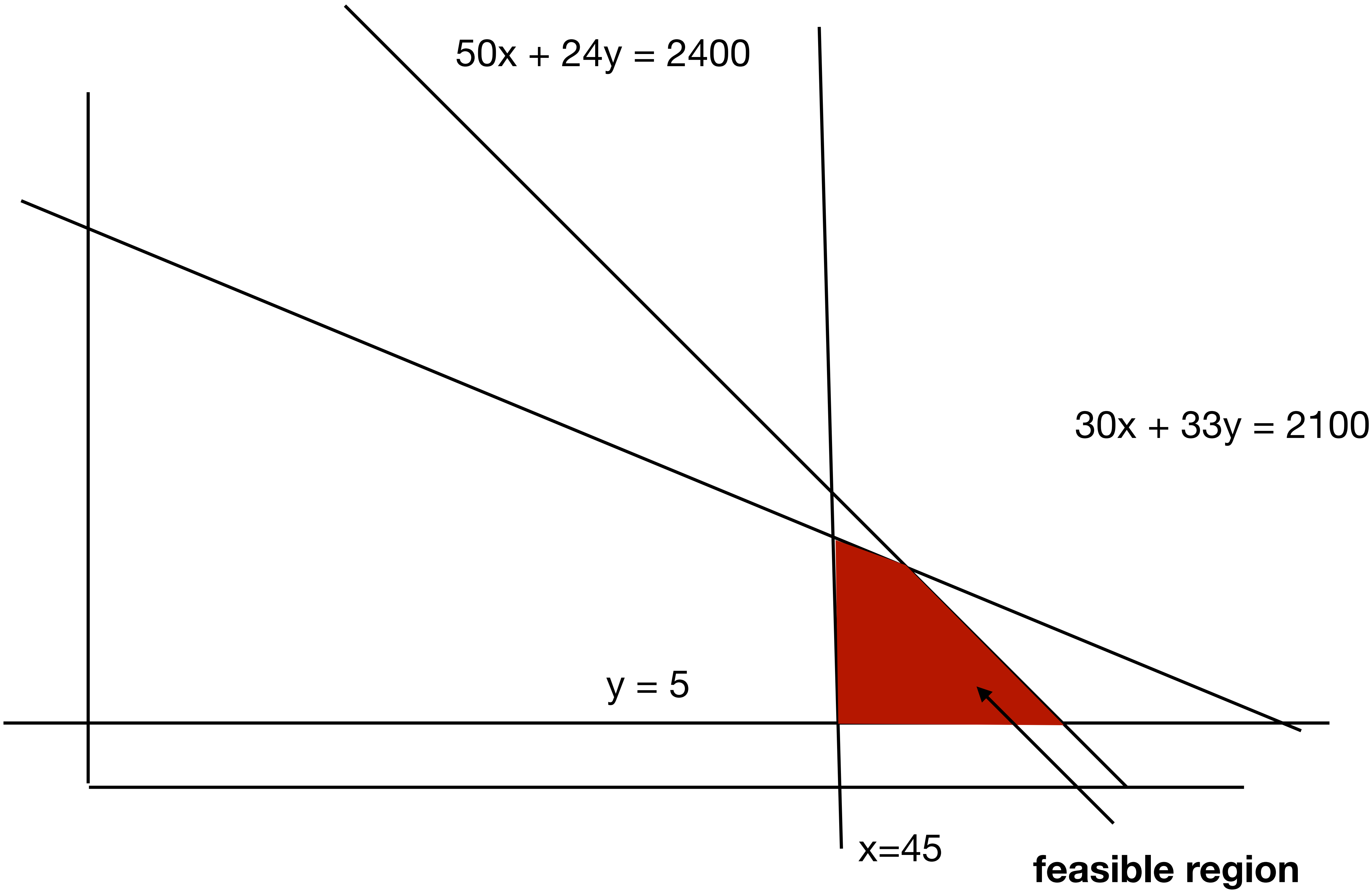

Do all LPs have feasible solutions?

Maximise $5x + 4y$

subject to
$$x + y \leq 2$$
$$-2x - 2y \leq -9$$
$$x, y \geq 0$$

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Maximise $5x + 4y$

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$$\boxed{\begin{aligned} x + y &\leq 2 \\ -2x - 2y &\leq -9 \end{aligned}}$$
one contradicts the other!
$$x, y \geq 0$$

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Solving the linear program

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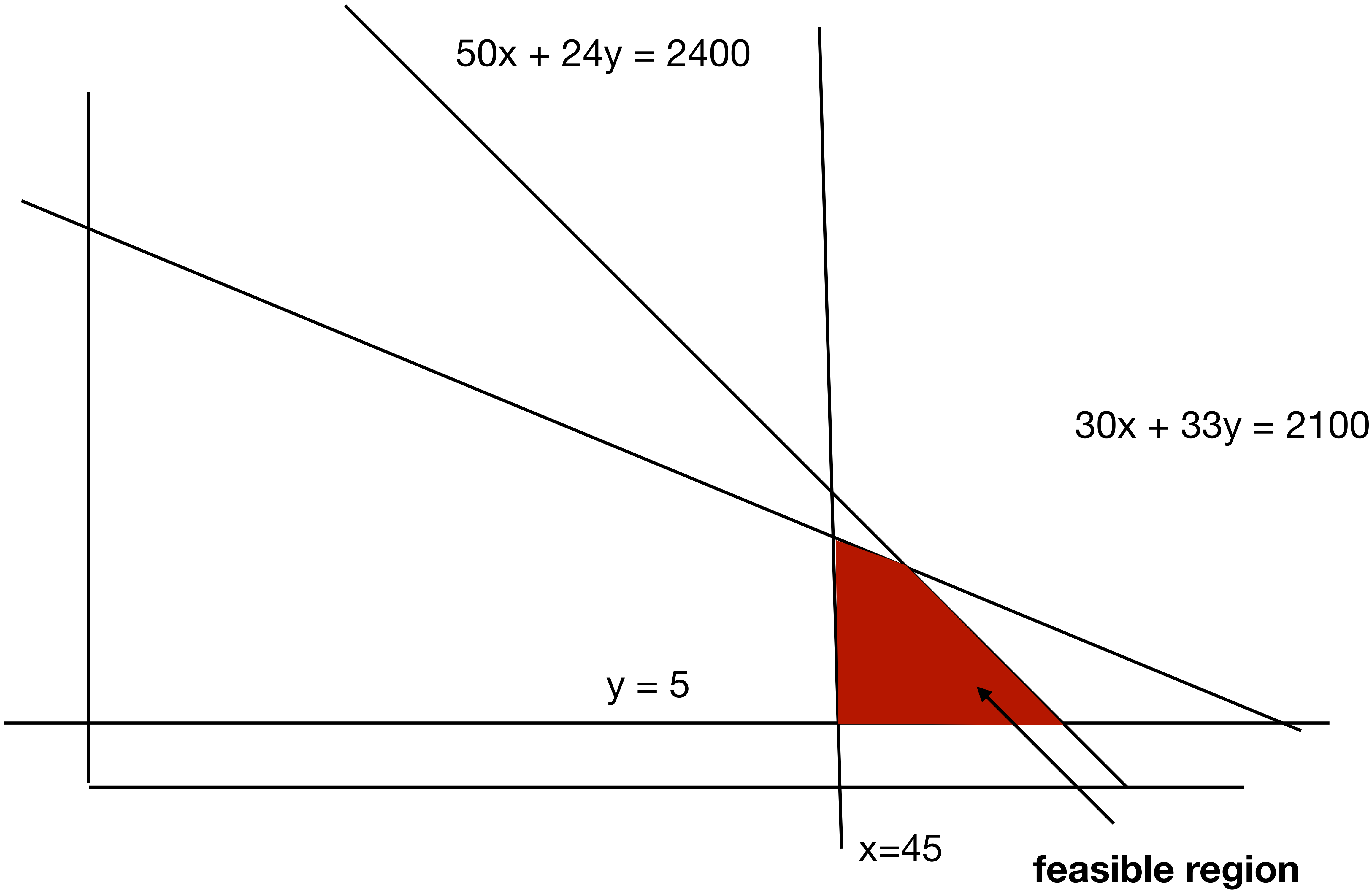
e.g., $50x+24y - 2400 = x - 45$

Solving the linear program



Diet Example

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Assume that we have two energy X and Y which provide calories, vitamin A and vitamin C daily.

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We would like to drink x bottles of X and y bottles of Y, to ensure that our daily intake is at least 300 calories, 36 units of vitamin A and 90 units of vitamin C.

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One bottle of X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C.

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One bottle of X costs £12, whereas one bottle of Y costs £15.

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One bottle of Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C.

One bottle of X costs £12, whereas one bottle of Y costs £15.

How do we maintain our diet goals at the lowest possible cost?

Diet Example

Minimise $12x + 15y$

subject to

$$60x + 60y \geq 300$$
$$12x + 6y \geq 36$$
$$10x + 30y \geq 90$$
$$x, y \geq 0$$

Diet Example

Minimise $12x + 15y$

subject to

$$x + y \geq 5$$

$$2x + y \geq 6$$

$$x + 3y \geq 9$$

$$x, y \geq 0$$

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$x + y - 5 = 2x + y - 6 \Rightarrow x = 1$ and $y = 4$

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$$x + y - 5 = 2x + y - 6 \Rightarrow x = 1 \text{ and } y = 4$$

$$12x + 15y = 72$$

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$x + y - 5 = x + 3y - 9 \Rightarrow y = 2$ and $y = 3$

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$$x + y - 5 = x + 3y - 9 \Rightarrow y = 2 \text{ and } y = 3$$

$$12x + 15y = 66$$

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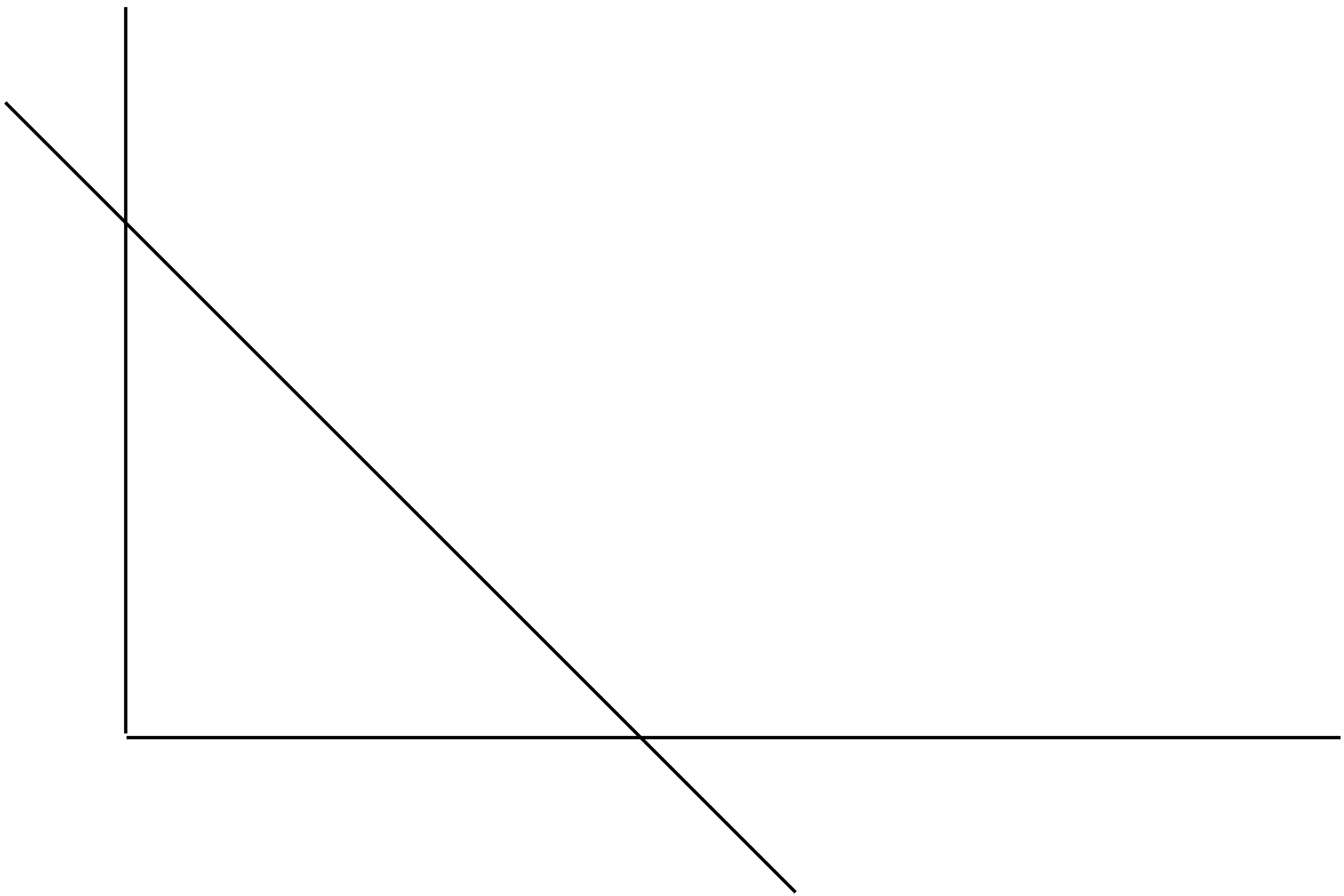

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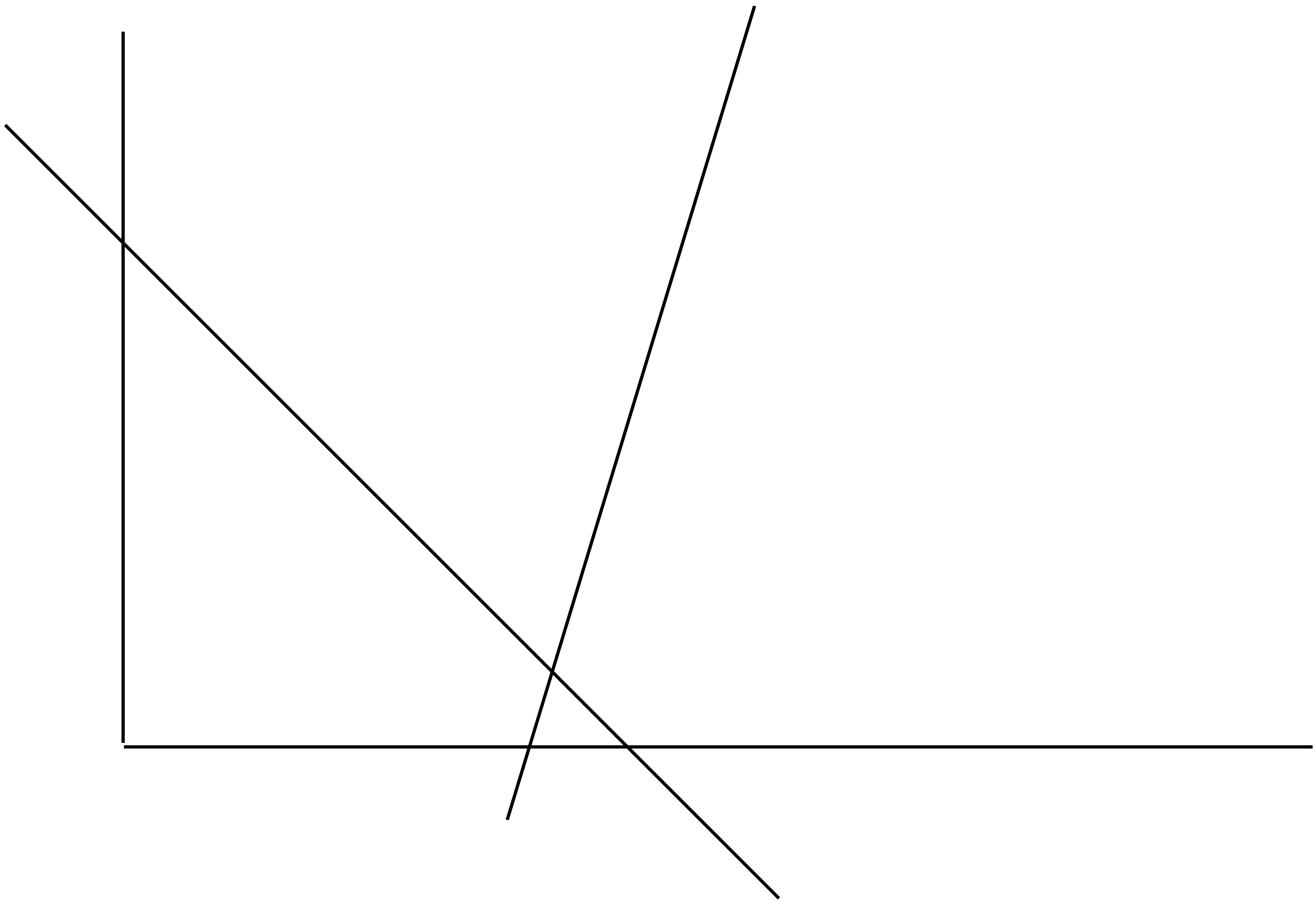

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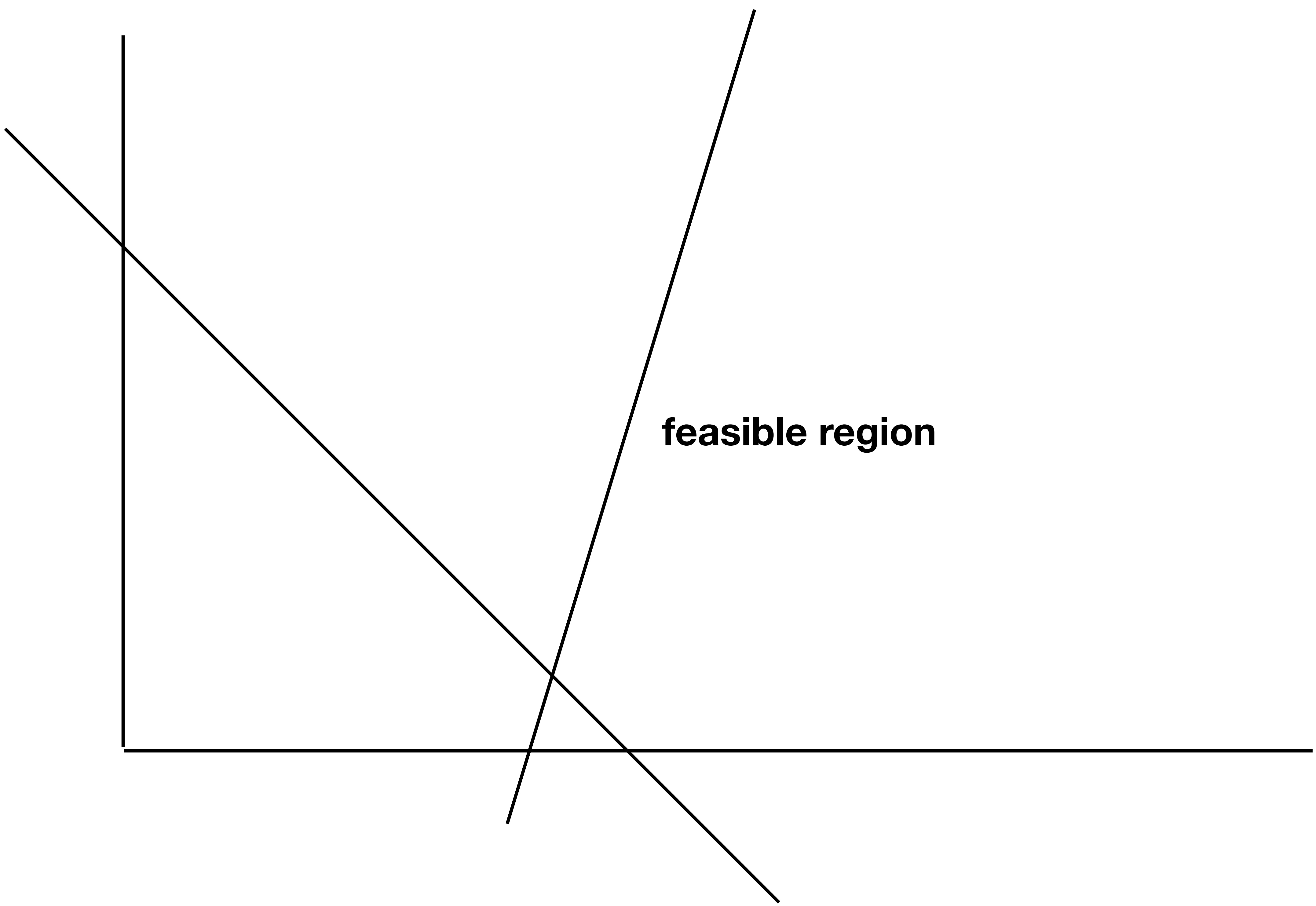

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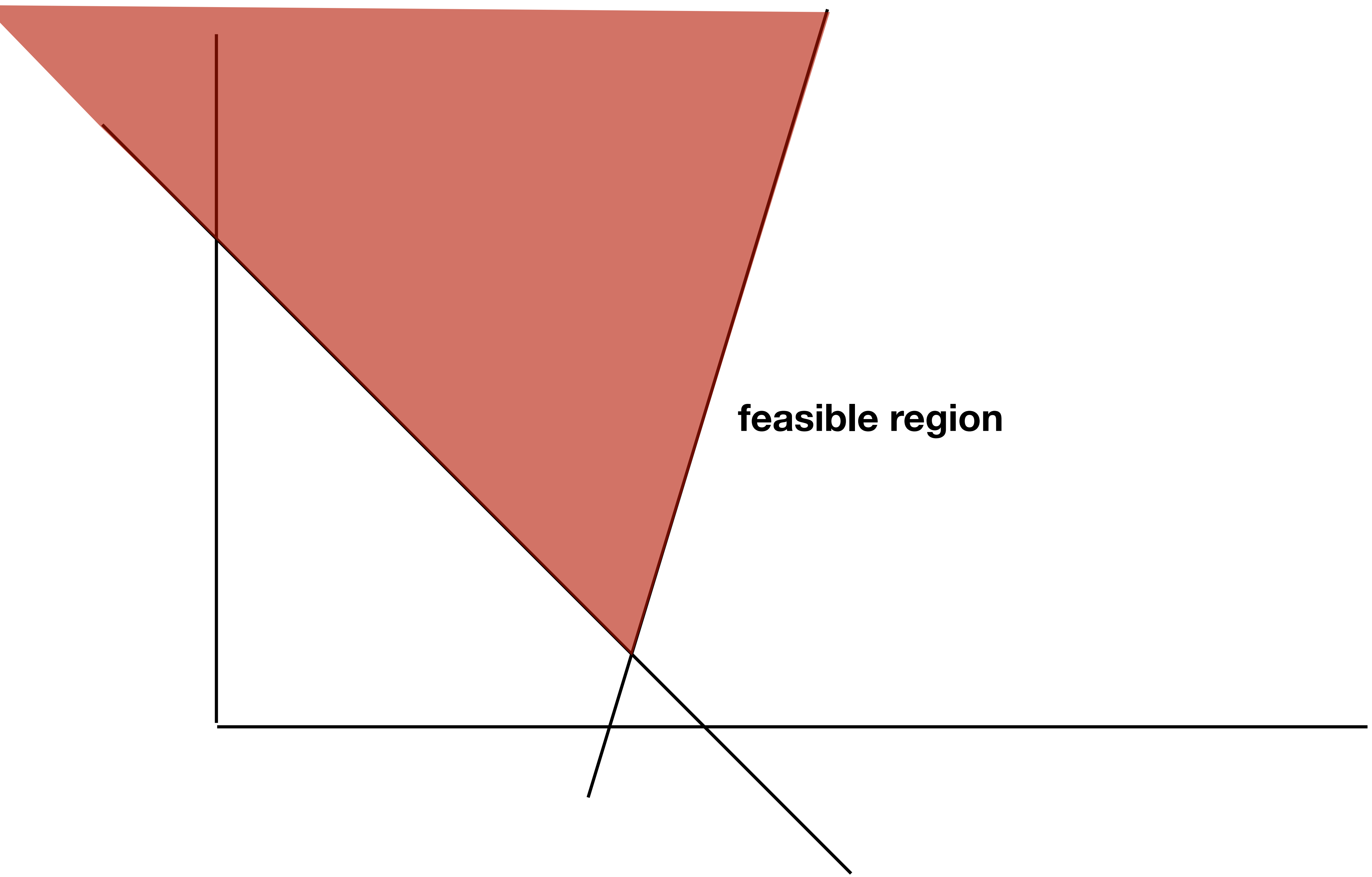

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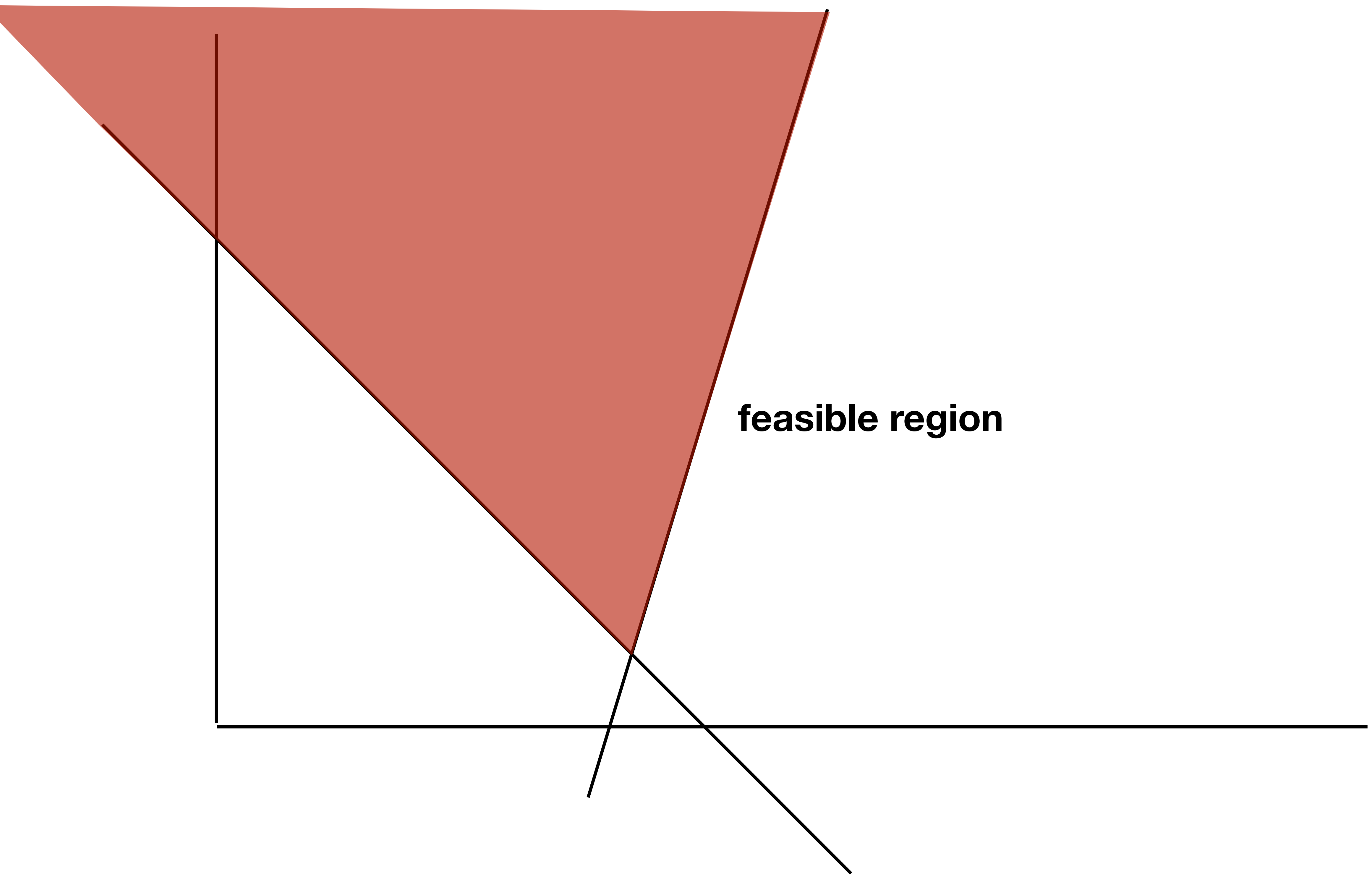
An LP is called unbounded if it has feasible solutions with arbitrarily large objective values.

Unbounded LPs



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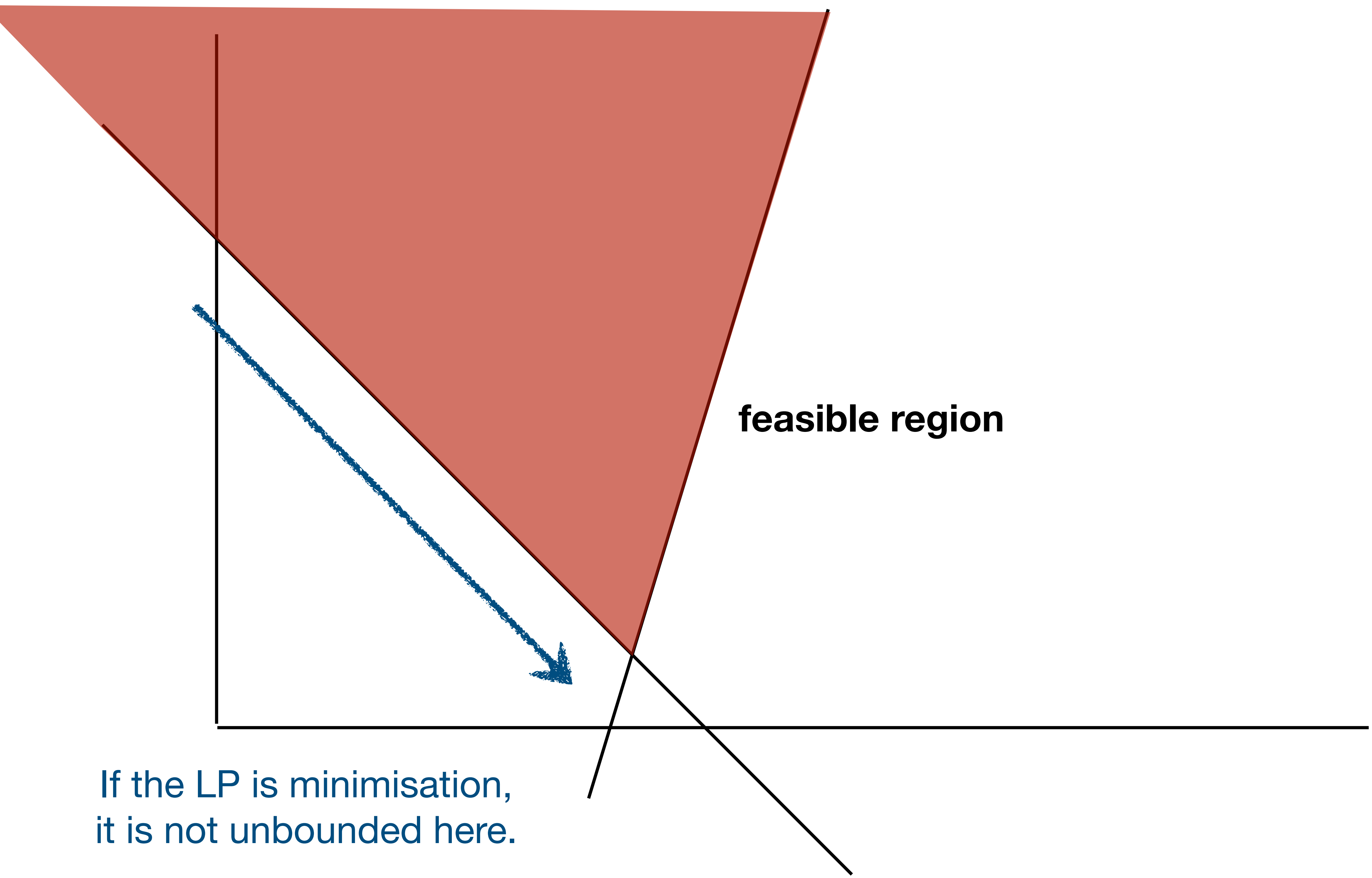

If the LP is minimisation,
it is not unbounded here.

More terminology

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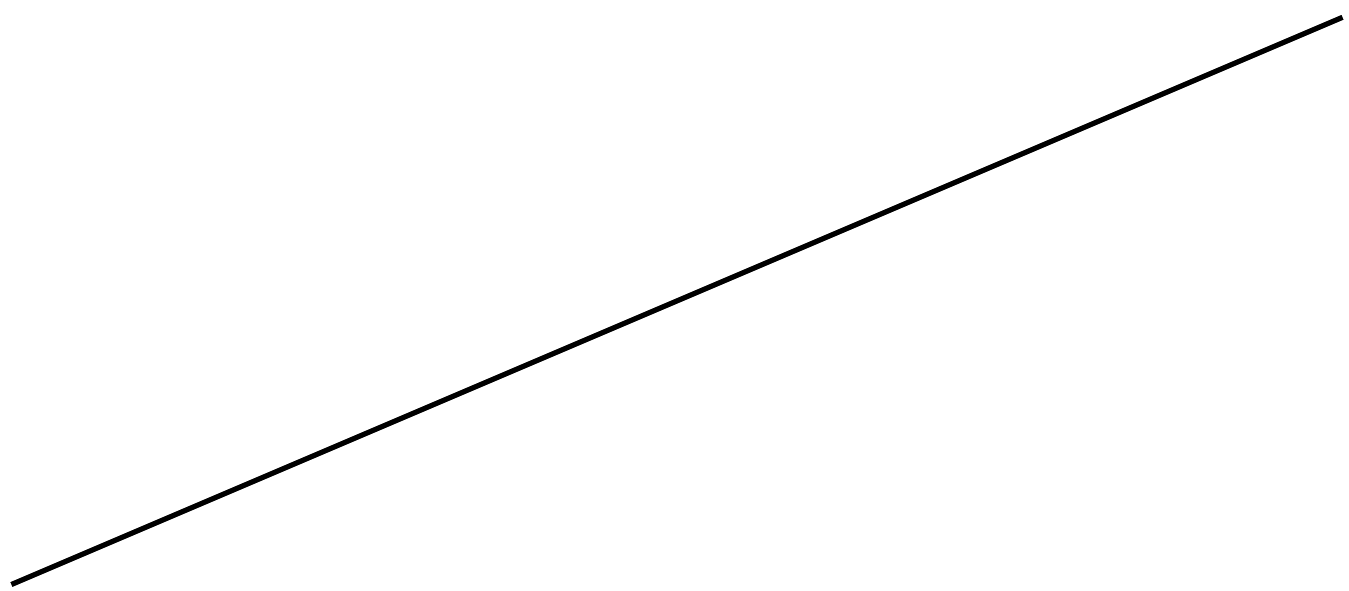

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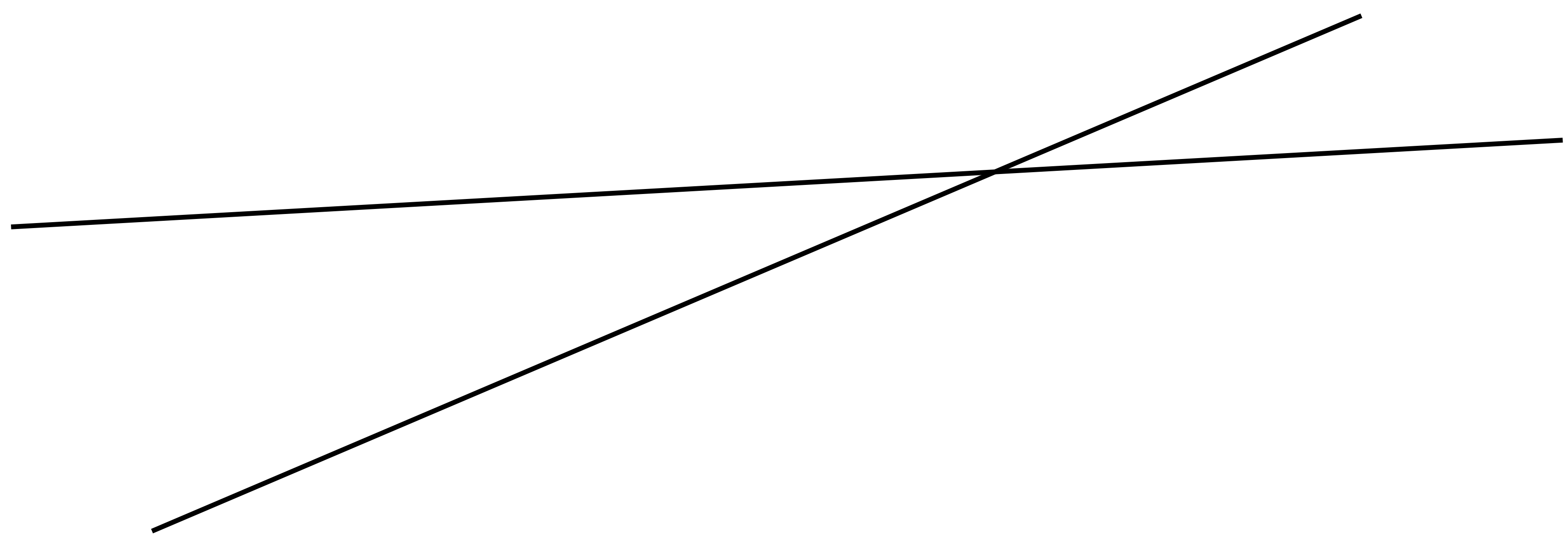

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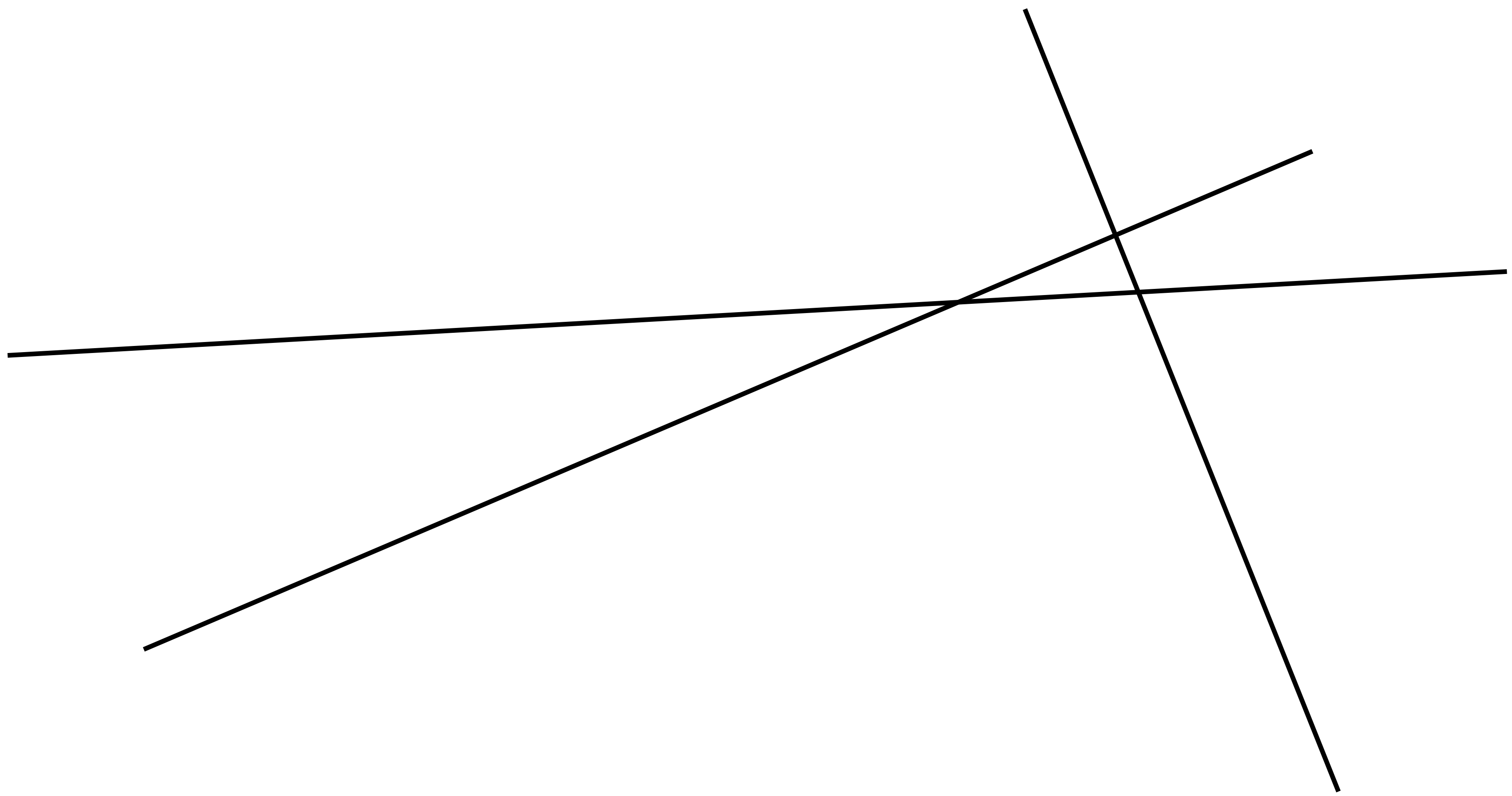

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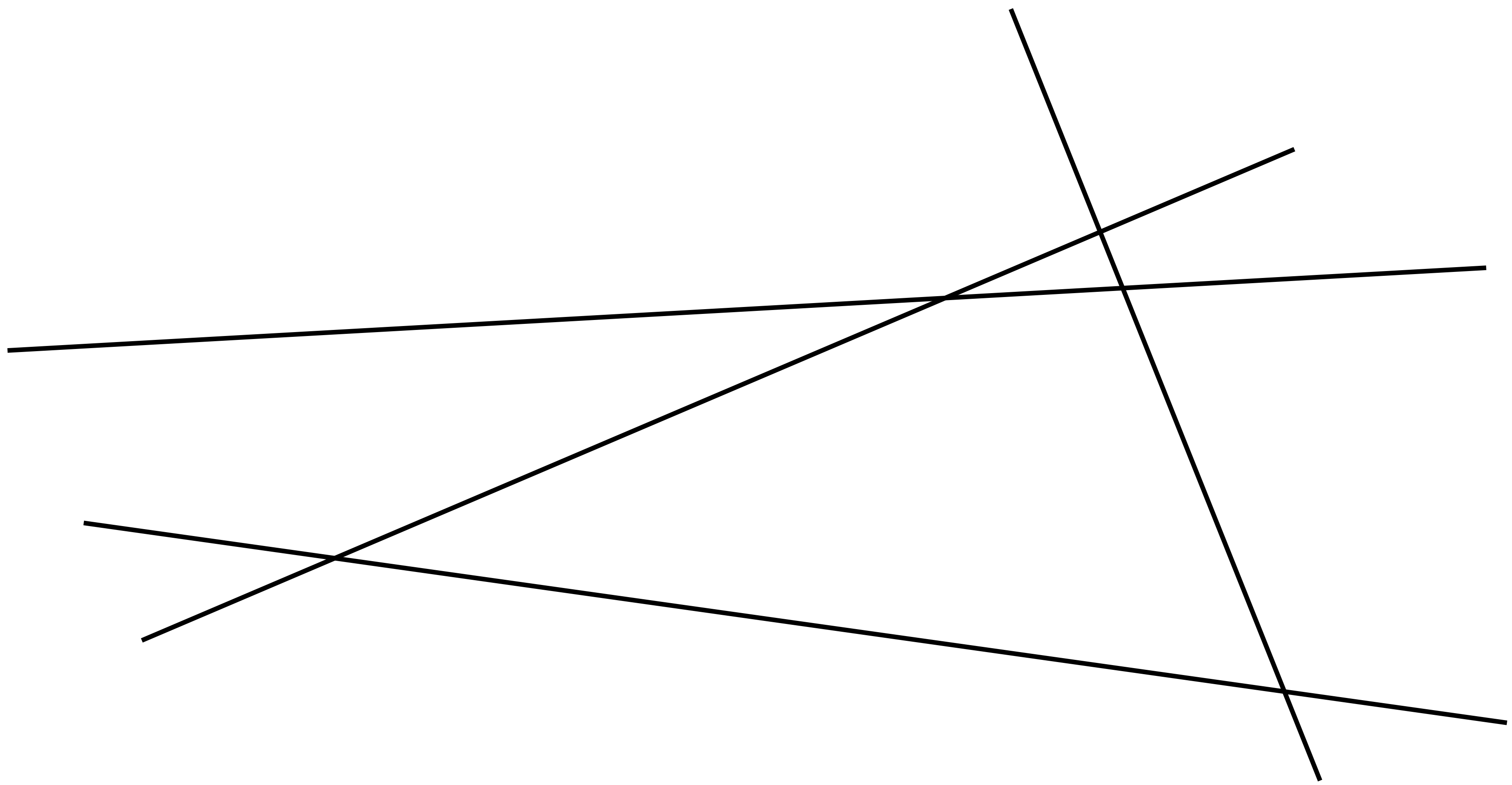

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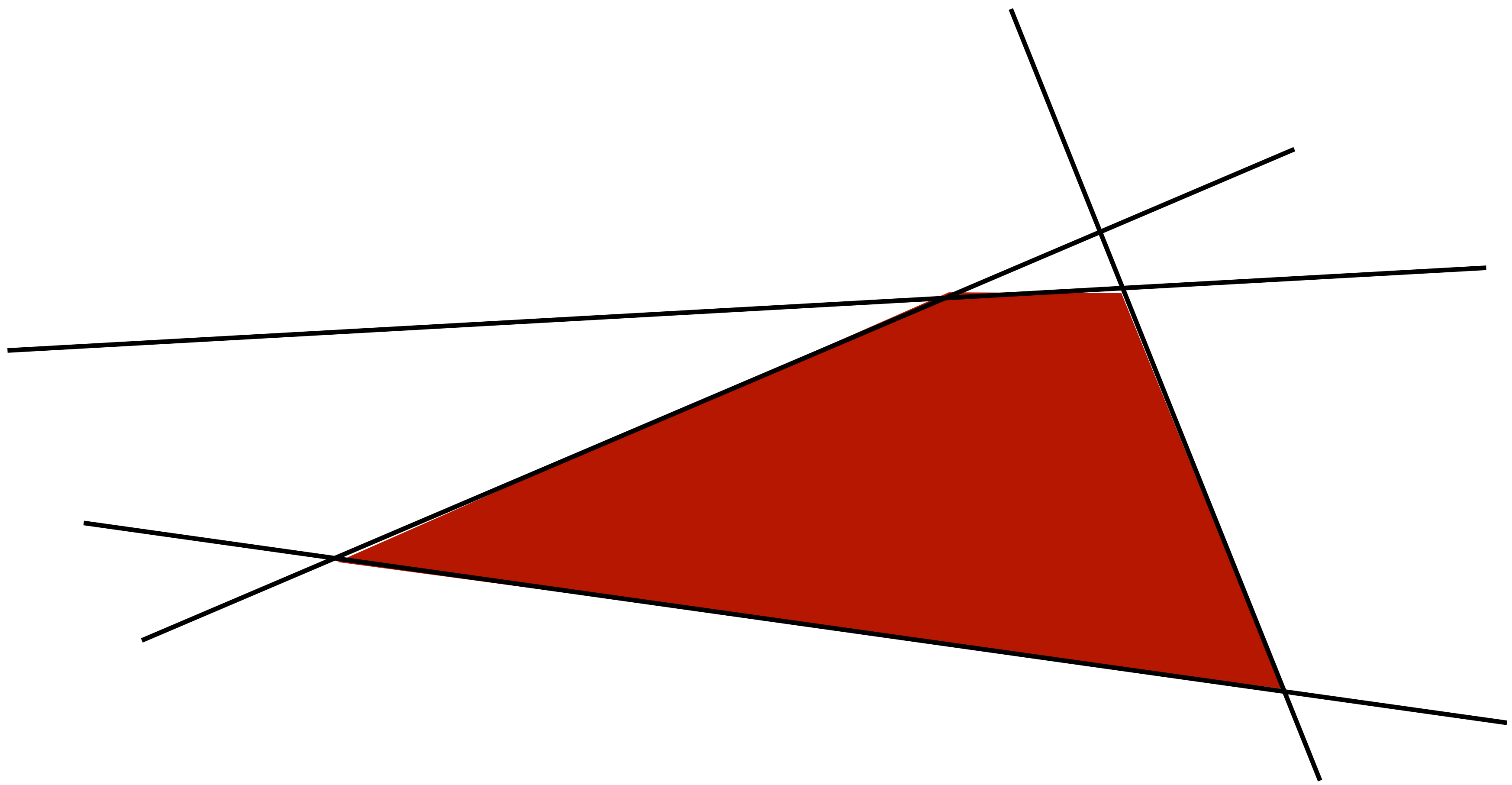

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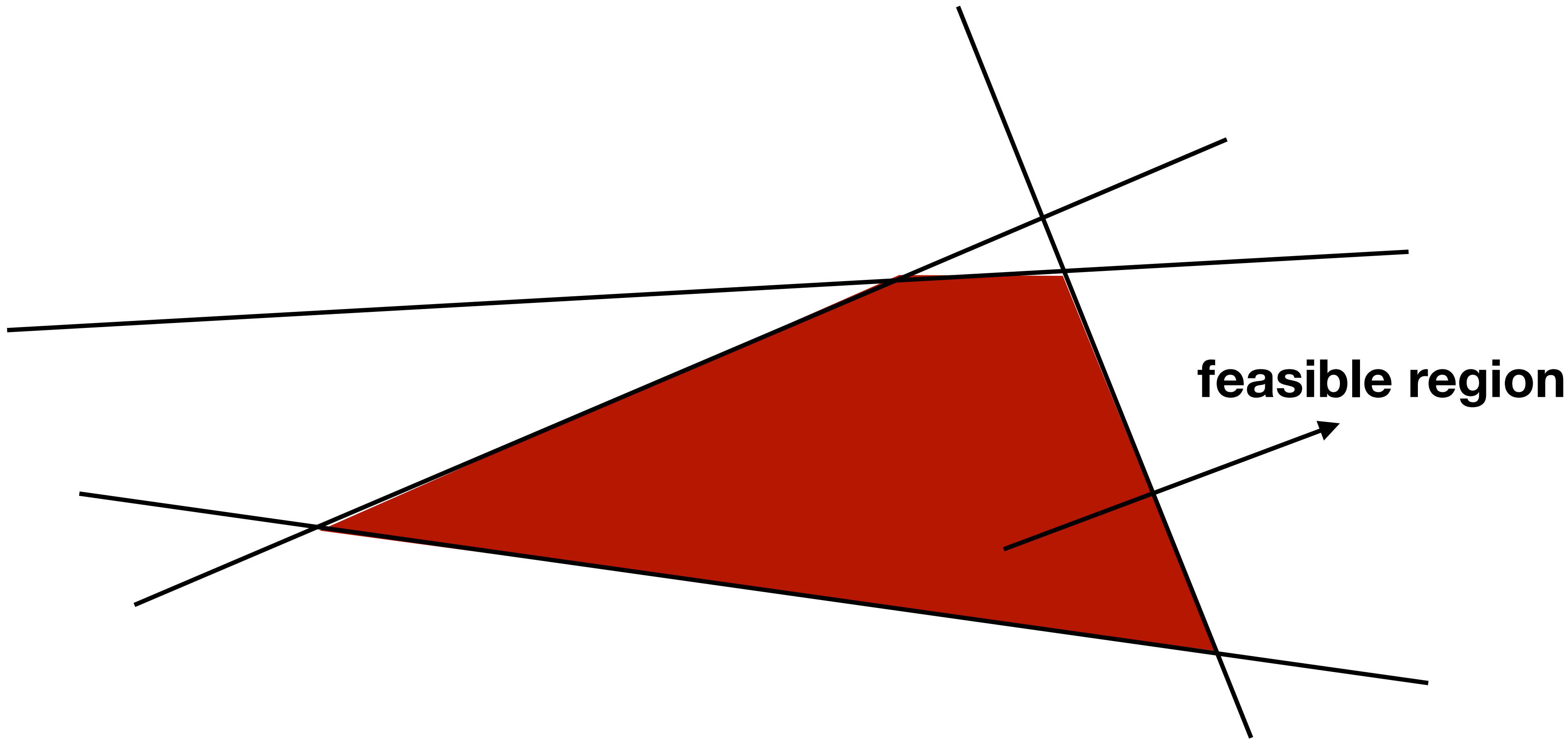

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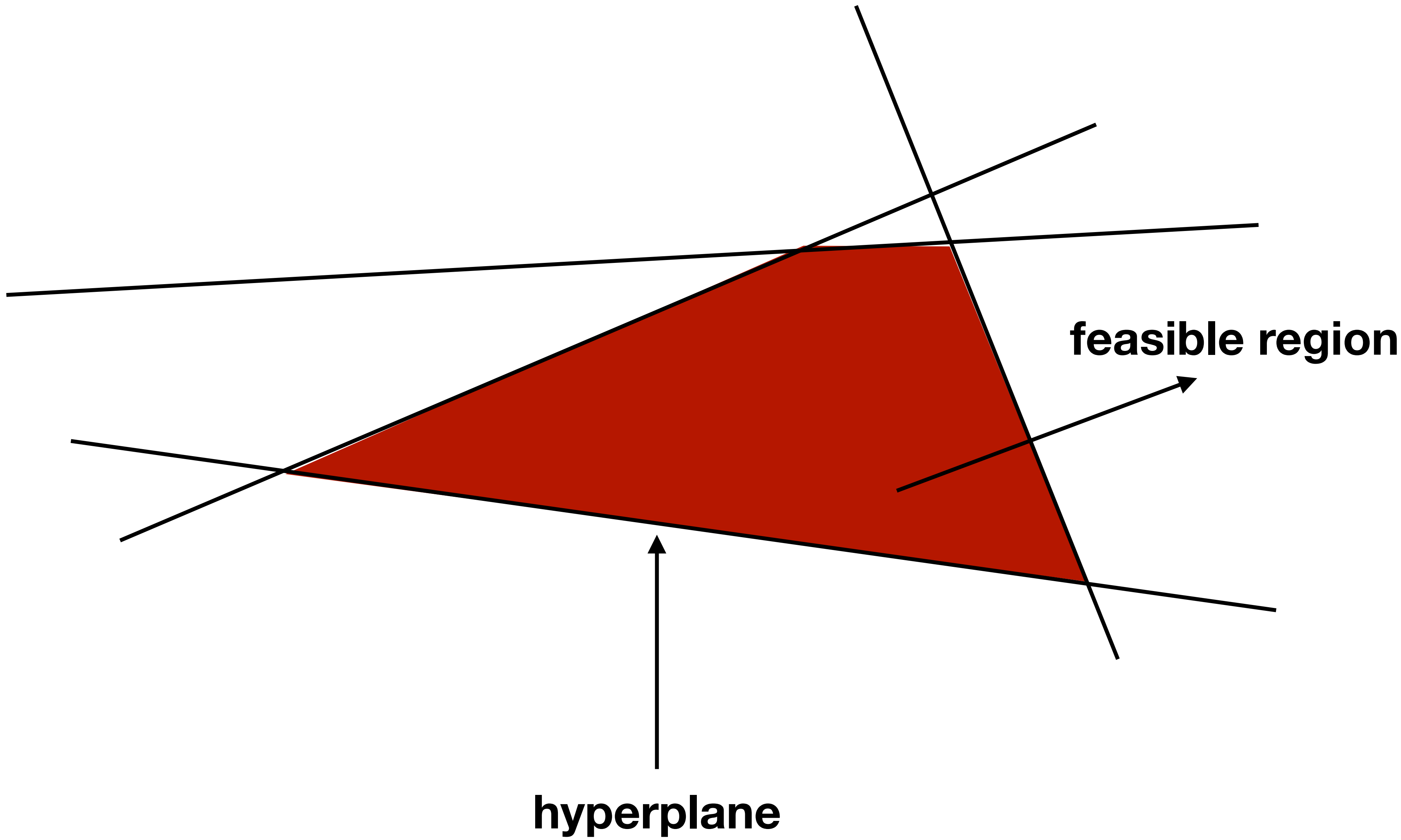

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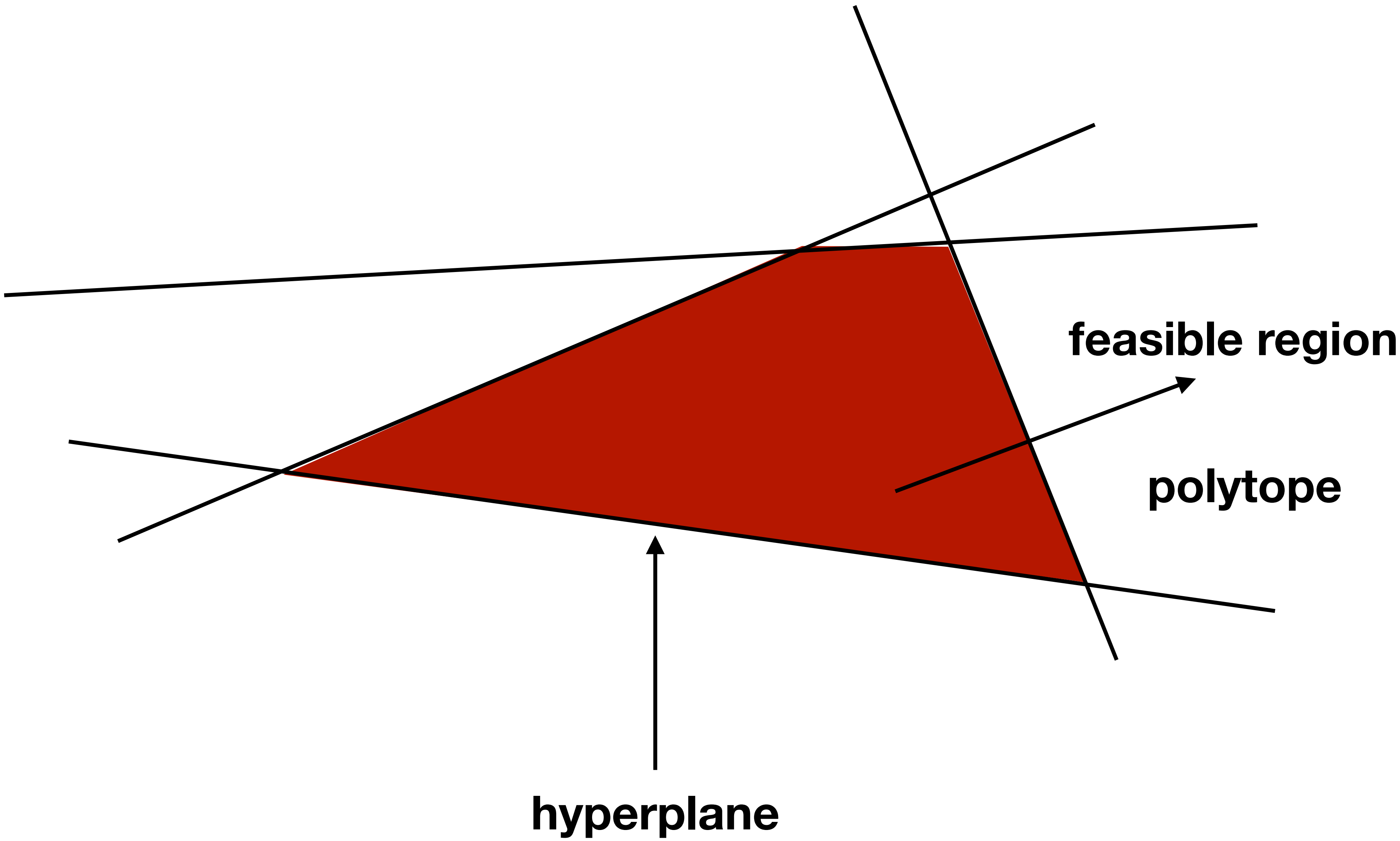

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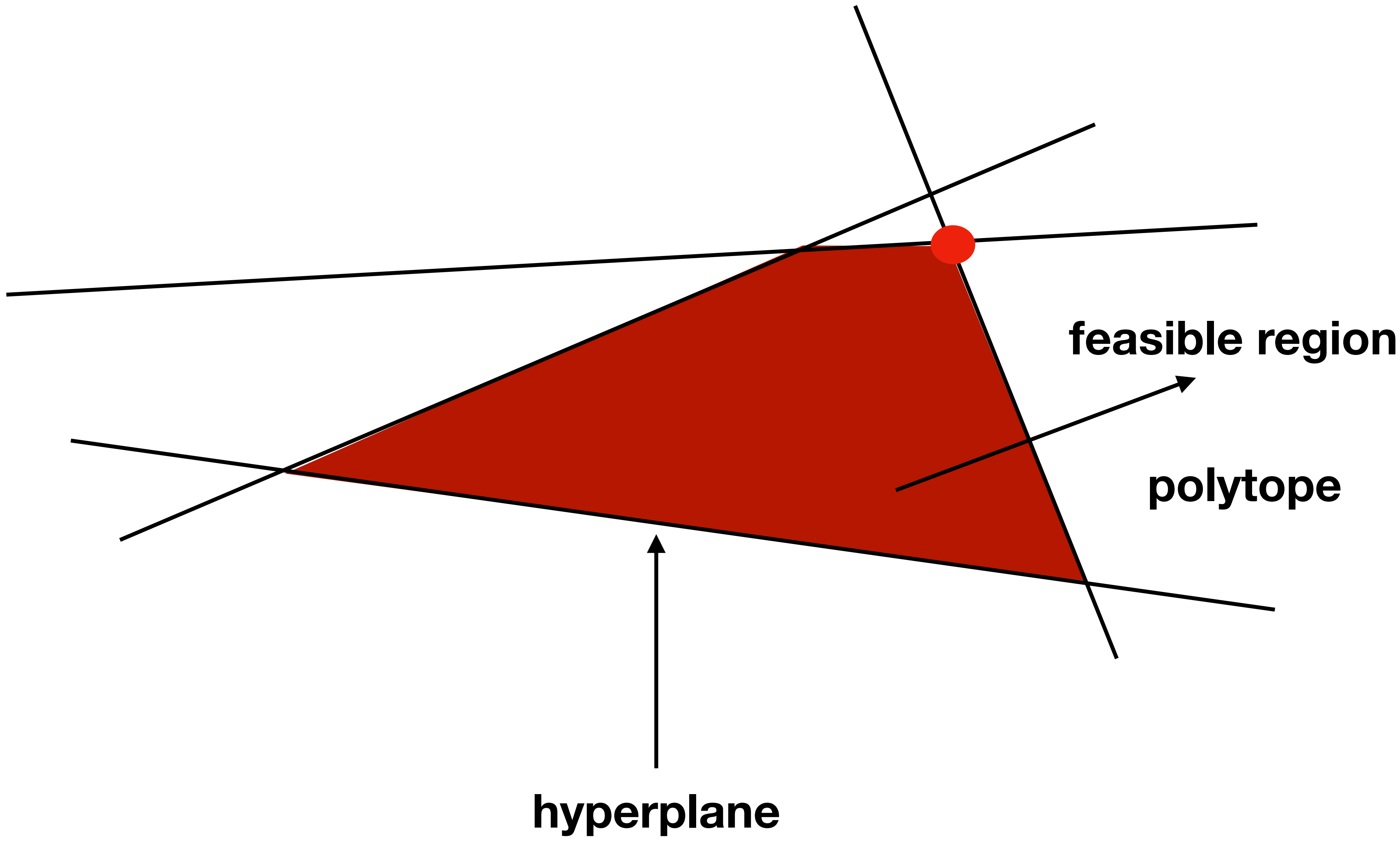

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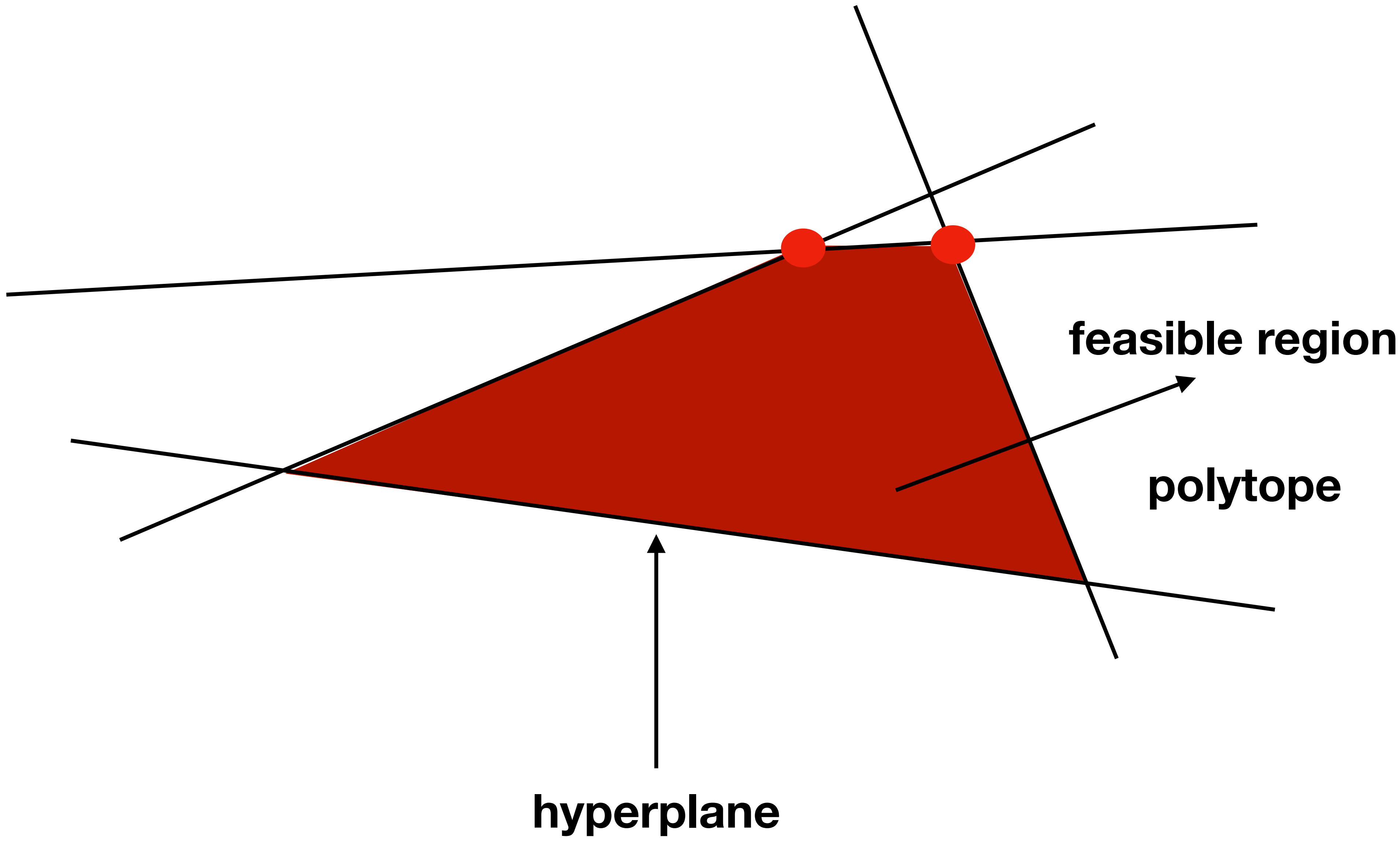

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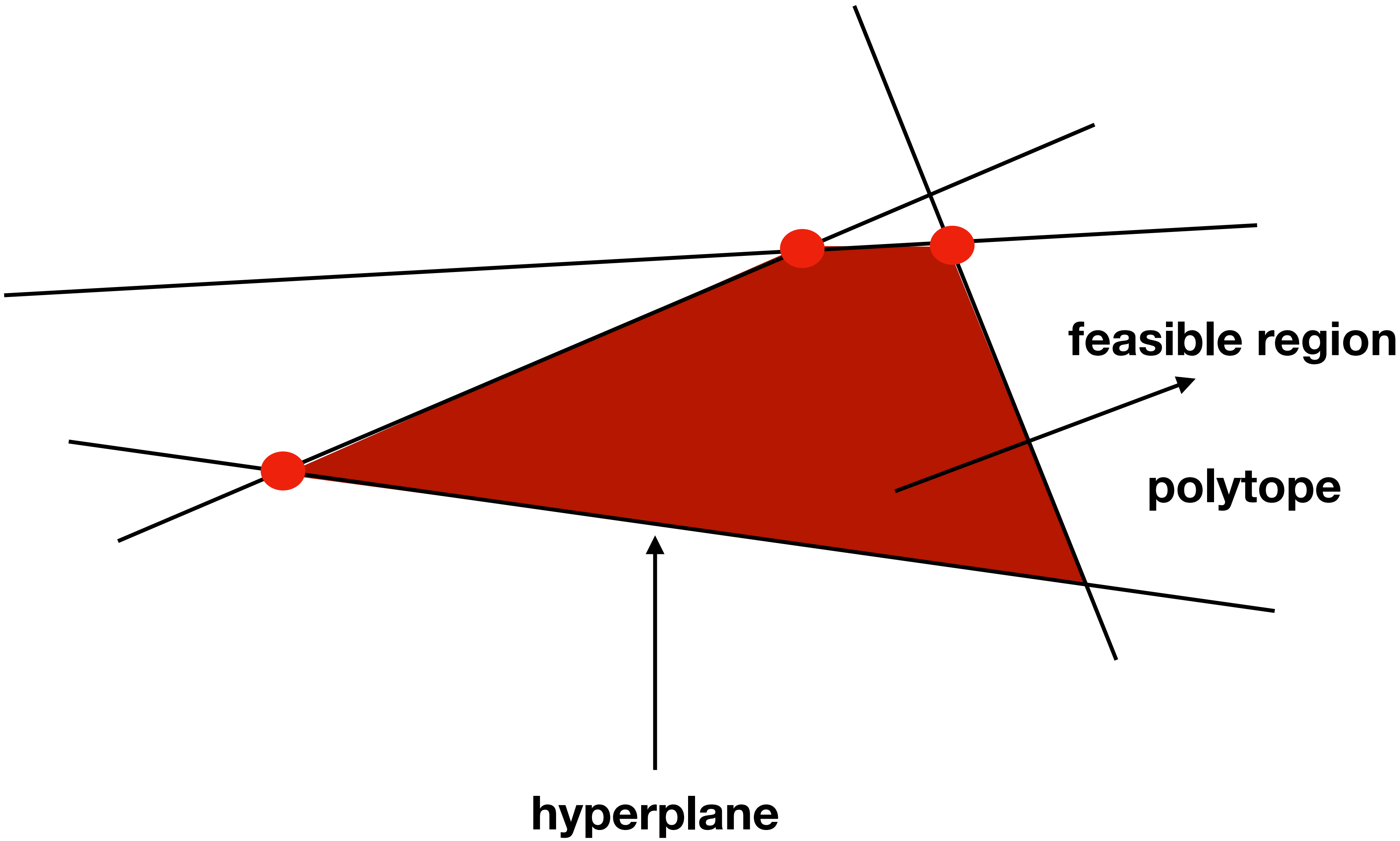

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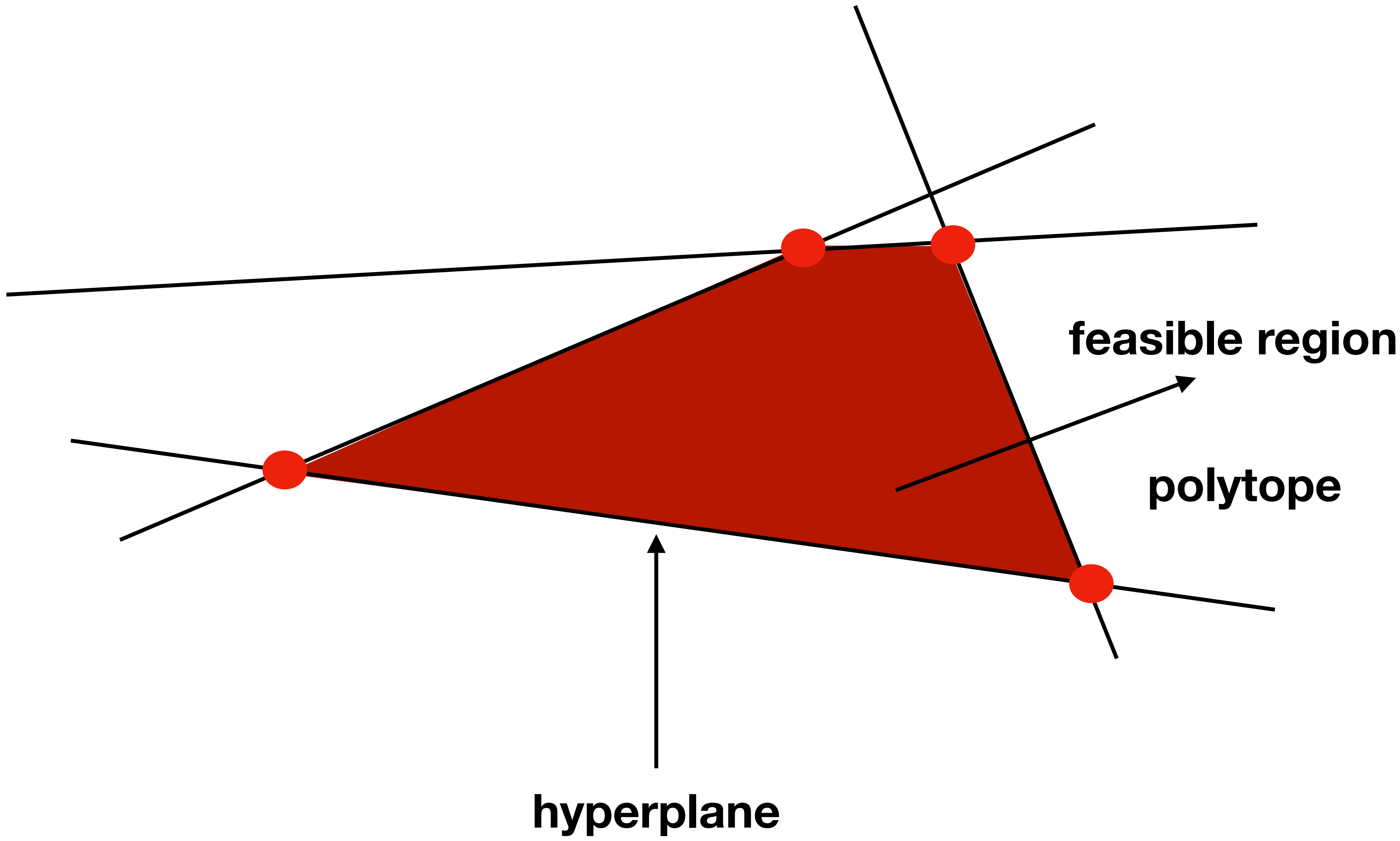

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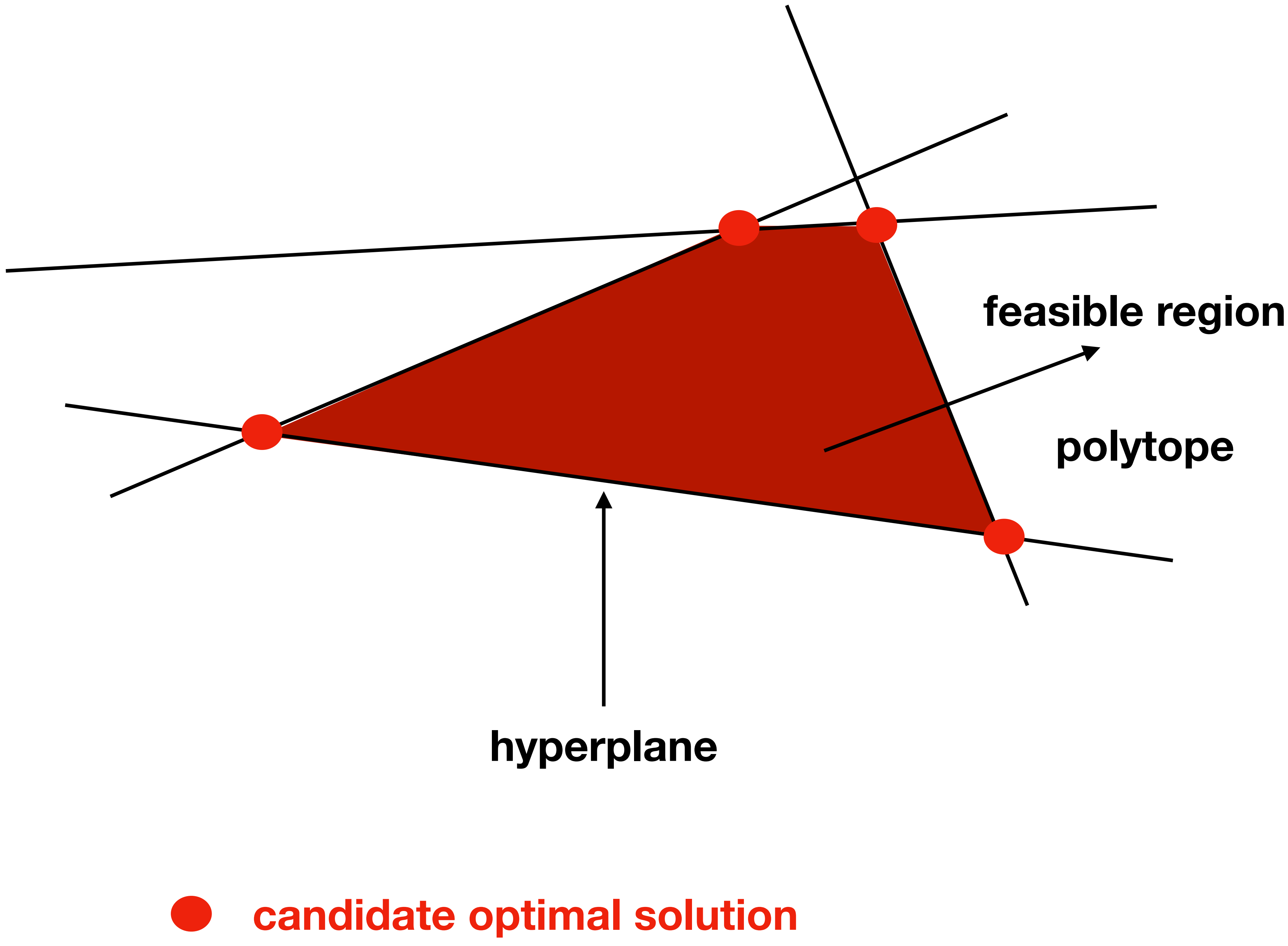

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What is the feasible region is empty, or the polytope is not bounded?

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We will consider valid solutions to say that *“the LP is infeasible”* or *“the LP is unbouded”*.

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Other algorithms for solving LPs:
Ellipsoid Method, Interior Point Methods

A simple but inefficient algorithm

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Idea: Elimination of variables

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Observation: Every inequality of the LP can be written in one of two forms

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$x_j \geq \alpha$ or $x_j \leq \beta$ for some α, β

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Repeat for the next variable, until we only have one variable.

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Substitute back to get the other variables.

A simple but inefficient algorithm (example)

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$$
\begin{aligned}
&x + y \geq 0 \\
&2x + y \geq 2 \\
&-x + y \geq 1 \\
&-x + 2y \geq -1
\end{aligned}
$$

A simple but inefficient algorithm (example)

$$x + y \geq 0$$
$$2x + y \geq 2$$
$$-x + y \geq 1$$
$$-x + 2y \geq -1$$

"Solve" for x:

A simple but inefficient algorithm (example)

$$x + y \geq 0$$
$$2x + y \geq 2$$
$$-x + y \geq 1$$
$$-x + 2y \geq -1$$

"Solve" for x:

$$x \geq -y$$
$$x \geq 1 - \frac{y}{2}$$
$$x \leq -1 + y$$
$$x \leq 1 + 2y$$

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The above implies:

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The above implies:

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$$-1 + y \geq 1 - \frac{y}{2}$$
$$1 + 2y \geq -y$$
$$1 + 2y \geq 1 - \frac{y}{2}$$

Simplifying:

$$y \geq 1/2$$
$$y \geq 4/3$$
$$y \geq -1/3$$
$$y \geq 0$$

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$y \geq 1/2$
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Pick a feasible y,
e.g., $y = 2$.

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A simple but inefficient algorithm (example)

Simplifying:

$$y \geq 1/2$$
$$y \geq 4/3$$
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$$y \geq 0$$

Pick a feasible y, e.g., $y = 2$.

We can find a feasible x using our inequalities:

$$x \geq -y$$
$$x \geq 1 - \frac{y}{2}$$
$$x \leq -1 + y$$
$$x \leq 1 + 2y$$

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How do we find an optimal solution?

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Observation: Given a linear objective function, we can substitute it with a variable x_0 (how?)

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Minimise $12x + 15y$

subject to

$$x + y \geq 5$$
$$2x + y \geq 6$$
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$$x, y \geq 0$$

Diet Example

Minimise x_0

subject to

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$$2x + y \geq 6$$
$$x + 3y \geq 9$$
$$x, y \geq 0$$
$$12x + 15y = x_0$$

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Observation: Given a linear objective function, we can substitute it with a variable x_0 (how?)

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Pick the x_0 that optimises the objective function.

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Observation: Given a linear objective function, we can substitute it with a variable x_0 (how?)

Eliminate to find inequalities for x_0.

Pick the x_0 that optimises the objective function.

Work out feasible $x_1, \ldots, x_n$ for the rest of the variables.

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Thus for k elimination steps we can have $\Omega\left(m^{2^k}\right)$ constraints.

A nice consequence of FME

If the LP has an optimal feasible solution, then it has a rational optimal feasible solution x^* and the objective function value $f(x^*)$ is also rational.

Linear programming (LP)

$$\text{maximise} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1\in}^{n} \alpha_{ij} x_j \leq b_i, \quad i = 1, ..., m$$

$$x_j \geq 0, \quad j = 1, ..., n$$

Integer Linear programming

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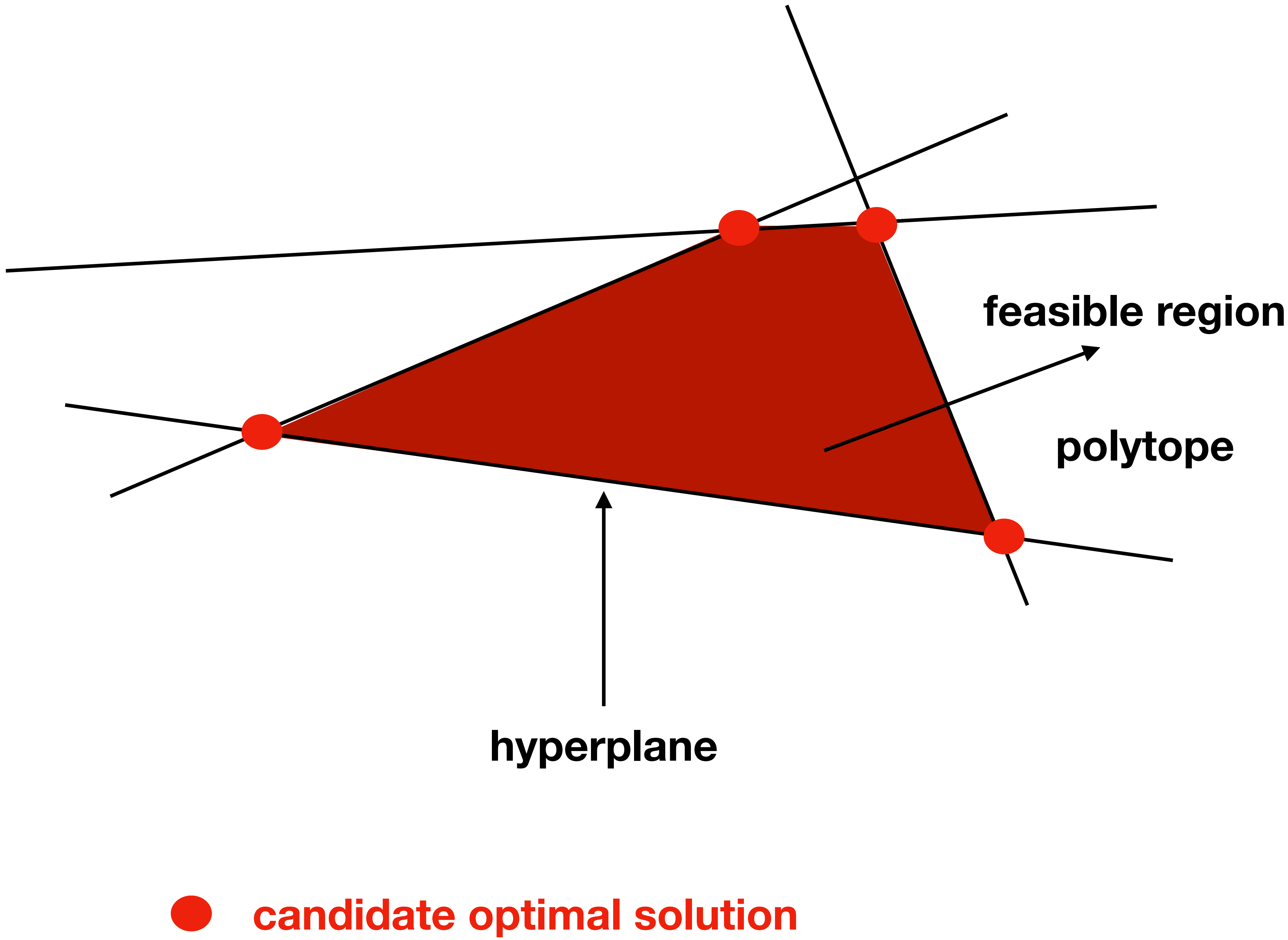
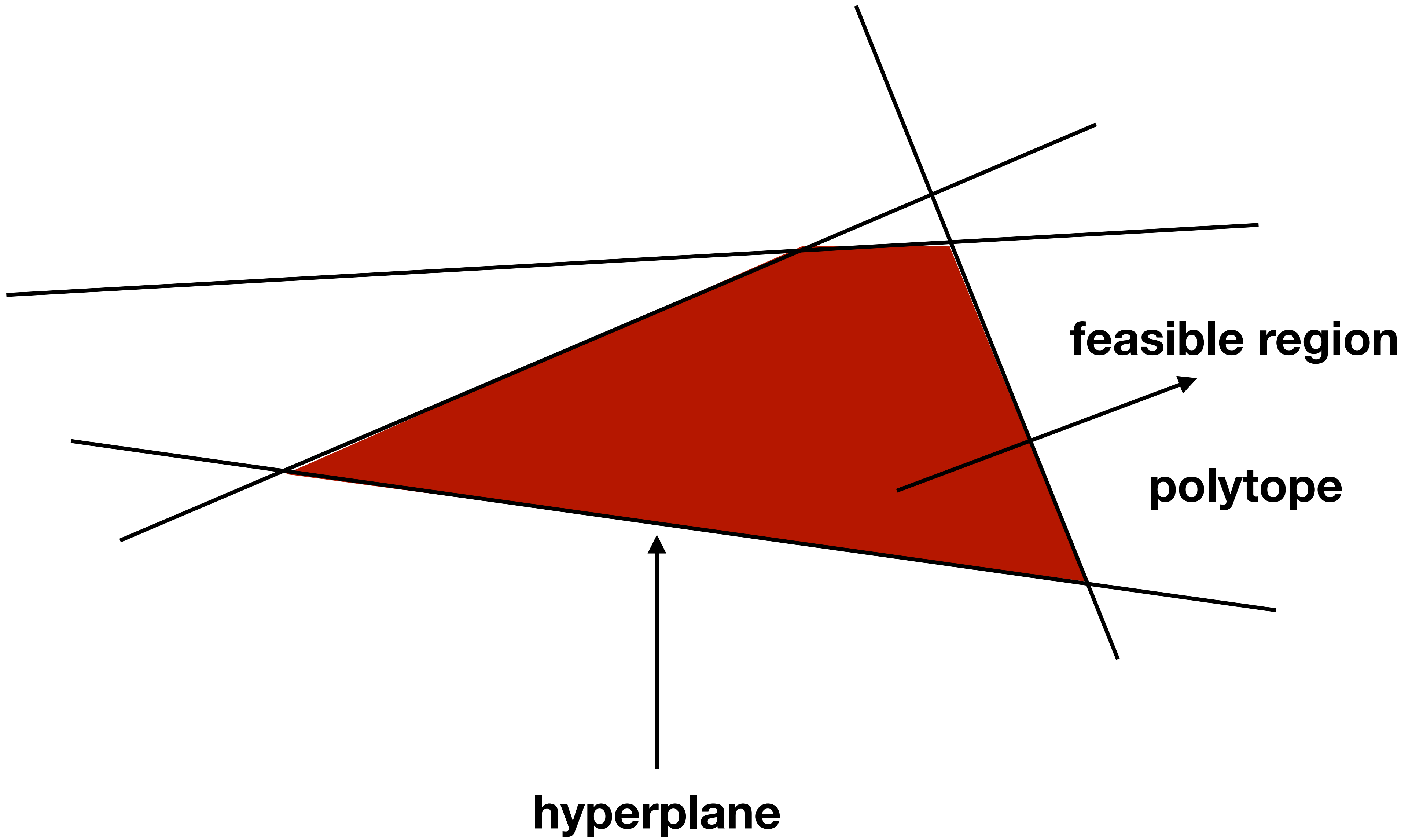
Feasible region



Feasible region



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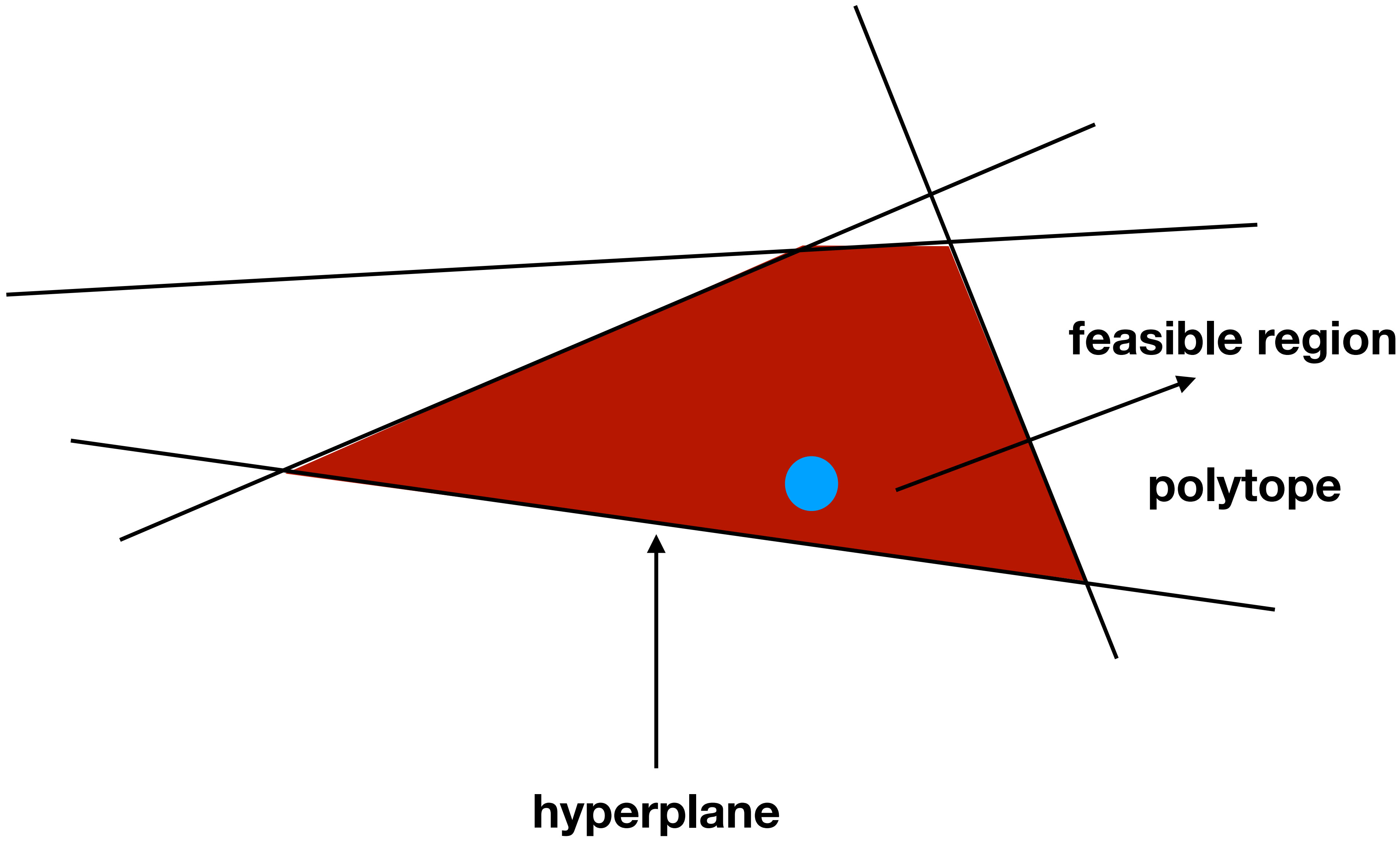

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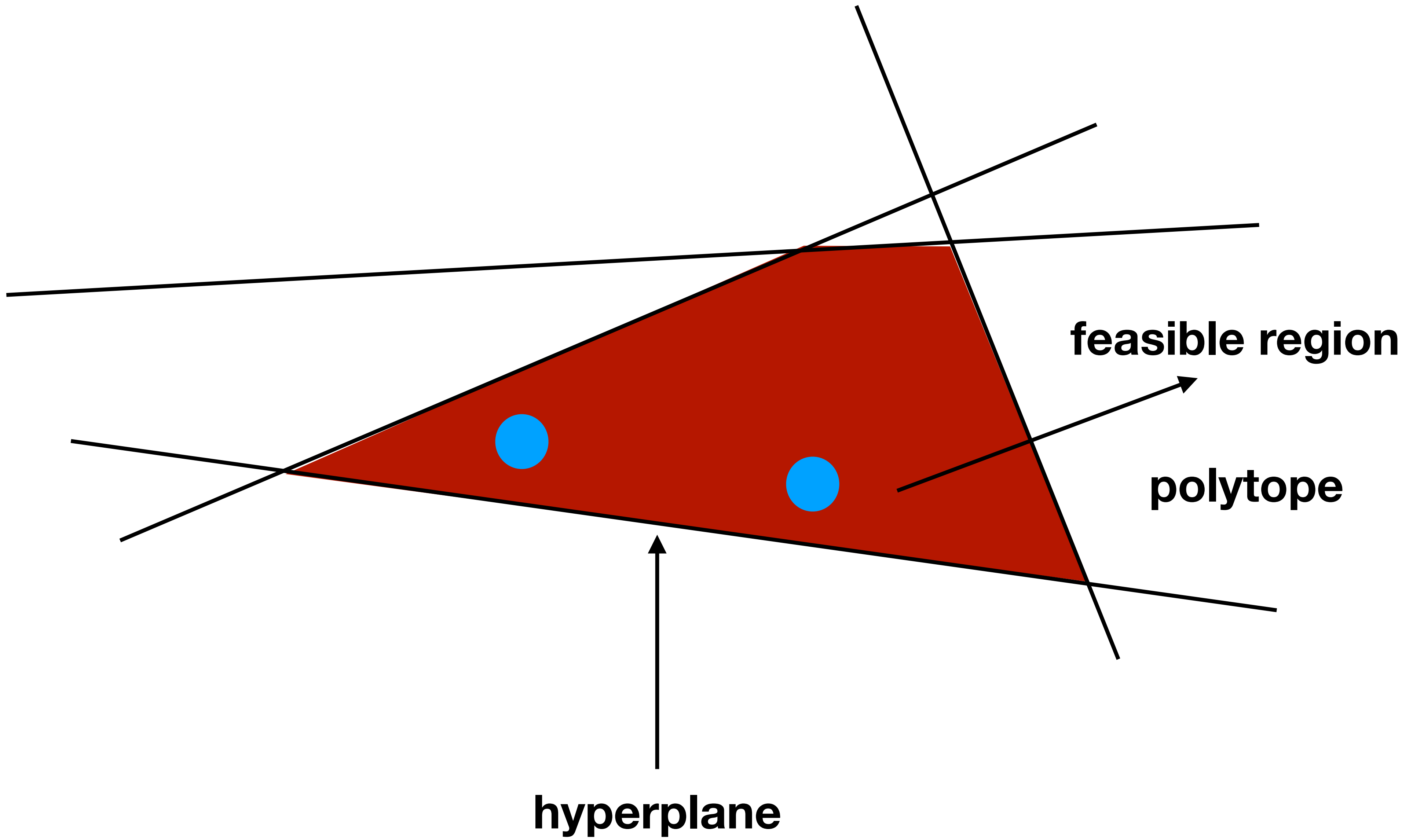

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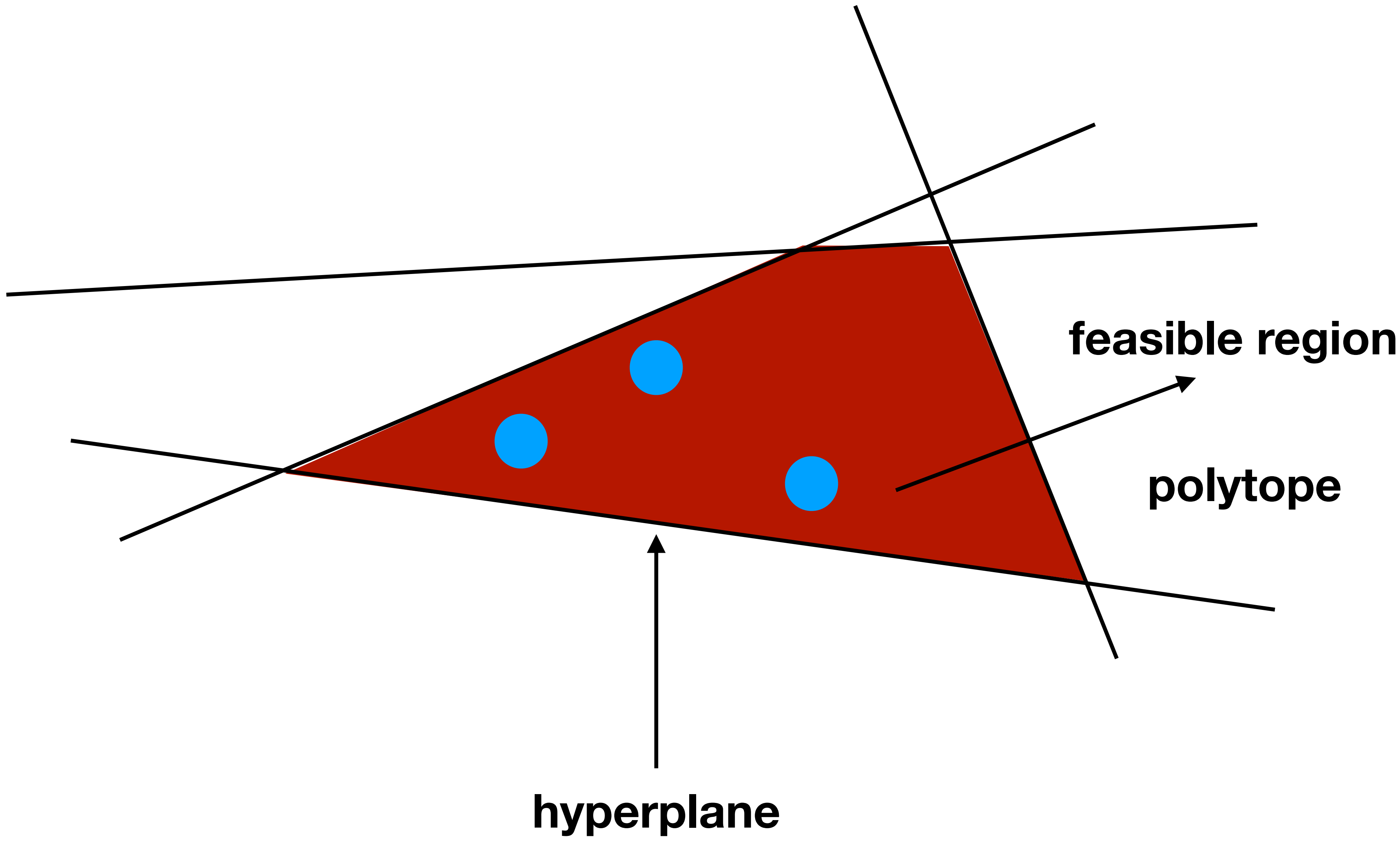

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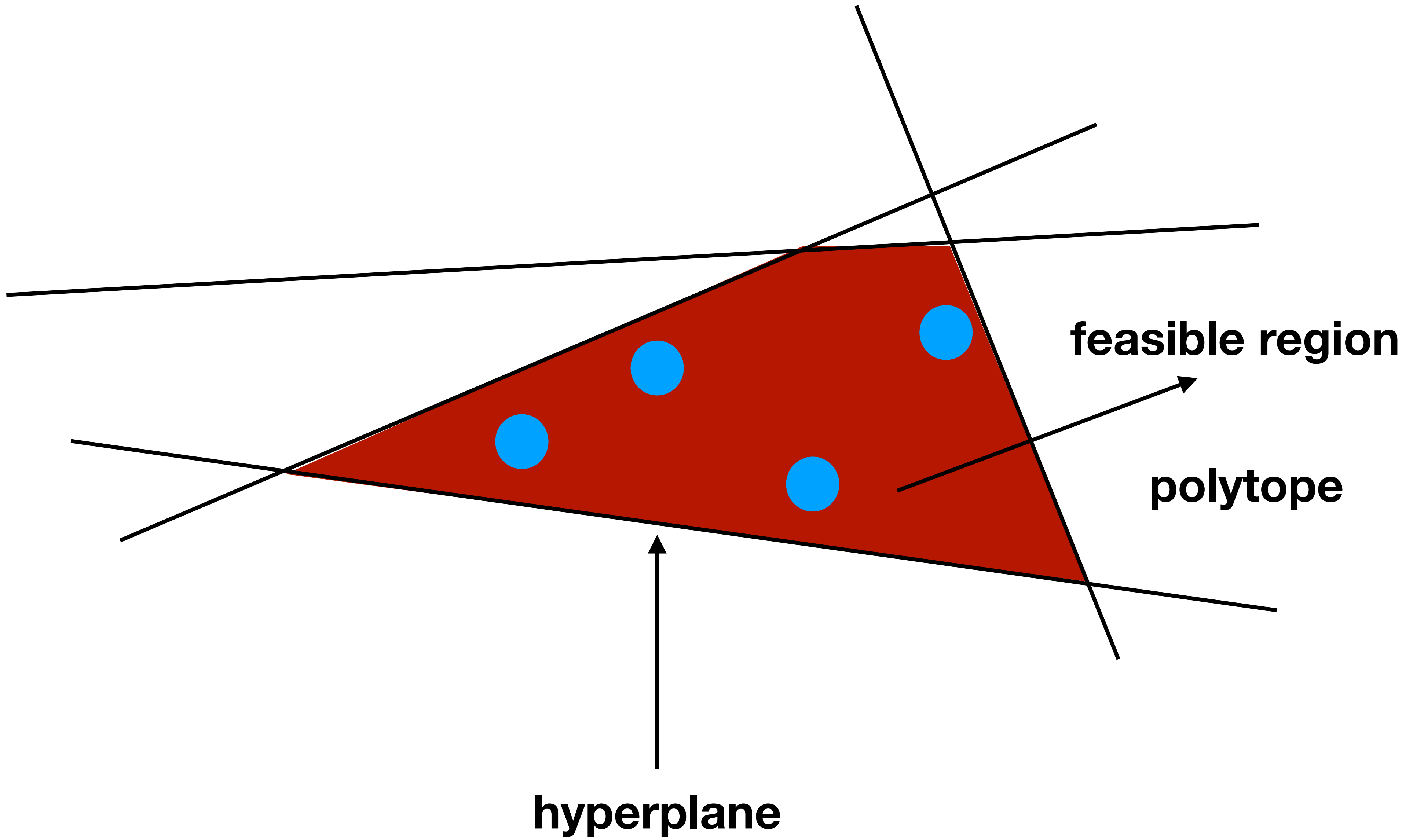

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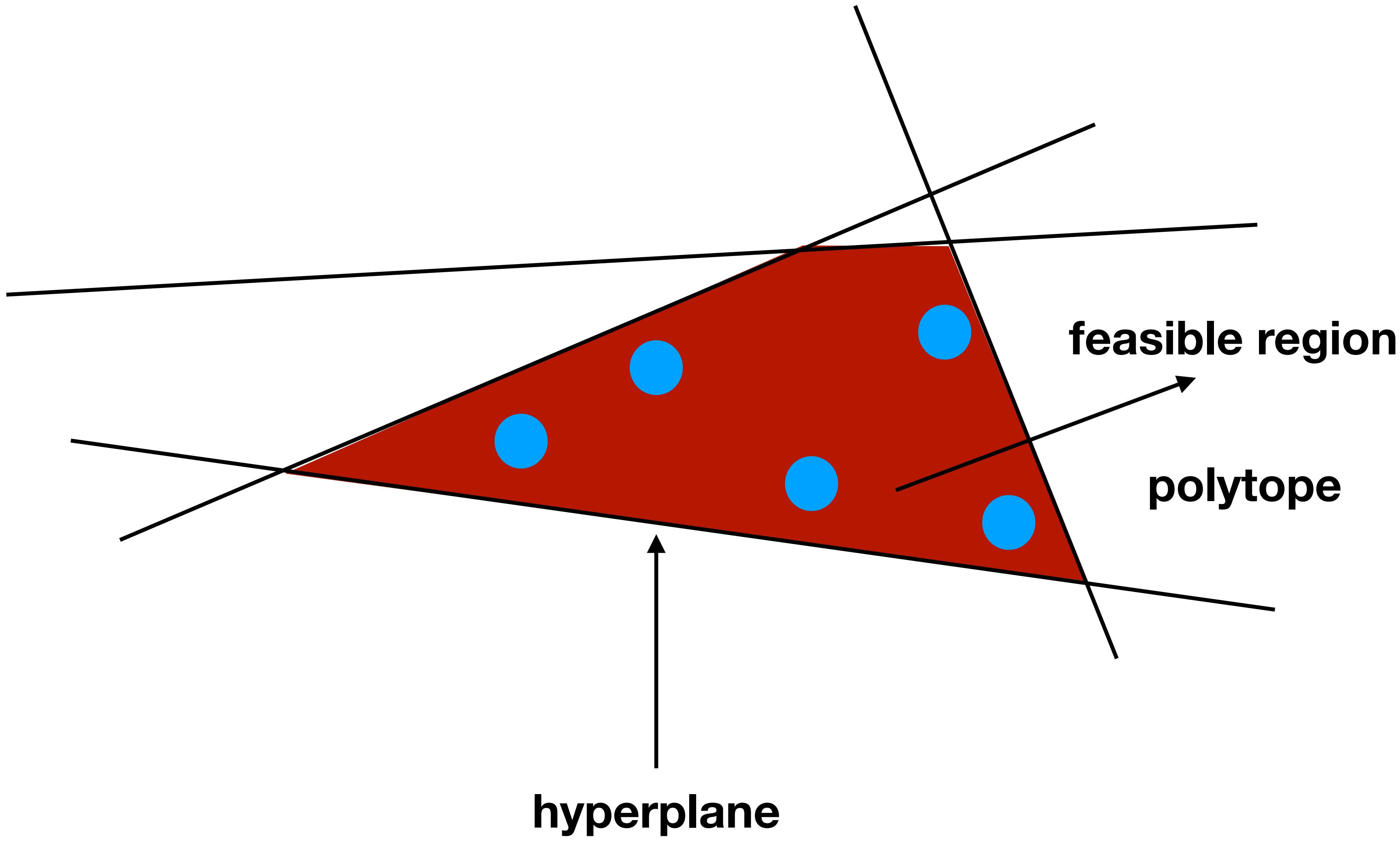

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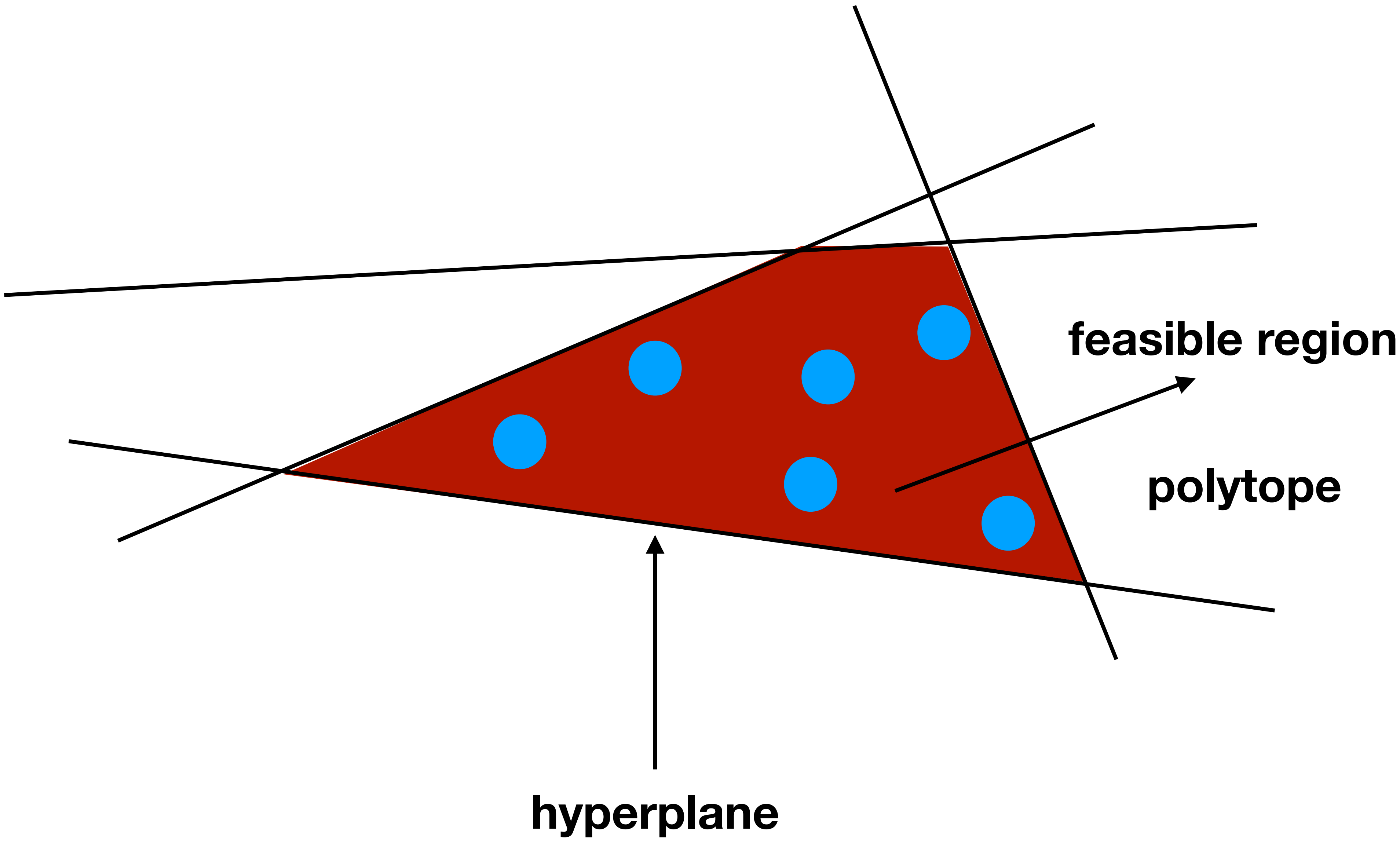

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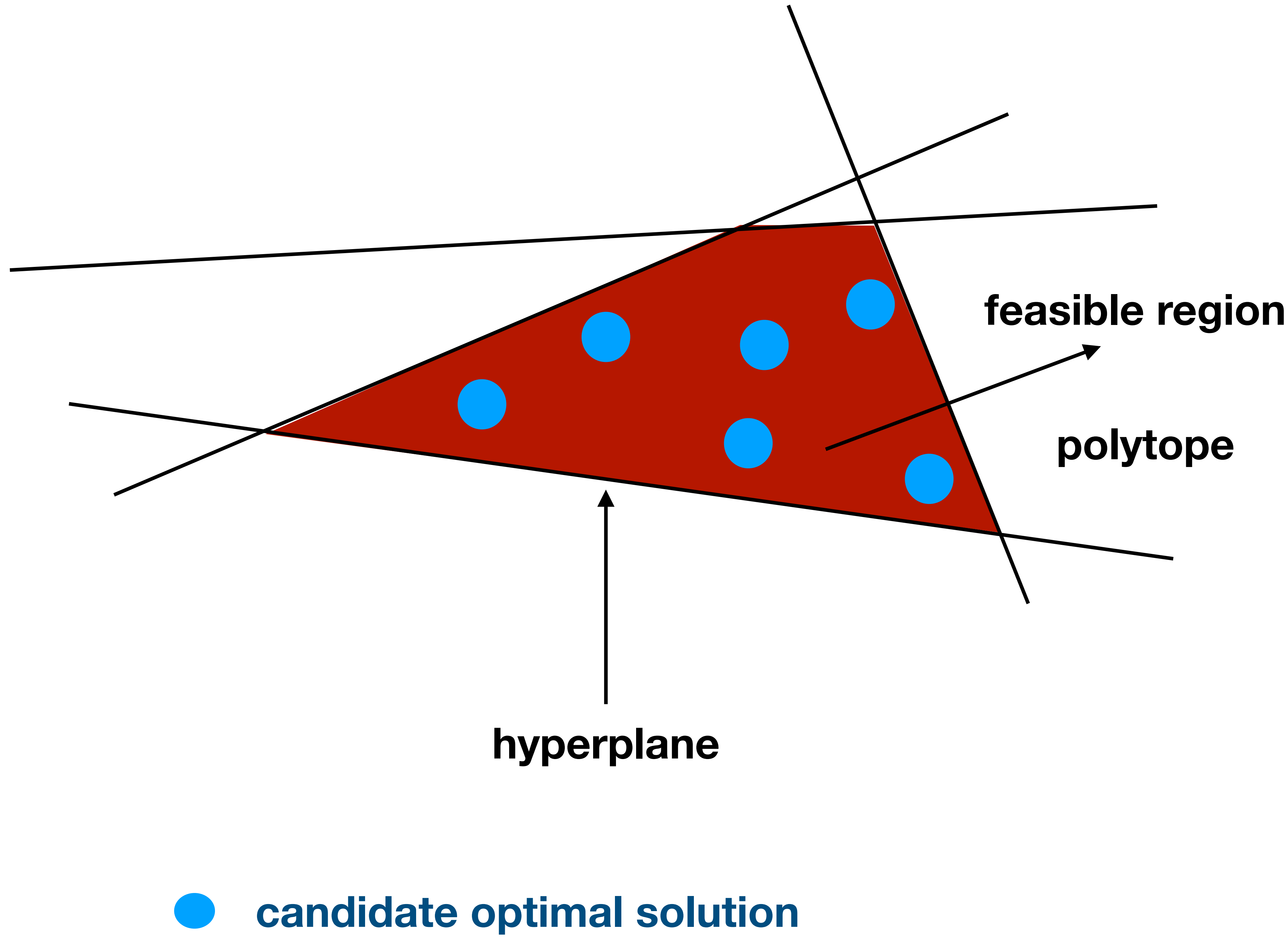

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Generally speaking, ILP solving is NP-hard.

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Integer Linear Programs generally cannot be solved in polynomial time (unless P=NP).