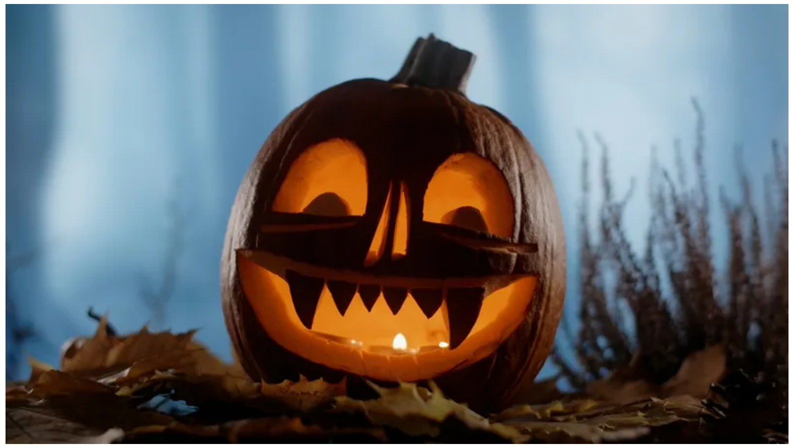

Algorithms and Data Structures

The Simplex Method

Linear Programs in Standard Form

$$\text{maximise} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1\in}^{n} \alpha_{ij} x_j \leq b_i, \quad i = 1, ..., m$$

$$x_j \geq 0, \quad j = 1, ..., n$$

The Linear Programming problem

Given a Linear Program (LP) in standard form:

Return an optimal solution (i.e., a feasible solution that maximises the objective function), or

Return that the LP is infeasible , or

Return that the LP is unbounded.

The Simplex Method

It does *not* run in polynomi

There are other algorithms
(e.g., the ellipsoid method,

Smoothed Analysis of Algorithms: Why the Simplex Algorithm Usually Takes Polynomial Time

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Then why study it? 🤔

It provides very useful insights into linear programming.

It runs quite fast in practice.

Polynomial time is still not out of the question.

The Simplex Method (explained via example)

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$$2x_1 + 3x_2 + x_3 \leq 5$$
$$4x_1 + x_2 + 2x + 3 \leq 11$$
$$3x_1 + 4x_2 + 2x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

Step 1: Slack Variables

For each constraint we introduce a *slack variable*:

e.g., for the constraint $2x_1 + 3x_2 + x_3 \leq 5$, we introduce variable w_1 and we write

$$w_1 = 5 - 2x_1 - 3x_2 - x_3$$

Step 1: Slack Variables

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$$2x_1 + 3x_2 + x_3 \leq 5$$
$$4x_1 + x_2 + 2x + 3 \leq 11$$
$$3x_1 + 4x_2 + 2x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

Step 1: Slack Variables

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$$w_1 = 5 - 2x_1 + 3x_2 + x_3$$
$$w_2 = 11 - 4x_1 + x_2 + 2x + 3$$
$$w_3 = 8 - 3x_1 + 4x_2 + 2x_3$$
$$x_1, x_2, x_3 \geq 0$$

Is this equivalent to the original LP?

Step 1: Slack Variables

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$$w_1 = 5 - 2x_1 + 3x_2 + x_3$$
$$w_2 = 11 - 4x_1 + x_2 + 2x + 3$$
$$w_3 = 8 - 3x_1 + 4x_2 + 2x_3$$
$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

Step 1: Slack Variables

For each constraint we introduce a *slack variable*:

e.g., for the constraint $2x_1 + 3x_2 + x_3 \leq 5$, we introduce variable w_1 and we write

$$w_1 = 5 - 2x_1 - 3x_2 - x_3$$

We also introduce a slack variable ζ for the objective function.

Step 1: Slack Variables

Maximise $\zeta = 5x_1 + 4x_2 + 3x_3$

subject to

$$
\begin{aligned}
w_1 &= 5 - 2x_1 + 3x_2 + x_3 \\
w_2 &= 11 - 4x_1 + x_2 + 2x + 3 \\
w_3 &= 8 - 3x_1 + 4x_2 + 2x_3 \\
&x_1, x_2, x_3, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

Dictionaries

Maximise $\zeta = \quad +5\, x_1 \quad +4\, x_2 \quad +3\, x_3$

subject to

$$
\begin{array}{rcrrrrrr}
w_1 & = & 5 & -2\, x_1 & -3\, x_2 & - & x_3 \\
w_2 & = & 11 & -4\, x_1 & - & x_2 & -2\, x_3 \\
w_3 & = & 8 & -3\, x_1 & -4\, x_2 & -2\, x_3
\end{array}
$$

nonbasic variables

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

basic variables

Step 1: Slack Variables

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$$w_1 = 5 - 2x_1 + 3x_2 + x_3$$
$$w_2 = 11 - 4x_1 + x_2 + 2x + 3$$
$$w_3 = 8 - 3x_1 + 4x_2 + 2x_3$$
$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

The Simplex Method (strategy)

Start with a feasible solution $x_1, x_2, x_3, w_1, w_2, w_3$

Improve this solution to some $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3$ such that $5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 > 5x_1 + 4x_2 + 3x_3$

Continue until no further improvement is possible (in that case we are at an optimal solution).

Does this remind you of something?

Dictionaries

Maximise $\zeta = \quad +5\,x_1 \quad +4\,x_2 \quad +3\,x_3$

subject to

$$\begin{aligned}
w_1 &= 5 && -2\,x_1 && -3\,x_2 && -\,x_3 \\
w_2 &= 11 && -4\,x_1 && -\,x_2 && -2\,x_3 \\
w_3 &= 8 && -3\,x_1 && -4\,x_2 && -2\,x_3
\end{aligned}$$

nonbasic variables

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

basic variables

Start with a feasible solution $x_1, x_2, x_3, w_1, w_2, w_3$

Suggestions?

Dictionaries

Maximise $\zeta = \quad +5\ x_1 \quad +4\ x_2 \quad +3\ x_3$

subject to

$$\begin{aligned} \boxed{w_1} &= 5 & -2\ x_1 & \quad -3\ x_2 & \quad -\ x_3 \\ \boxed{w_2} &= 11 & -4\ x_1 & \quad -\ x_2 & \quad -2\ x_3 \\ \boxed{w_3} &= 8 & -3\ x_1 & \quad -4\ x_2 & \quad -2\ x_3 \end{aligned}$$

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

nonbasic variables

basic variables

A solution obtained by setting all the nonbasic variables to 0 is called a basic feasible solution.

$x_1 = x_2 = x_3 = 0 \qquad w_1 = 5, w_2 = 11, w_3 = 8$

Step 2: Improving the solution

Maximise $\zeta = \quad +5x_1 \quad +4x_2 \quad +3x_3$

subject to

$$
\begin{aligned}
w_1 &= 5 \quad -2x_1 - 3x_2 - x_3 \\
w_2 &= 11 \quad -4x_1 - x_2 - 2x_3 \\
w_3 &= 8 \quad -3x_1 - 4x_2 - 2x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = x_2 = x_3 = 0 \qquad w_1 = 5, w_2 = 11, w_3 = 8$$

We can increase the value of some nonbasic variable, e.g., x_1

We should not violate any constraints though!

We don't want any of the slack variables to become negative.

Step 2: Improving the solution

Maximise $\zeta = \quad +5x_1 \quad +4x_2 \quad +3x_3$

subject to

$$\begin{aligned} w_1 &= 5 && -2x_1 && -3x_2 && -x_3 \\ w_2 &= 11 && -4x_1 && -x_2 && -2x_3 \\ w_3 &= 8 && -3x_1 && -4x_2 && -2x_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = 5/2, \;\; x_2 = x_3 = 0 \qquad w_1 = 0, \; w_2 = 1, \; w_3 = 1/2$$

For w_1, x_1 can become as large as $5/2 = 30/12$.

For w_2, x_1 can become as large as $11/4 = 33/12$.

For w_3, x_1 can become as large as $32/12$.

Step 2: Improving the solution

Maximise $\zeta = \quad +5\ x_1 \quad +4\ x_2 \quad +3\ x_3$

entering variable

subject to

$$\begin{aligned} w_1 &= 5 && -2\ x_1 && -3\ x_2 && -\ x_3 \\ w_2 &= 11 && -4\ x_1 && -\ x_2 && -2\ x_3 \\ w_3 &= 8 && -3\ x_1 && -4\ x_2 && -2\ x_3 \end{aligned}$$

leaving variable

just rearranging

what about here?

"row operations"

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = 5/2, \quad x_2 = x_3 = 0 \qquad w_1 = 0,\ w_2 = 1,\ w_3 = 1/2$$

We need to rearrange the inequalities, so that x_1 now only appears on the left.

This gives rise to a new dictionary, where x_1 is now basic and w_1 is nonbasic.

Step 2: Improving the solution

Maximise $\zeta = \quad +5\,x_1 \quad +4\,x_2 \quad +3\,x_3$

subject to

$$
\begin{aligned}
w_1 &= 5 && -2\,x_1 && -3\,x_2 && -\,x_3 \\
w_2 &= 11 && -4\,x_1 && -\,x_2 && -2\,x_3 \\
w_3 &= 8 && -3\,x_1 && -4\,x_2 && -2\,x_3
\end{aligned}
$$

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

just rearranging

what about here?

"row operations"

Notice that $w_2 - 2w_1 = 11 - 4x_1 - x_2 - 2x_3 - 10 + 4x_1 + 6x_2 + 2x_3$

$\Rightarrow w_2 = 1 + 2w_1 + 5x_2$ $\quad$ x_1 has been eliminated

The New Dictionary

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

the objective function value has increased

subject to

$$\begin{aligned} x_1 &= 2.5 - 0.5 w_1 - 1.5\, x_2 - 0.5\, x_3 \\ w_2 &= 1 + 2\, w_1 + 5\, x_2 \\ w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3 \end{aligned}$$

$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$

nonbasic variables

basic variables

$$w_1 = 0,\ x_2 = 0\ x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

Which variable should we try to increase next?

Step 3: Improving the solution

Maximise $\zeta = 12.5 - 2.5\,w_1 - 3.5\,x_2 + 0.5\,x_3$

subject to

$$
\begin{aligned}
x_1 &= 2.5 - 0.5 w_1 - 1.5\,x_2 - 0.5\,x_3 \\
w_2 &= 1 + 2\,w_1 + 5\,x_2 \\
w_3 &= 0.5 + 1.5\,w_1 + 0.5\,x_2 - 0.5\,x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

leaving variable

entering variable

$$x_3 = 1, \quad w_1 = x_2 = 0 \qquad\qquad x_1 = 2,\ w_2 = 1,\ w_3 = 0$$

For x_1, x_3 can become as large as $2.5/0.5 = 5$.
For w_2, x_3 can become as large as ∞.
For w_3, x_3 can become as large as $0.5/0.5 = 1$.

The New Dictionary

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

the objective function value has increased

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

nonbasic variables

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

basic variables

$$w_1 = 0,\ x_2 = 0\ w_3 = 0 \qquad x_1 = 2,\ w_2 = 1,\ w_3 = 1$$

Which variable should we try to increase next?

We have computed an optimal solution!

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

5. Increase the value of the entering variable to be $x_k = \min_{i : \hat{a}_{ik} > 0} \hat{b}_i / \hat{a}_{ik}$

6. Compute the new dictionary making sure x_k only appears on the left.

Let’s do it again, “mechanically”

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$$2x_1 + 3x_2 + x_3 \leq 5$$
$$4x_1 + x_2 + 2x + 3 \leq 11$$
$$3x_1 + 4x_2 + 2x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

Maximise $\zeta = 5x_1 + 4x_2 + 3x_3$

subject to

$$w_1 = 5 - 2x_1 + 3x_2 + x_3$$
$$w_2 = 11 - 4x_1 + x_2 + 2x + 3$$
$$w_3 = 8 - 3x_1 + 4x_2 + 2x_3$$
$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

2. Write the original dictionary.

Maximise $\zeta = \quad +5x_1 + 4x_2 + 3x_3$

subject to

$$
\begin{aligned}
w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\
w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\
w_3 &= 8 - 3x_1 - 4x_2 - 2x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$
\begin{array}{llrrrr}
\textbf{Maximise} & \zeta = & & +5\,x_1 & +4\,x_2 & +3\,x_3 \\
\textbf{subject to} & w_1 = & 5 & -2\,x_1 & -3\,x_2 & -\,x_3 \\
& w_2 = & 11 & -4\,x_1 & -\,x_2 & -2\,x_3 \\
& w_3 = & 8 & -3\,x_1 & -4\,x_2 & -2\,x_3 \\
\end{array}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = x_2 = x_3 = 0 \qquad w_1 = 5, w_2 = 11, w_3 = 8$$

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

entering variable

$$
\begin{array}{lrcrlrlrl}
\textbf{Maximise} & \zeta = & & +5 & x_1 & +4 & x_2 & +3 & x_3 \\
\textbf{subject to} & w_1 = & 5 & -2 & x_1 & -3 & x_2 & - & x_3 \\
 & w_2 = & 11 & -4 & x_1 & - & x_2 & -2 & x_3 \\
 & w_3 = & 8 & -3 & x_1 & -4 & x_2 & -2 & x_3
\end{array}
$$

leaving variable

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = x_2 = x_3 = 0 \qquad w_1 = 5, w_2 = 11, w_3 = 8$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

$5/2$ **vs** $11/4$ **vs** $8/3 \Rightarrow w_1$

5. Increase the value of the entering variable to be $x_k = \min_{i:\hat{a}_{ik}>0} \hat{b}_i/\hat{a}_{ik}$

$$
\begin{array}{llrrrr}
\textbf{Maximise} & \zeta = & & +5\ x_1 & +4\ x_2 & +3\ x_3 \\
\textbf{subject to} & w_1 = & 5 & -2\ x_1 & -3\ x_2 & -\ x_3 \\
 & w_2 = & 11 & -4\ x_1 & -\ x_2 & -2\ x_3 \\
 & w_3 = & 8 & -3\ x_1 & -4\ x_2 & -2\ x_3 \\
\end{array}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = 2.5,\ x_2 = 0,\ x_3 = 0$$

6. Compute the new dictionary making sure x_k only appears on the left.

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$
\begin{aligned}
x_1 &= 2.5 - 0.5\, w_1 - 1.5\, x_2 - 0.5\, x_3 \\
w_2 &= 1 + 2\, w_1 + 5\, x_2 \\
w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

5. Increase the value of the entering variable to be $x_k = \min_{i : \hat{a}_{ik} > 0} \hat{b}_i / \hat{a}_{ik}$

6. Compute the new dictionary making sure x_k only appears on the left.

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$
\begin{aligned}
x_1 &= 2.5 - 0.5\, w_1 - 1.5\, x_2 - 0.5\, x_3 \\
w_2 &= 1 + 2\, w_1 + 5\, x_2 \\
w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$
\begin{aligned}
x_1 &= 2.5 - 0.5 w_1 - 1.5\, x_2 - 0.5\, x_3 \\
w_2 &= 1 + 2\, w_1 + 5\, x_2 \\
w_3 &= 0.5 + 1.5 w_1 + 0.5 x_2 - 0.5\, x_3
\end{aligned}
$$

entering variable

leaving variable

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = x_3 = 0 \qquad x_1 = 2.5,\ w_2 = 1,\ w_3 = 0.5$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

$2.5/0.5$ **vs** ∞ **vs** $0.5/0.5 \Rightarrow w_3$

5. Increase the value of the entering variable to be $x_k = \min_{i:\hat{a}_{ik}>0} \hat{b}_i / \hat{a}_{ik}$

Maximise $\zeta = 12.5 - 2.5\, w_1 - 3.5\, x_2 + 0.5\, x_3$

subject to

$$
\begin{aligned}
x_1 &= 2.5 - 0.5\, w_1 - 1.5\, x_2 - 0.5\, x_3 \\
w_2 &= 1 + 2\, w_1 + 5\, x_2 \\
w_3 &= 0.5 + 1.5\, w_1 + 0.5\, x_2 - 0.5\, x_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$x_1 = 2.5,\ x_2 = 0,\ x_3 = 1$$

6. Compute the new dictionary making sure x_k only appears on the left.

$$
\begin{array}{llrlrlrl}
\textbf{Maximise} & \zeta = & 13 & - \; w_1 & -2 & x_2 & - & w_3 \\
\textbf{subject to} & x_1 = & 2 & -2 \; w_1 & -2 & x_2 & + & w_3 \\
 & w_2 = & 1 & +2 \; w_1 & +5 & x_2 & & \\
 & x_3 = & 1 & +3 \; w_1 & + & x_2 & -2 & w_3 \\
\end{array}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$\begin{aligned} x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = w_3 = 0 \qquad x_1 = 2,\ w_2 = 1,\ w_3 = 1$$

The Simplex Method

1. Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and ζ.

2. Write the original dictionary.

Repeat:

3. Find a basic feasible solution by setting the nonbasic variables to 0.

4. Choose a variable to enter the basis (entering variable) and a variable to leave the basis (leaving variable).

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

5. Increase the value of the entering variable to be $x_k = \min_{i : \hat{a}_{ik} > 0} \hat{b}_i / \hat{a}_{ik}$

6. Compute the new dictionary making sure x_k only appears on the left.

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = x_2 = w_3 = 0 \qquad x_1 = 2,\ w_2 = 1,\ w_3 = 1$$

We have computed an optimal solution!

Potential Problem

Consider the following LP:

Maximise $-2x_1 - x_2$

subject to

$$
\begin{aligned}
-x_1 + x_2 &\leq -1 \\
-x_1 - 2x_2 &\leq -2 \\
x_2 &\leq 1 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Corresponding dictionary

Maximise $\zeta = \quad -2x_1 - x_2$

subject to

$$
\begin{aligned}
w_1 &= -1 + x_1 - x_2 \\
w_2 &= -2 + x_1 + 2x_2 \\
w_3 &= 1 \phantom{{}+x_1} - x_2
\end{aligned}
$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$w_1 = x_2 = x_3 = 0 \qquad w_1 = -1,\ w_2 = -2,\ w_3 = 1$$

The dictionary is infeasible!

Initialisation

Consider the following LP:

Maximise $-2x_1 - x_2$

subject to

$$
\begin{aligned}
-x_1 + x_2 &\leq -1 \\
-x_1 - 2x_2 &\leq -2 \\
x_2 &\leq 1 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Initialisation

Consider the following alternative LP:

Maximise $-x_0$

subject to

$$
\begin{aligned}
-x_1 + x_2 - x_0 &\leq -1 \\
-x_1 - 2x_2 - x_0 &\leq -2 \\
x_2 - x_0 &\leq 1 \\
x_1, x_2, x_0 &\geq 0
\end{aligned}
$$

Initialisation

subject to
$$
\begin{aligned}
-x_1 + x_2 &\leq -1 \\
-x_1 - 2x_2 &\leq -2 \\
x_2 &\leq 1 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

The first LP is feasible if any only if the second LP has an optimal solution of value 0.

Maximise $-x_0$

subject to
$$
\begin{aligned}
-x_1 + x_2 - x_0 &\leq -1 \\
-x_1 - 2x_2 - x_0 &\leq -2 \\
x_2 - x_0 &\leq 1 \\
x_1, x_2, x_0 &\geq 0
\end{aligned}
$$

Initialisation

Consider the following alternative LP:

Maximise $- x_0$

subject to

$$
\begin{aligned}
-x_1 + x_2 - x_0 &\leq -1 \\
-x_1 - 2x_2 - x_0 &\leq -2 \\
x_2 - x_0 &\leq 1 \\
x_1, x_2, x_0 &\geq 0
\end{aligned}
$$

Auxiliary problem dictionary

Maximise $\zeta = \qquad - x_0$

subject to

$$
\begin{aligned}
w_1 &= -1 + x_1 - x_2 + x_0 \\
w_2 &= -2 + x_1 + 2x_2 + x_0 \\
w_3 &= 1 - x_2 + x_0
\end{aligned}
$$

leaving variable

entering variable

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

3. Find a basic feasible solution by setting the nonbasic variables to 0.

The dictionary is infeasible!

Entering variable: x_0

Leaving variable: the one that is "most infeasible"

6. Compute the new dictionary making sure x_0 only appears on the left.

The new auxiliary problem dictionary

$$
\begin{array}{lrl}
\textbf{Maximise} & \zeta = & -2 + x_1 + 2x_2 - w_2 \\
\textbf{subject to} & w_1 = & 1 - 3x_2 + w_2 \\
 & x_0 = & 2 - x_1 - 2x_2 + w_2 \\
 & w_3 = & 3 - x_1 - 3x_2 + w_2 \\
 & & x_1, x_2, w_1, w_2, w_3, x_0 \geq 0
\end{array}
$$

The dictionary is feasible, we can apply the simplex method.

steps…

The final auxiliary problem dictionary

Maximise $\zeta = -x_0$

subject to

$$
\begin{aligned}
x_2 &= 0.33 & & & -0.33\,w_1 & +0.33\,w_2 \\
x_1 &= 1.33 & - & x_0 & +0.67\,w_1 & +0.33\,w_2 \\
w_3 &= 2 & + & x_0 & +0.33\,w_1 & +0.33\,w_2
\end{aligned}
$$

$$x_1, x_2, w_1, w_2, w_3, x_0 \geq 0$$

Remove x_0 from the constraints and substitute the original objective function.

The first dictionary of our original problem

Maximise $\zeta = -2x_1 - x_2$

subject to

$$x_2 = 0.33 - 0.33w_1 + 0.33w_2$$

$$x_1 = 1.33 + 0.67w_1 + 0.33w_2$$

$$w_3 = 2 + 0.33w_1 + 0.33w_2$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

We should have only nonbasic variables in the objective function.

Easy Fix

Maximise $\zeta = \quad -2\ x_1 \quad - x_2$

subject to

$$\begin{aligned} w_1 &= -1 + x_1 - x_2 \\ w_2 &= -2 + x_1 + 2\ x_2 \\ w_3 &= 1 - x_2 \end{aligned}$$

$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

We have $\zeta = -2x_1 - x_2 = -3 - w_1 - w_2$

The first dictionary of our original problem

Maximise $\zeta = -3w_1 - w_2$

subject to

$$\begin{aligned} x_2 &= 0.33 - 0.33\,w_1 + 0.33\,w_2 \\ x_1 &= 1.33 + 0.67\,w_1 + 0.33\,w_2 \\ w_3 &= 2 + 0.33\,w_1 + 0.33\,w_2 \end{aligned}$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

We have found an optimal solution!

We were lucky: we can only expect to find a feasible solution.

3. Find a basic feasible solution by setting the nonbasic variables to 0.

$$w_1 = w_2 = 0 \qquad x_1 = 1.33,\ x_2 = 0.33,\ w_3 = 2$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

What if we have this dictionary?

entering variable

Maximise $\zeta = 5 + x_3 - x_1$

subject to

$$
\begin{aligned}
x_2 &= 5 + 2x_3 - 3x_1 \\
x_4 &= 7 - 4x_1 \\
x_5 &= x_1
\end{aligned}
$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Entering variable: Any variable with positive coefficient in the objective function. If none exists, `break`;

Leaving variable: The variable with the smallest ratio $\hat{b}_i / \hat{a}_{ik}$ (for the constraint $\hat{b}_i - \hat{a}_{ik} x_k \geq 0$).

What if we have this dictionary?

entering variable

Maximise $\zeta = 5 + x_3 - x_1$

subject to

$$
\begin{aligned}
x_2 &= 5 + 2x_3 - 3x_1 \\
x_4 &= 7 - 4x_1 \\
x_5 &= x_1
\end{aligned}
$$

The LP is unbounded!

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

We can increase the value of some nonbasic variable, here x_3

We should not violate any constraints though!

We don't want any of the slack variables to become negative.

This does not happen regardless of how much we increase x_3.

What about this dictionary?

Maximise $\zeta = 3 - 0.5\,x_1 + 2\,x_2 - 1.5\,w_1$

subject to

$$x_3 = 1 - 0.5\,x_1 - 0.5\,w_1$$

$$w_2 = x_1 - x_2 + w_1$$

$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

leaving variable

entering variable

We can increase the value of some nonbasic variable, here x_2

We should not violate any constraints though!

We don't want any of the slack variables to become negative.

x_2 cannot be increased! Are we stuck?

Degeneracy! Next lecture

Historic Note

The Simplex Method was invented by George Dantzig in 1947.

It is still being used today in most of the LP-solvers.

> The origins of the simplex method go back to one of two famous unsolved problems in mathematical statistics proposed by Jerzy Neyman, which I mistakenly solved as a homework problem; it later

Dantzig. Origins of the Simplex Method. In *A History of Scientific Computing*, 1990.