Algorithms and Data Structures

Minimum Spanning Trees - Greedy Algorithms Running Time

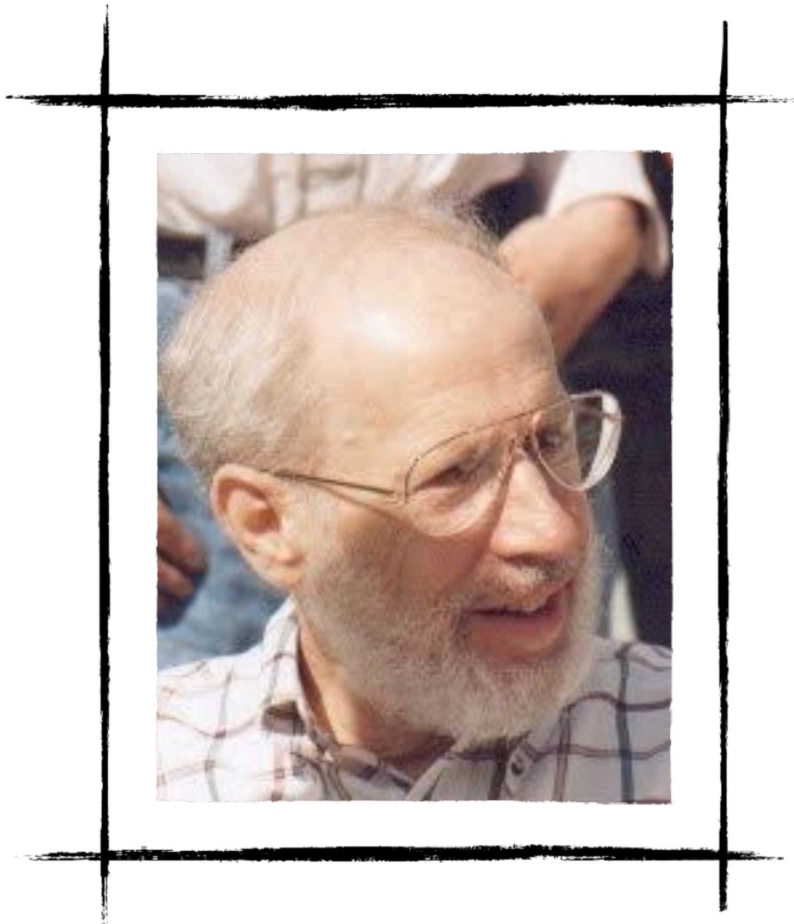
Minimum Spanning Tree

$G'=(V, T)$ is a spanning tree and the problem is called the Minimum Spanning Tree problem.

Consider a *connected* graph $G=(V, E)$, such that for every edge $e=\{v,w\}$ of E, there is an associated positive cost c_e.

Goal: Find a subset T of E so that the graph $G'=(V, T)$ is connected and the total cost $\sum_{e \in T} c_e$ is minimised.

Kruskal's Algorithm



Start with an empty set of edges T.

Add one edge to T.

Which one?

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We continue like this.

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Only if we don't introduce any cycles.

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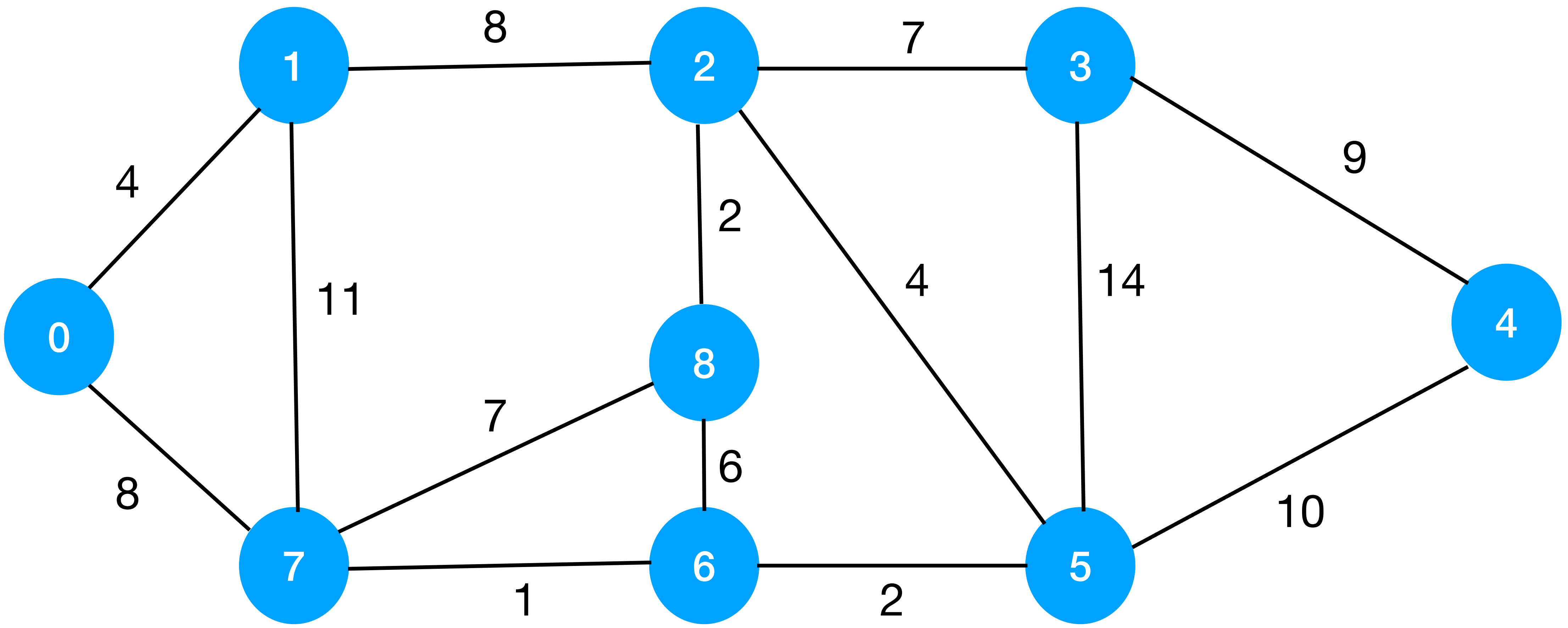

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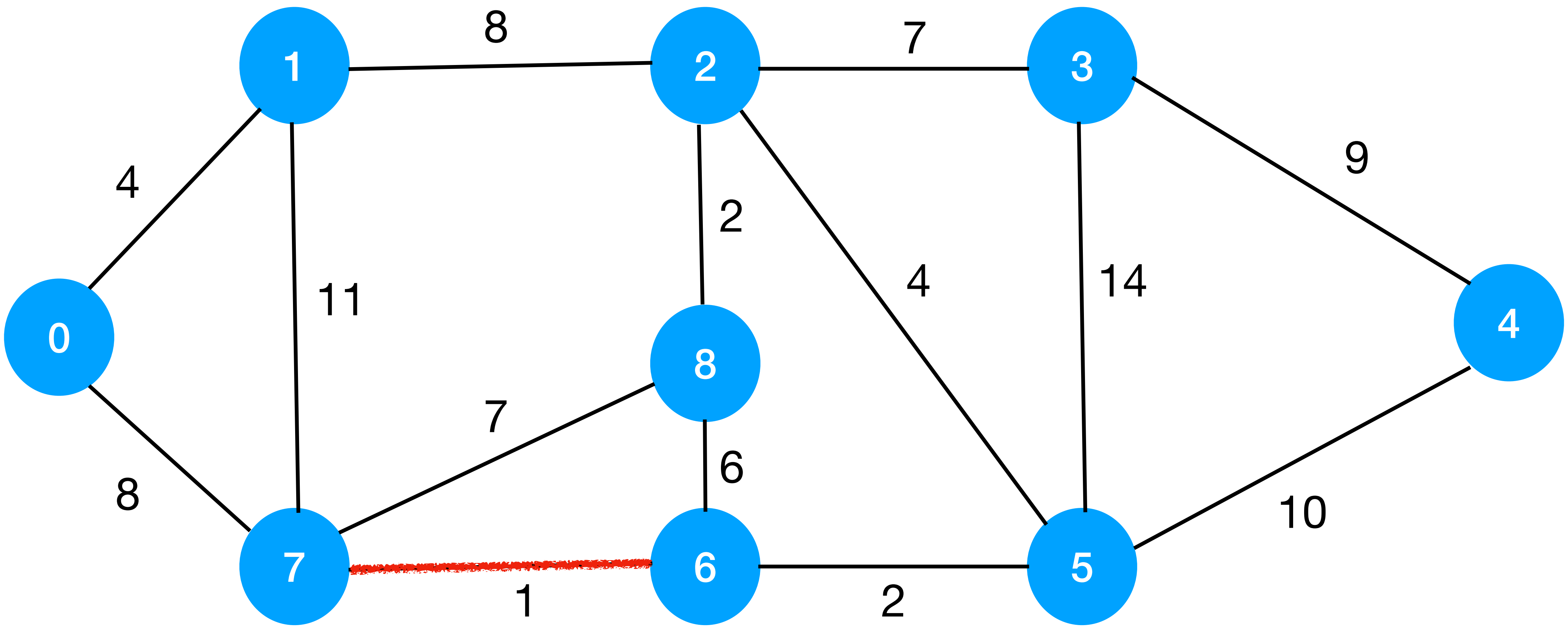

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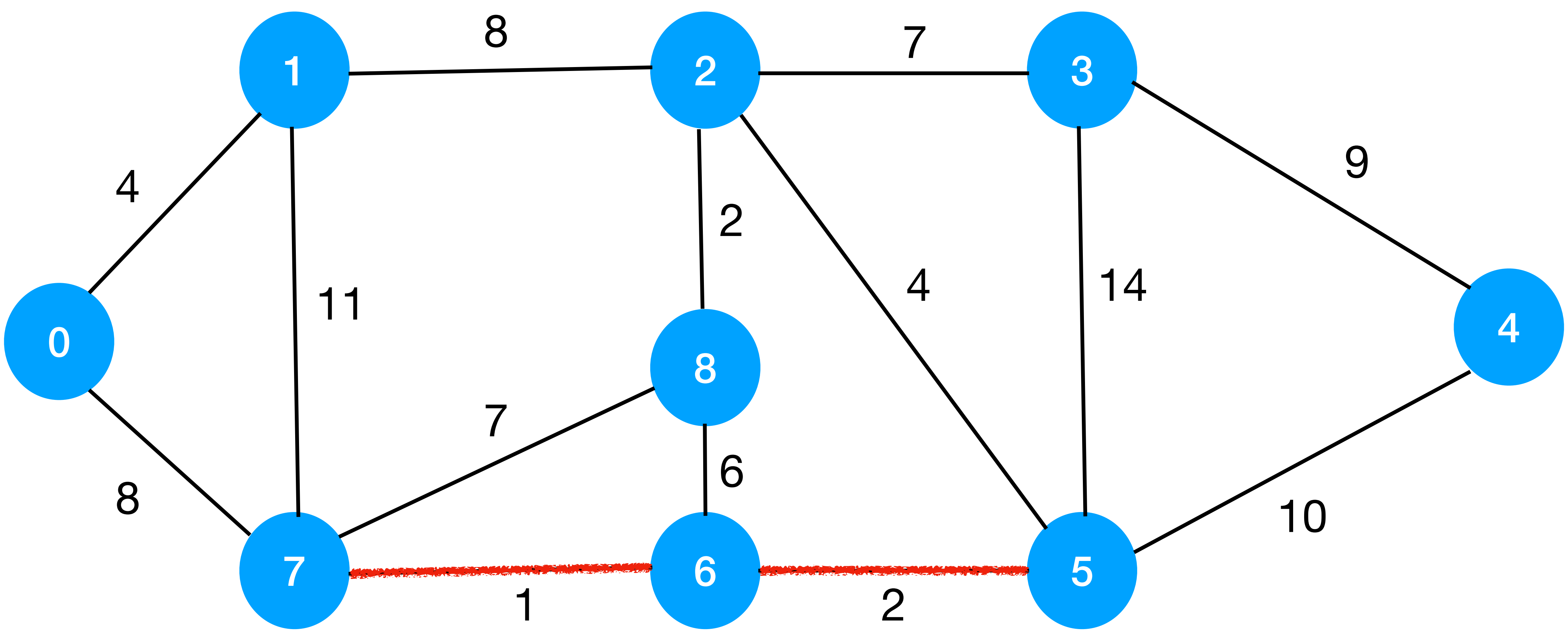

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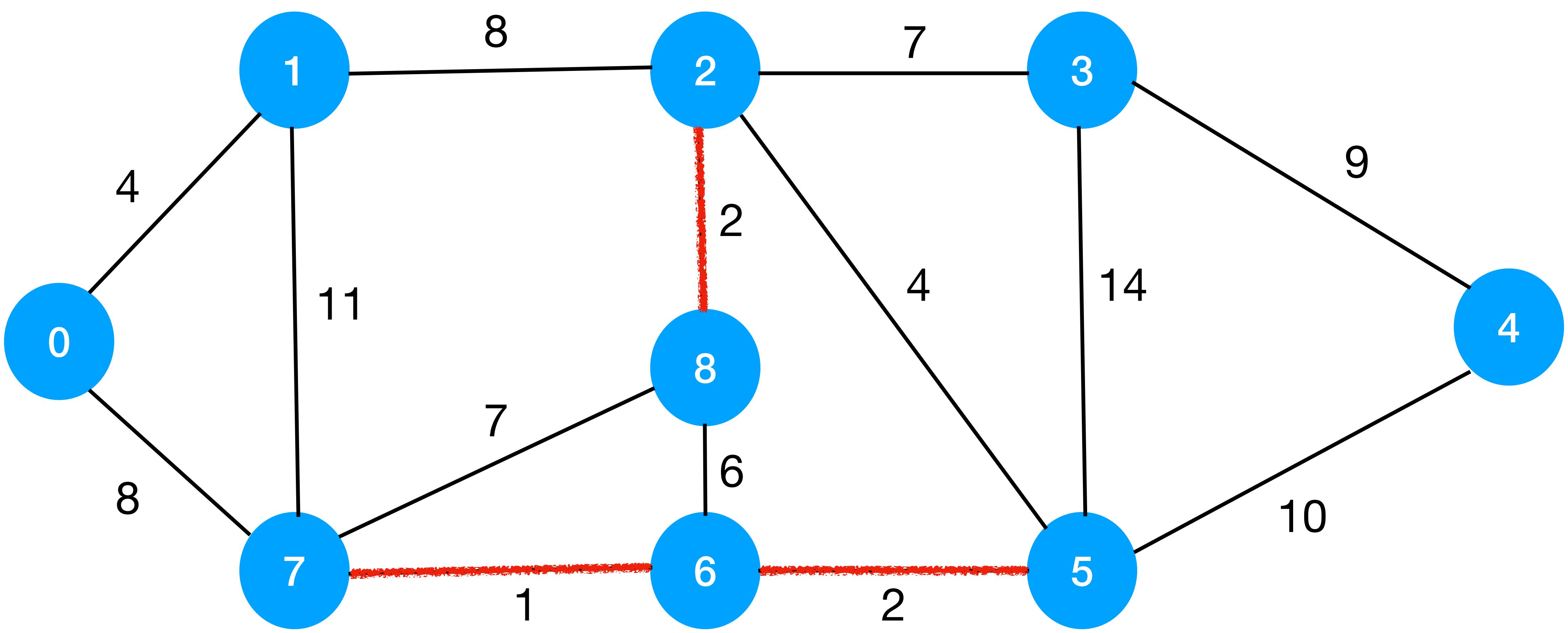

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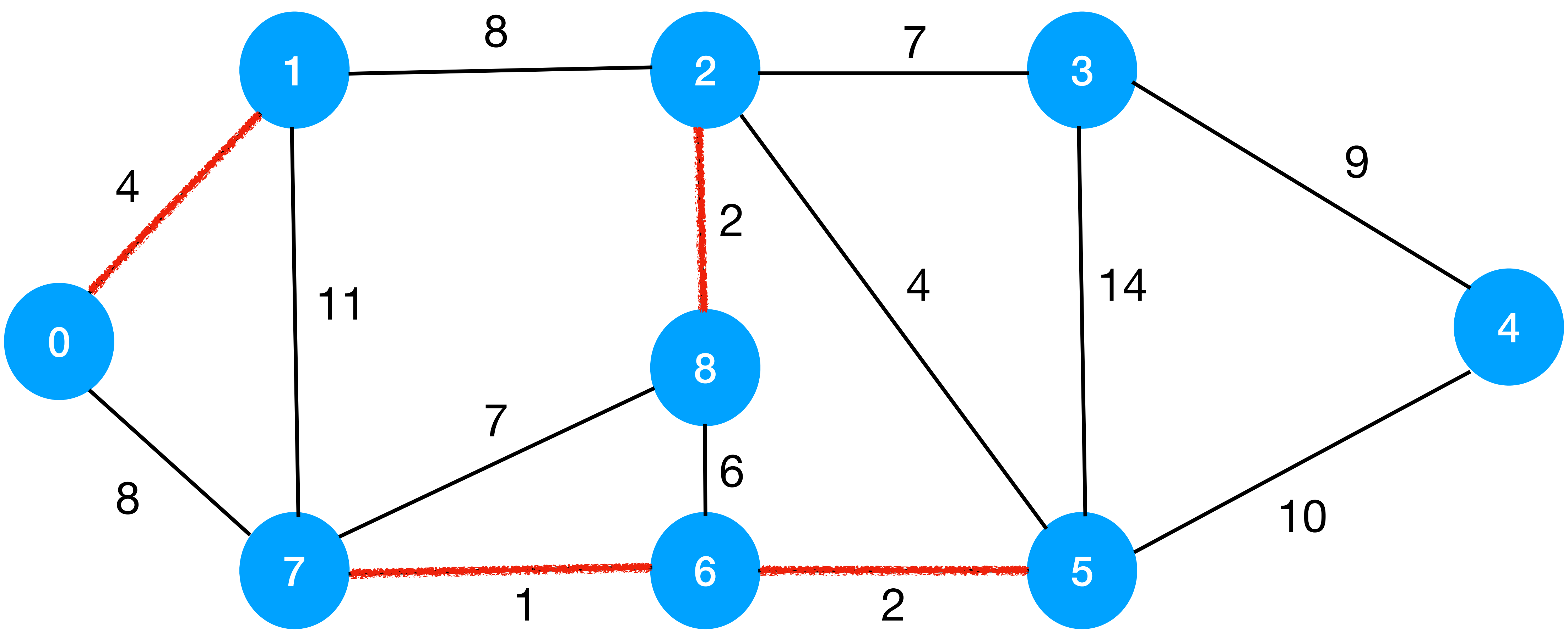

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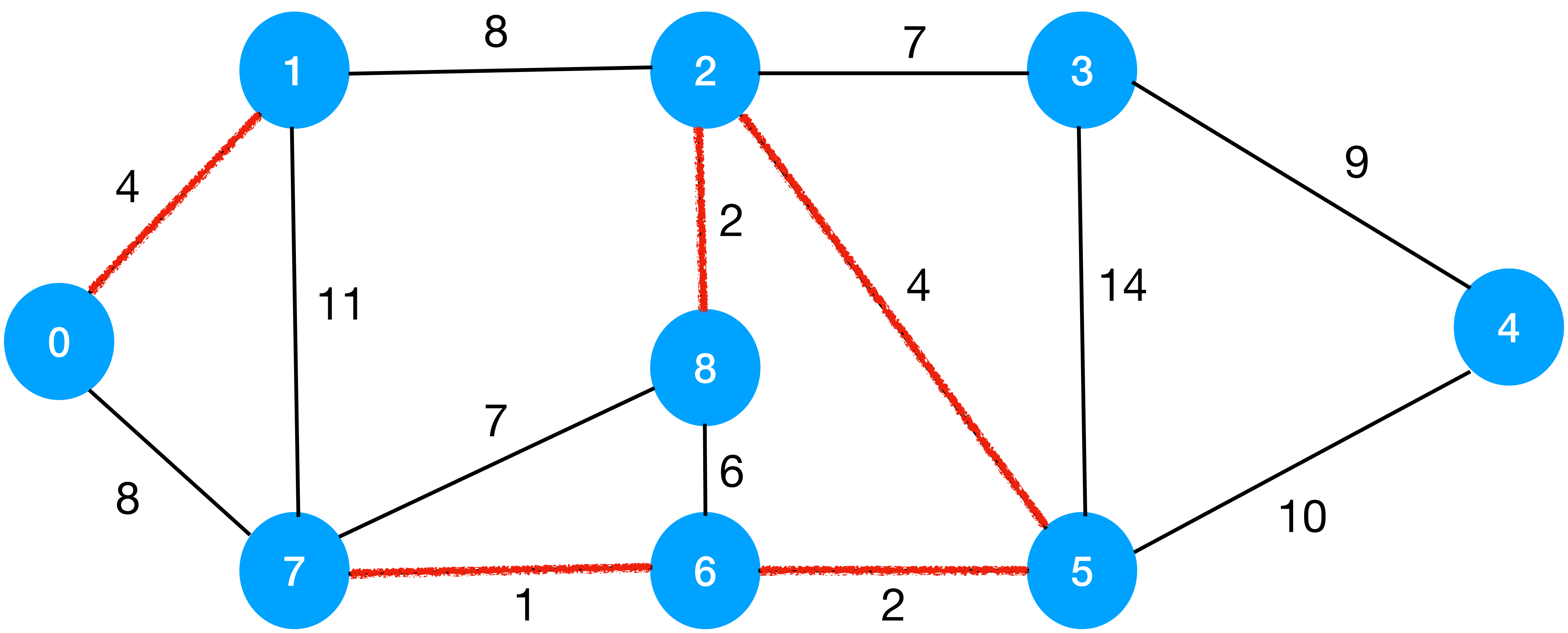

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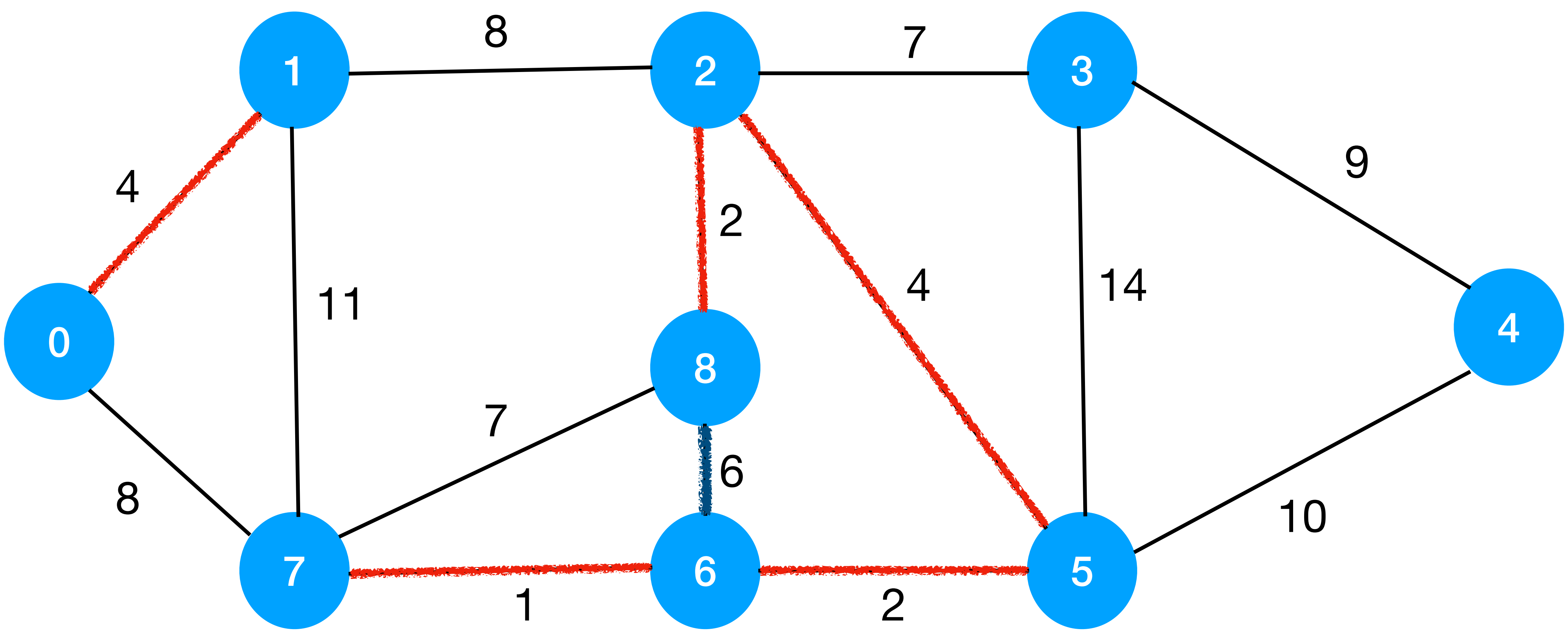

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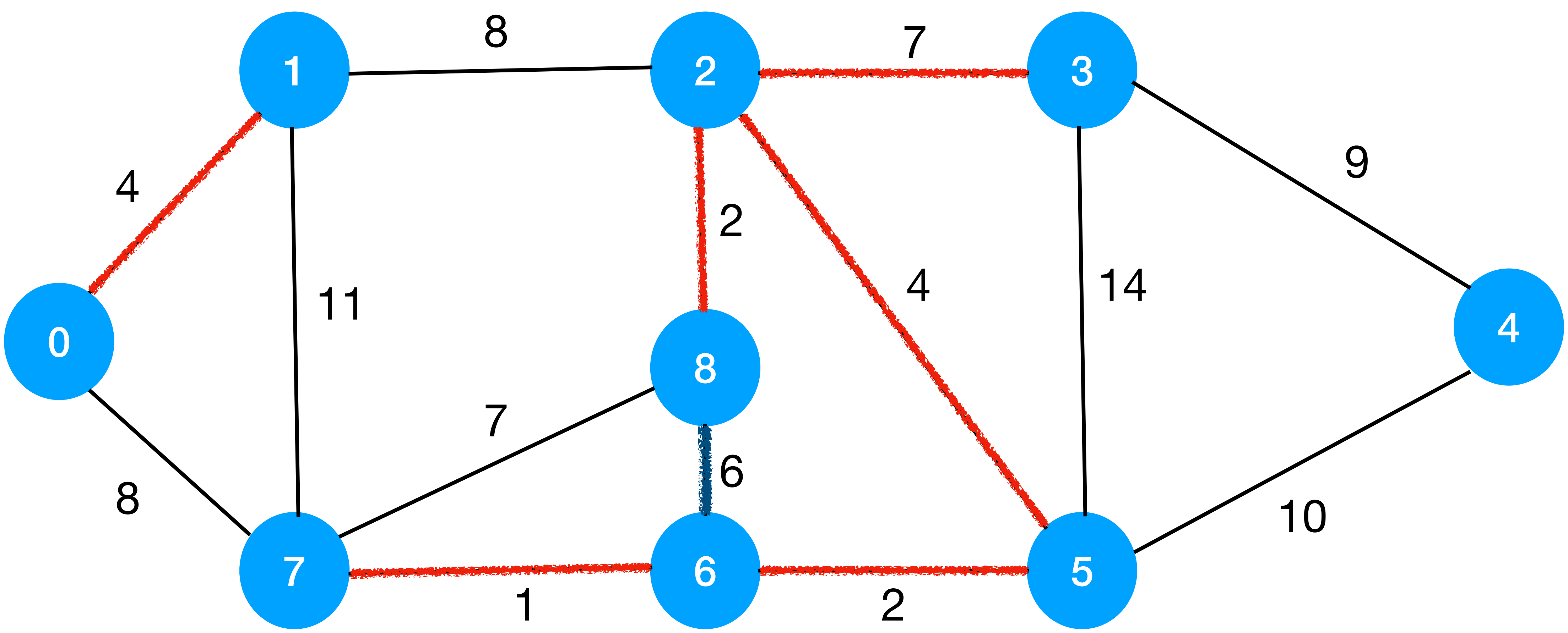

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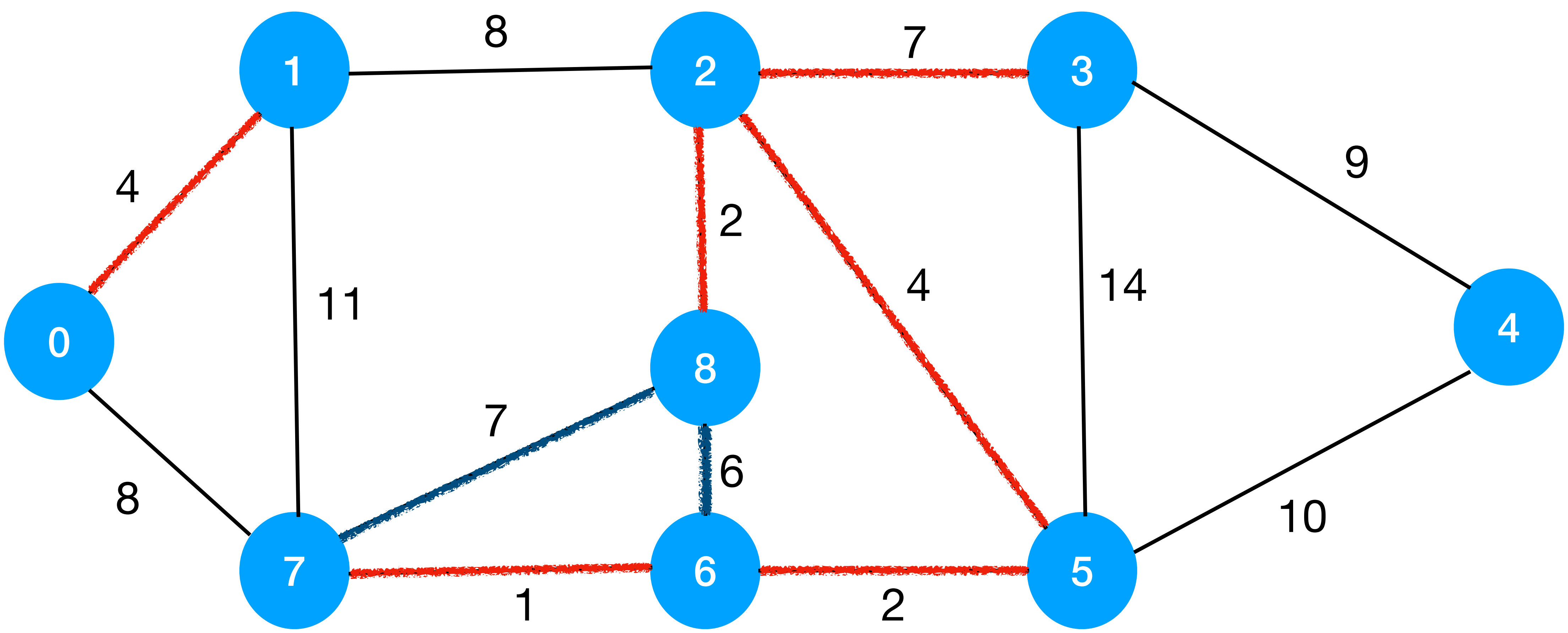

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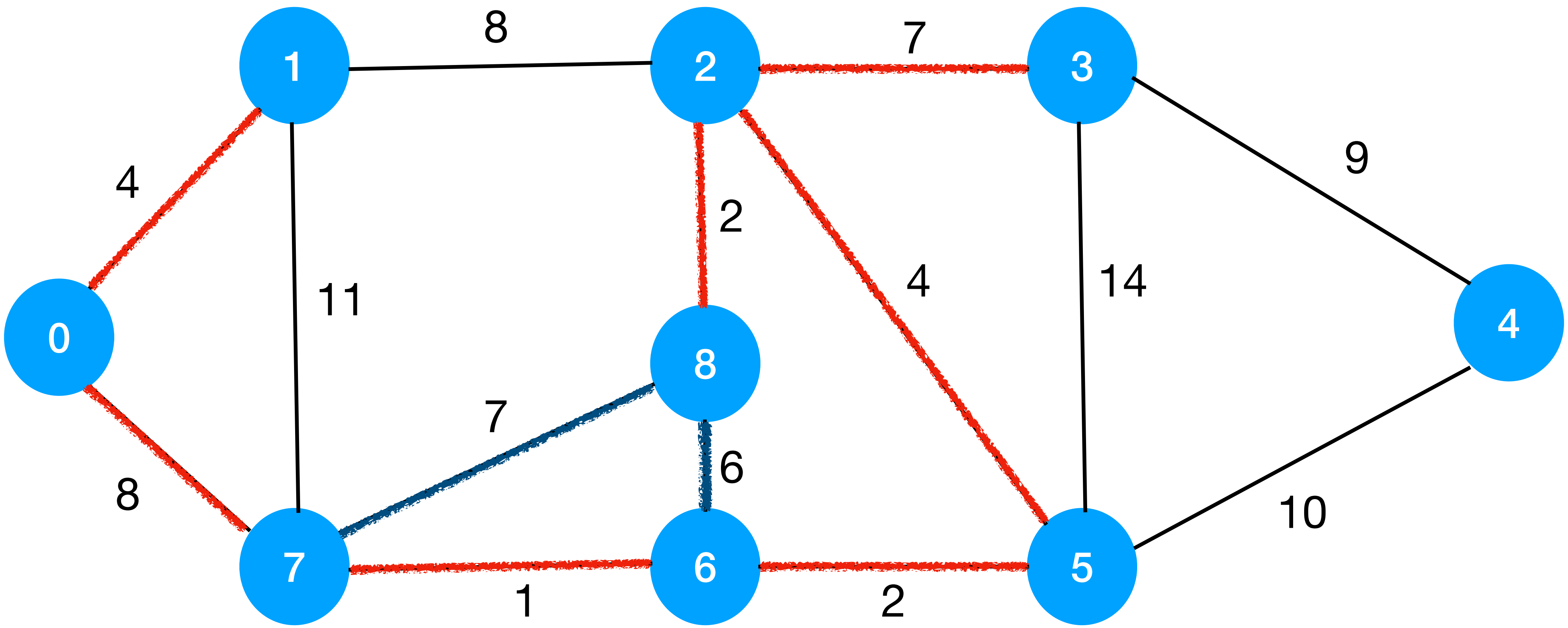

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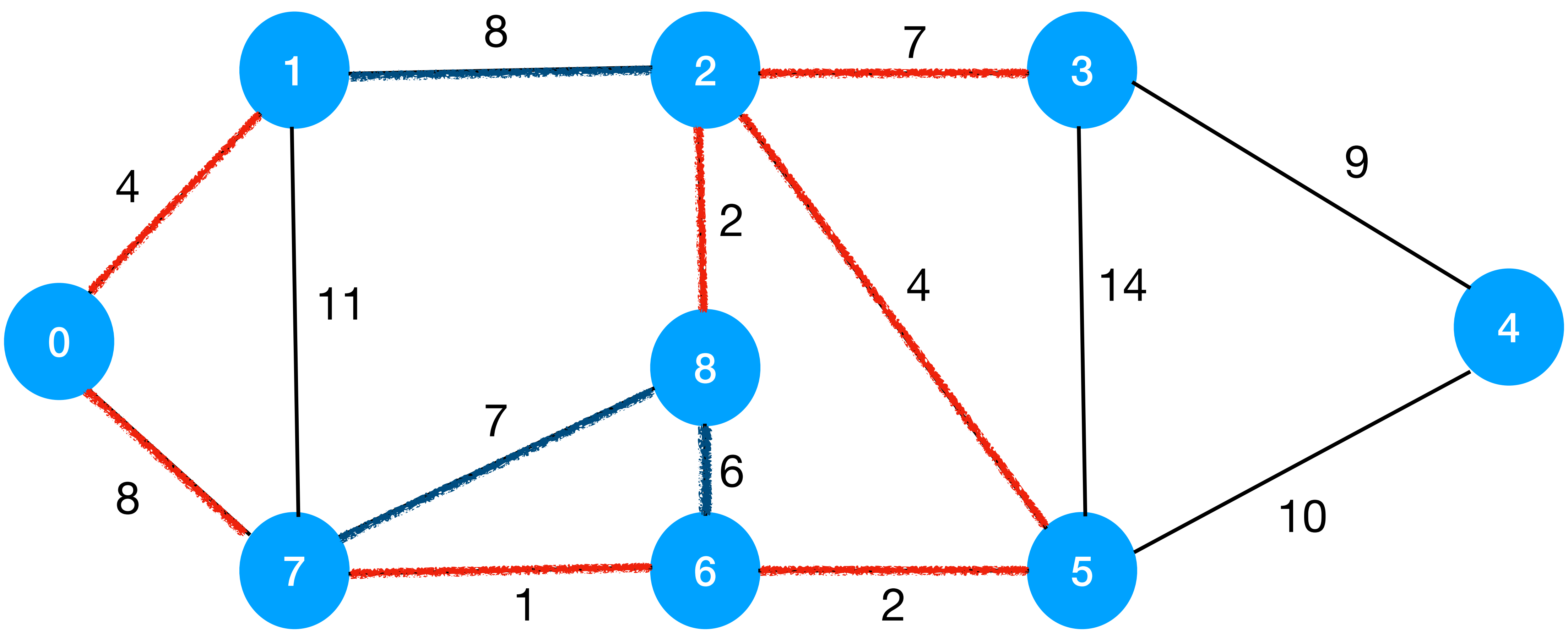

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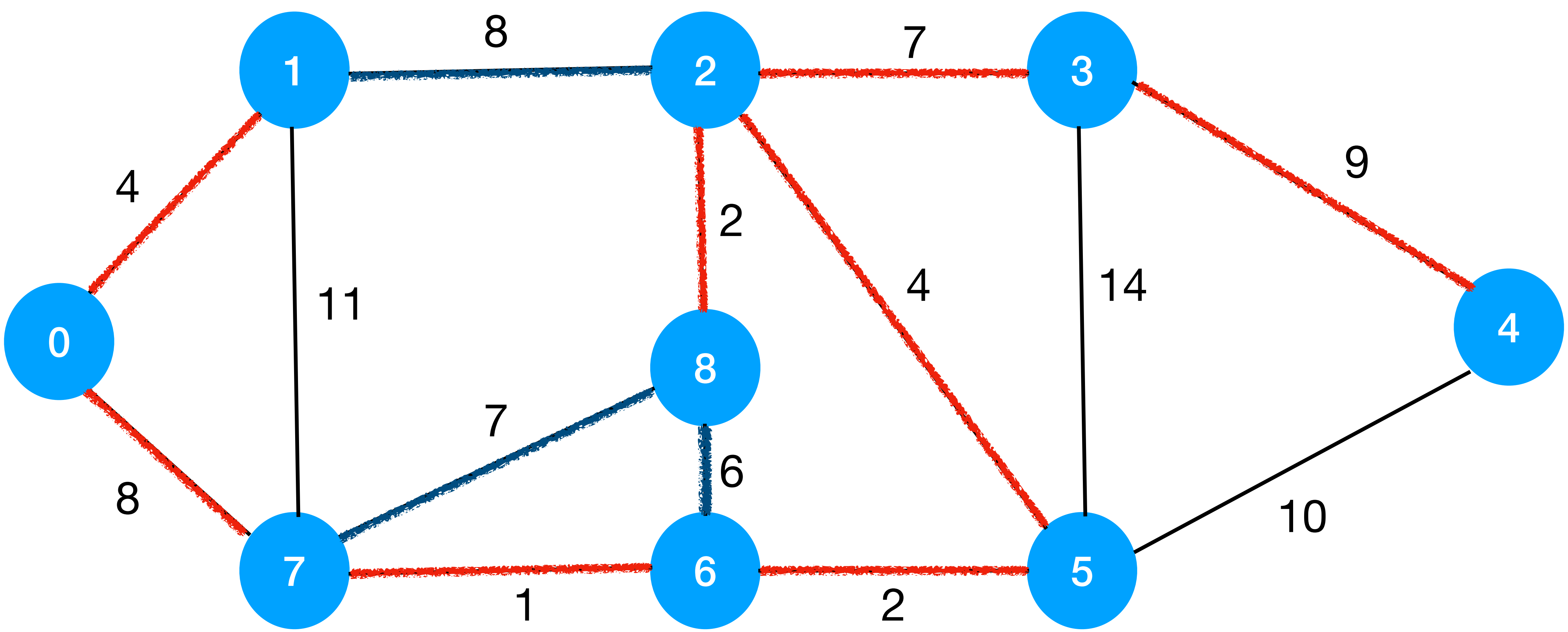

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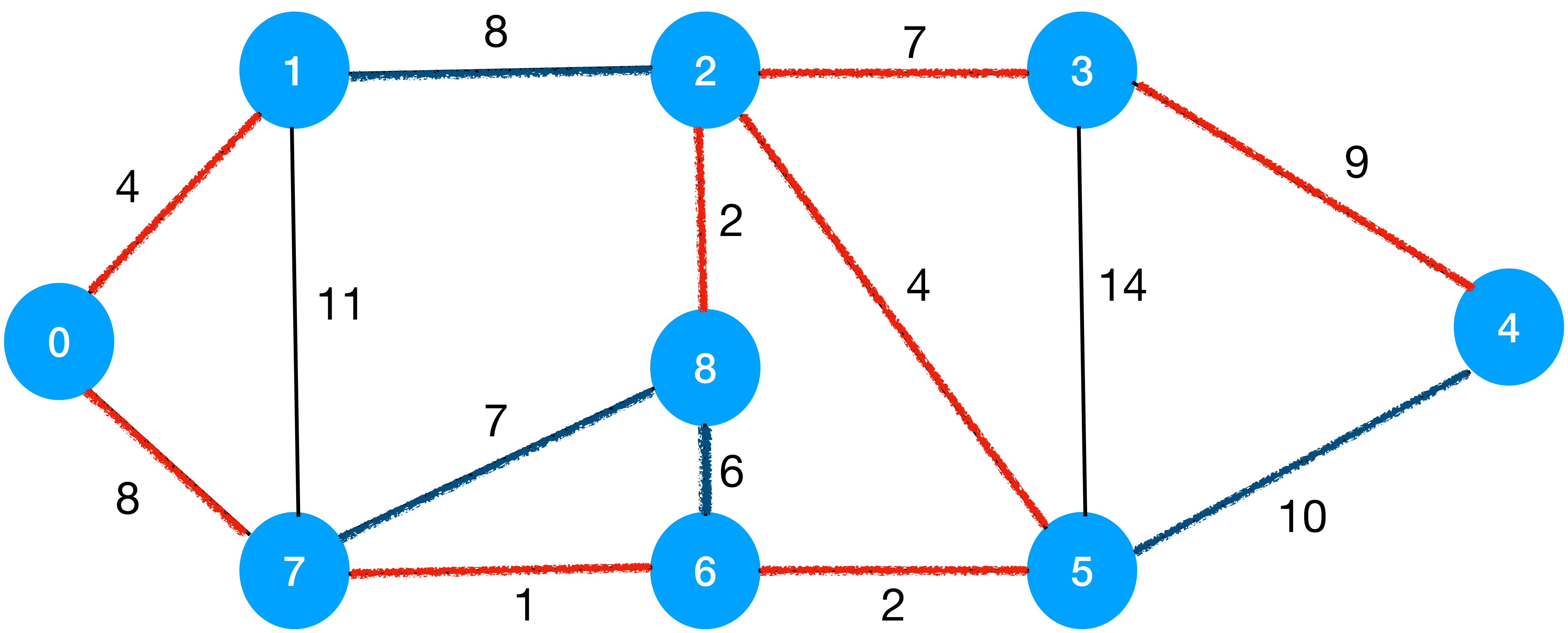

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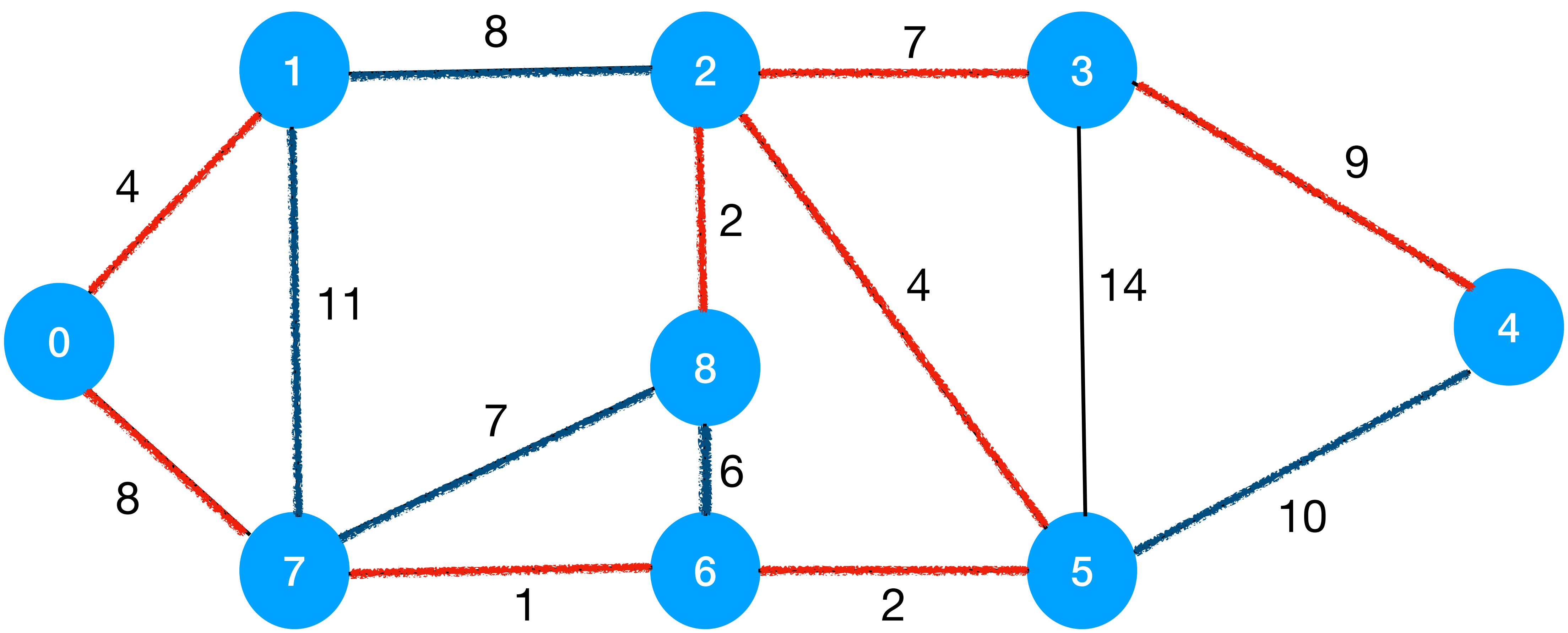

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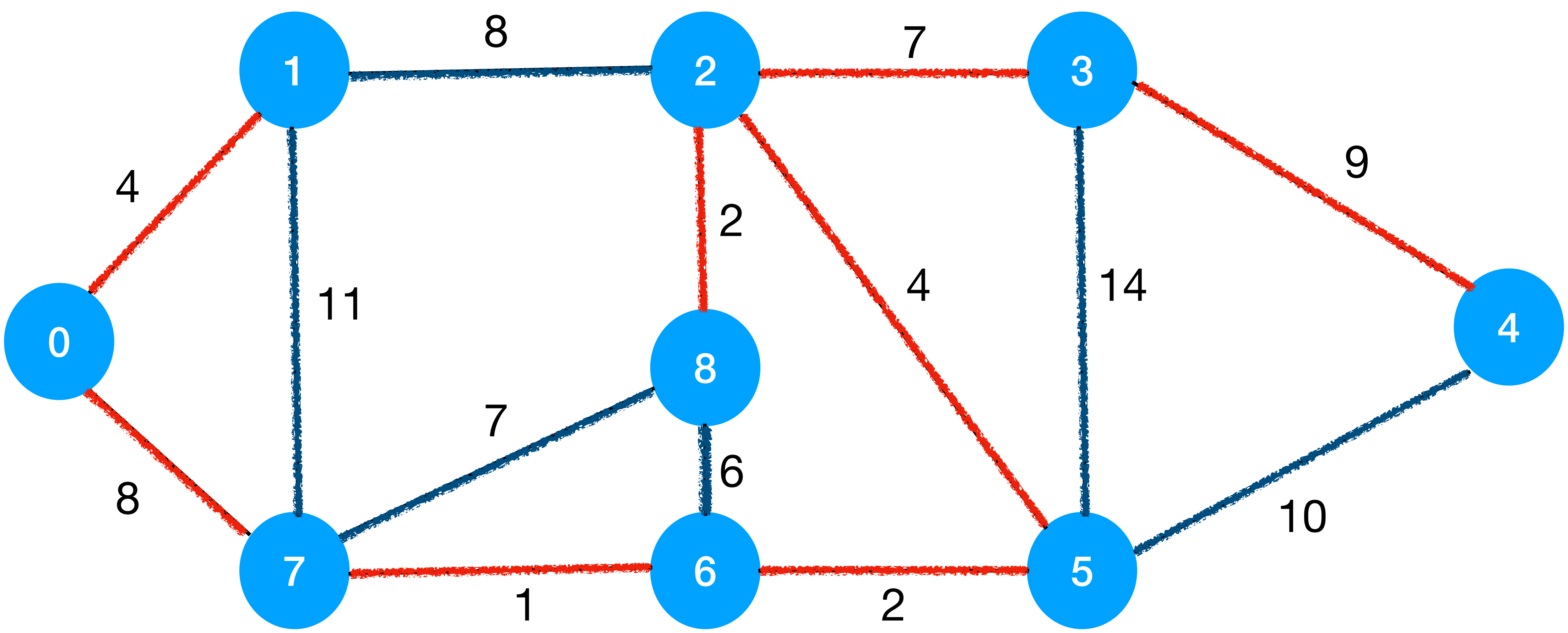

Prim's Algorithm

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Start with a node s.

Add an edge $e=\{s,w\}$ to T.

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The one with the minimum cost c_e.

We continue like this.

We only consider edges to neighbours that are not in the spanning tree.

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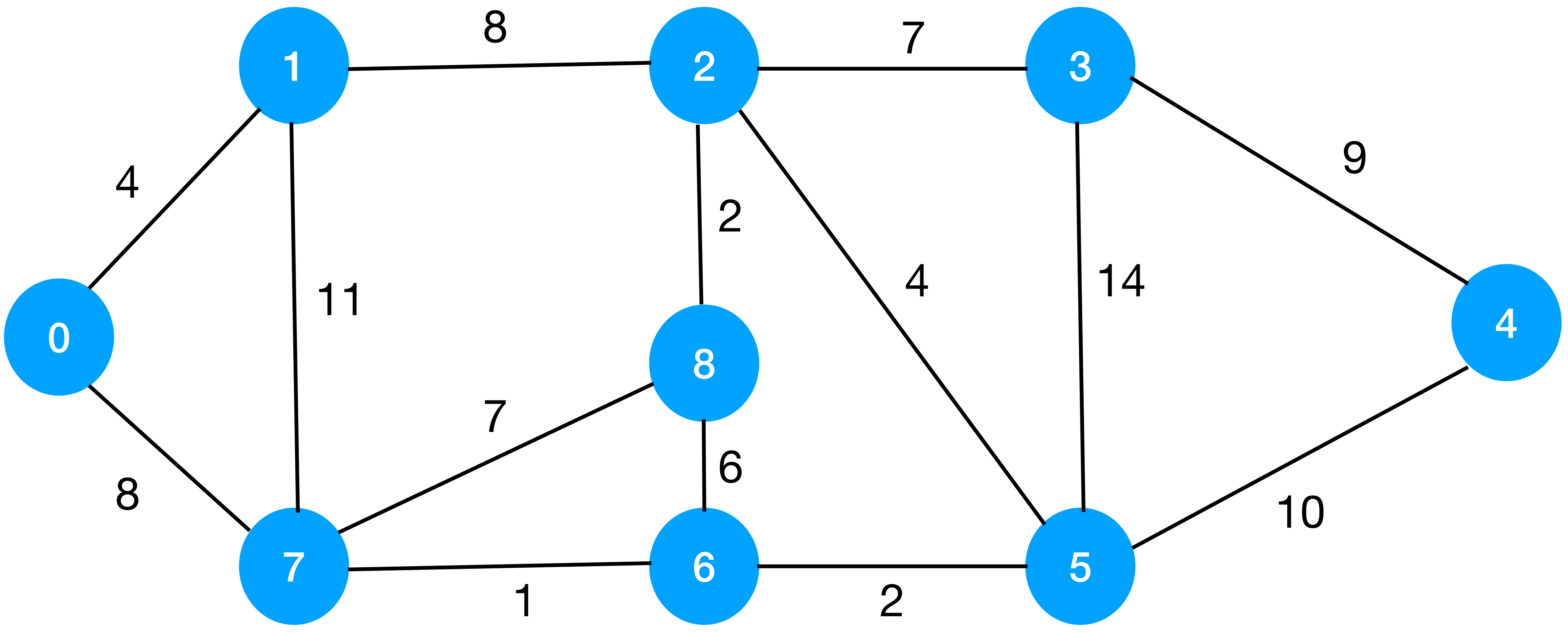

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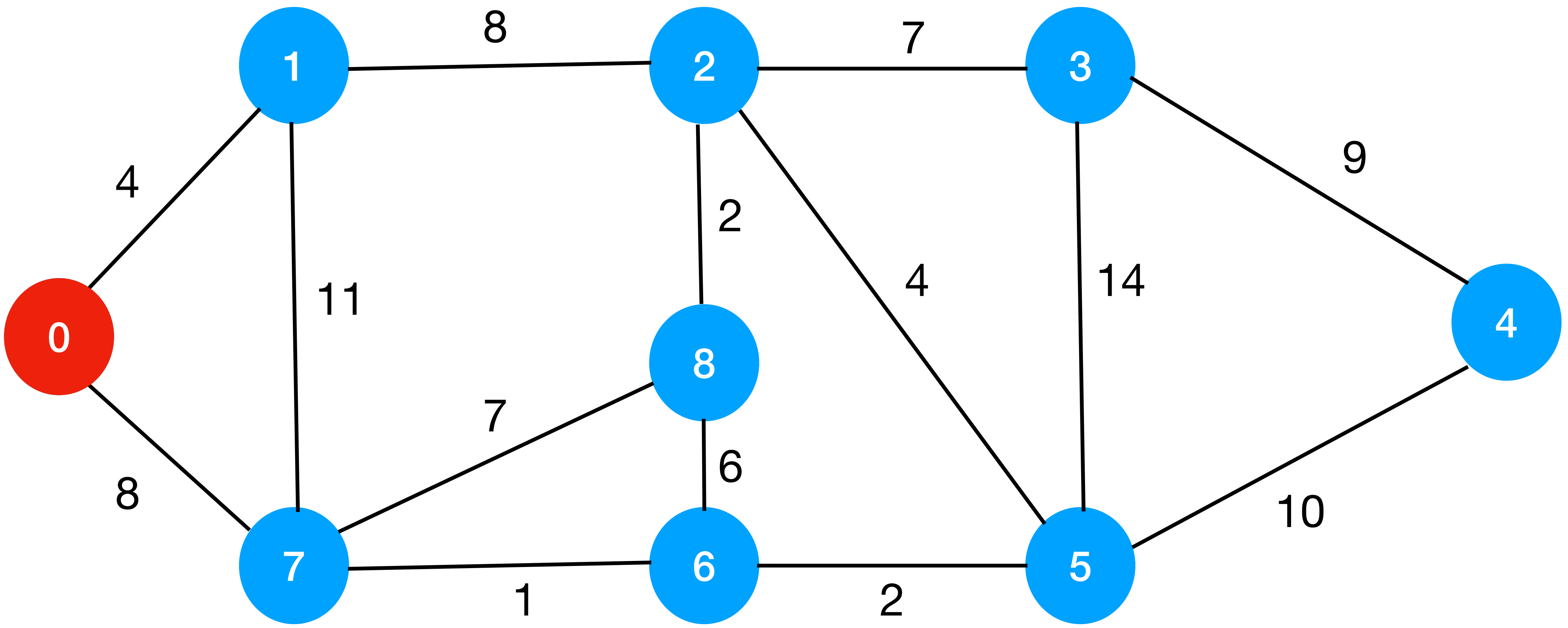

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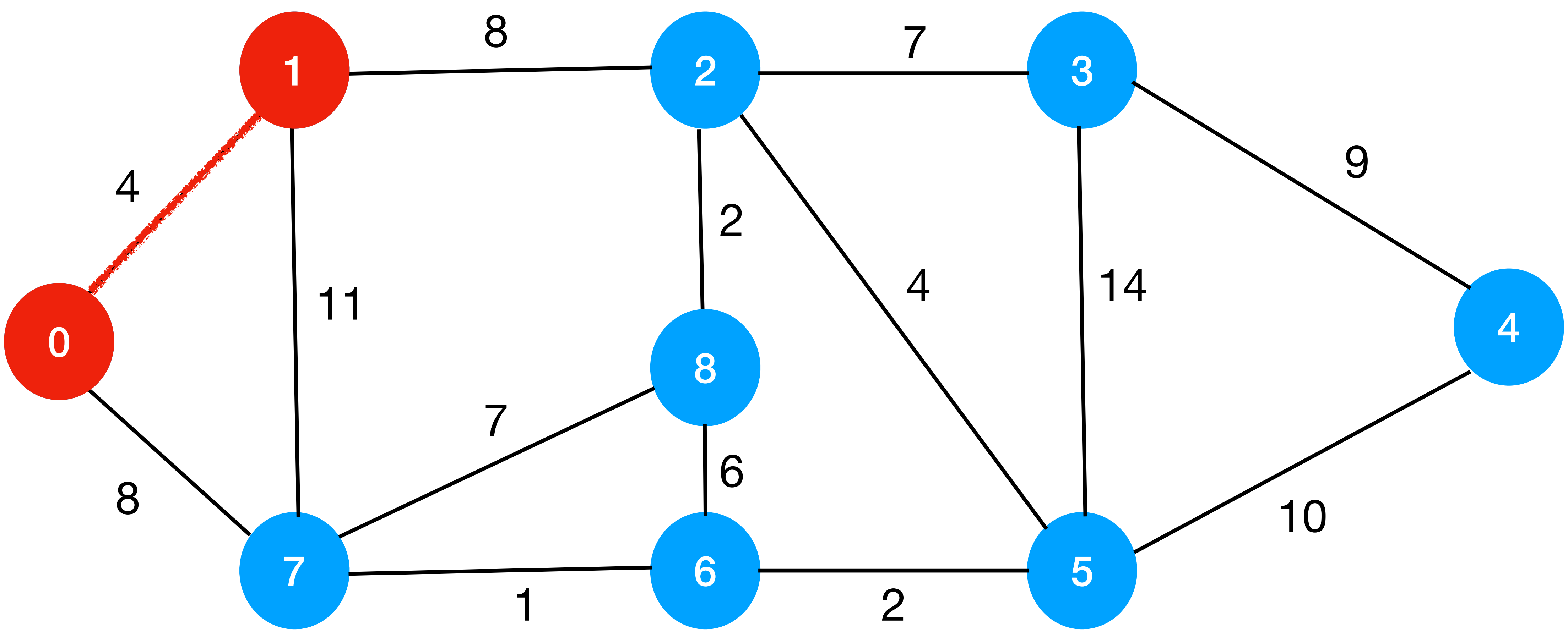

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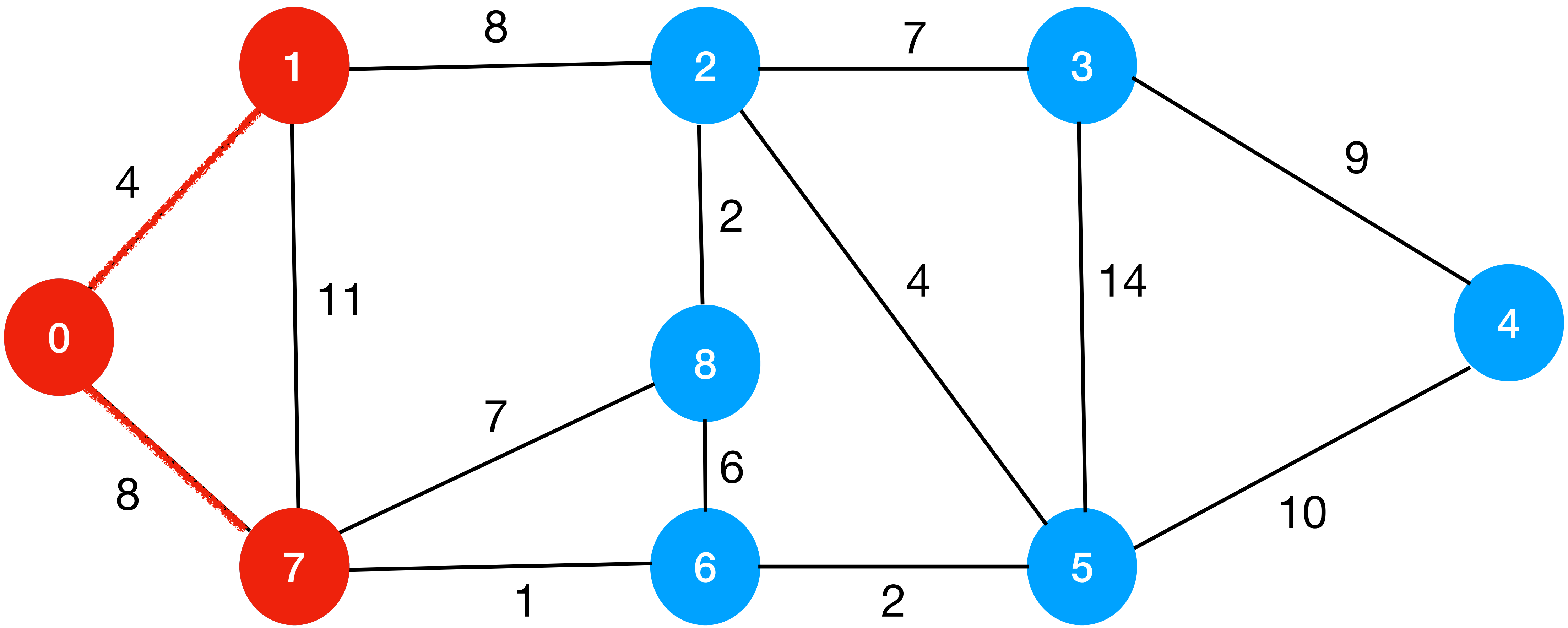

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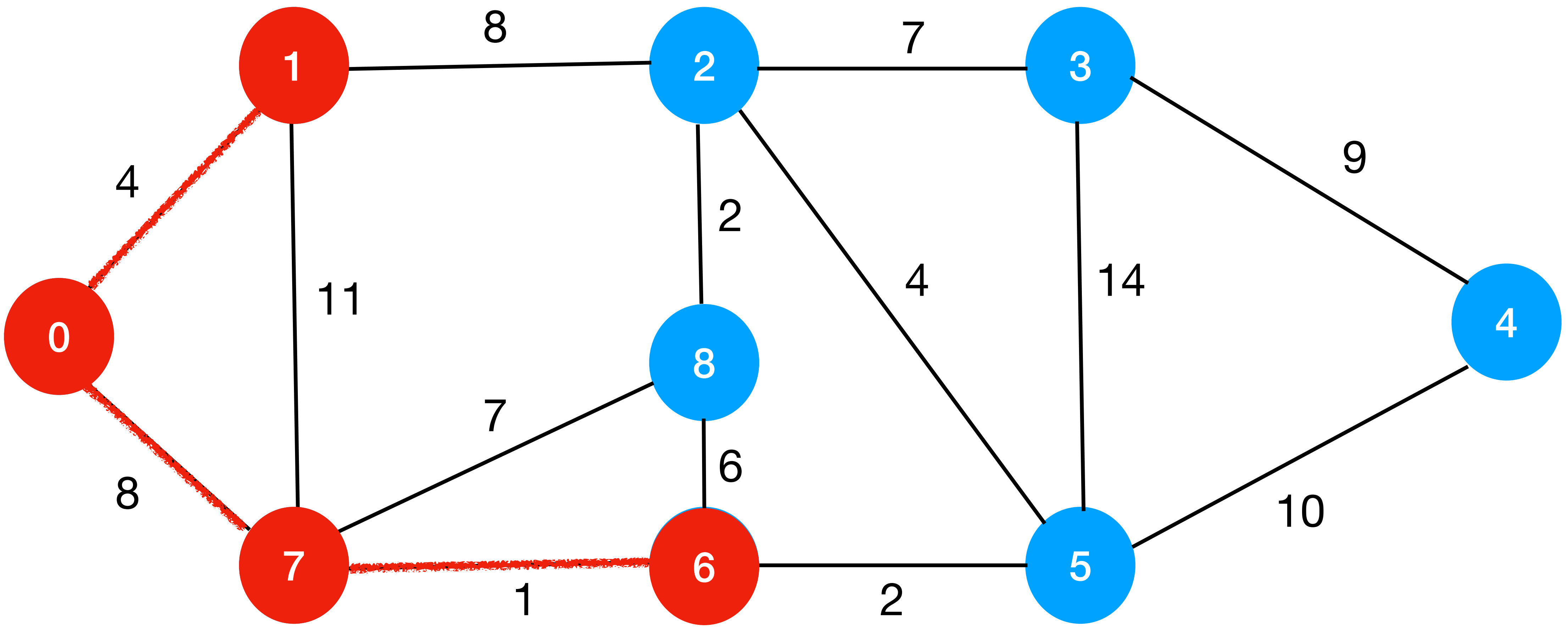

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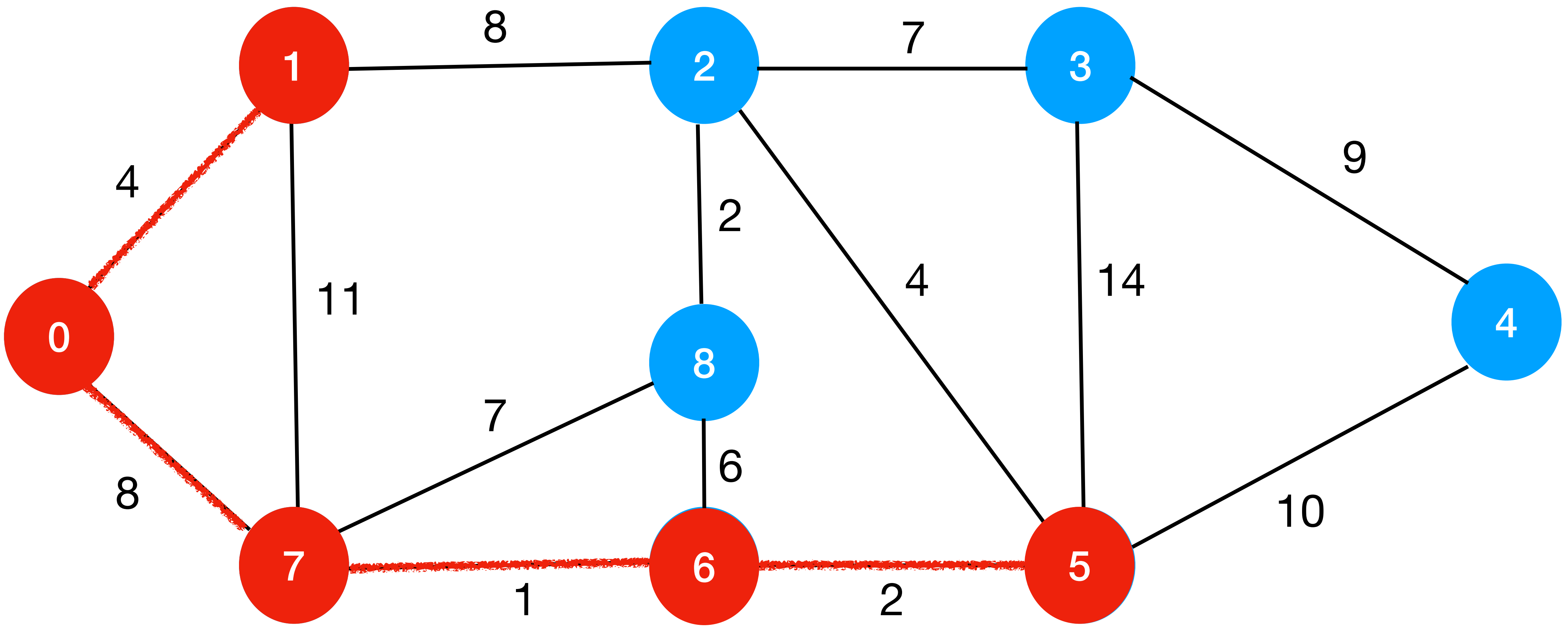

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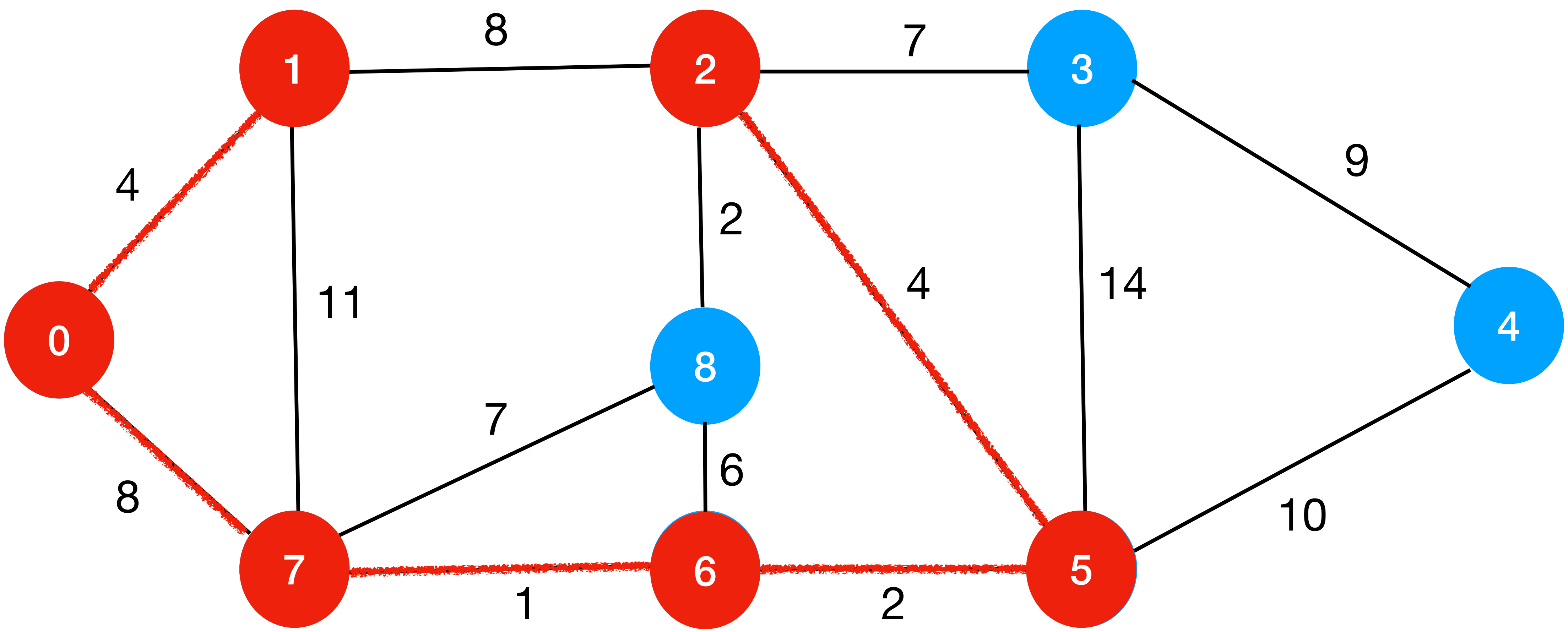

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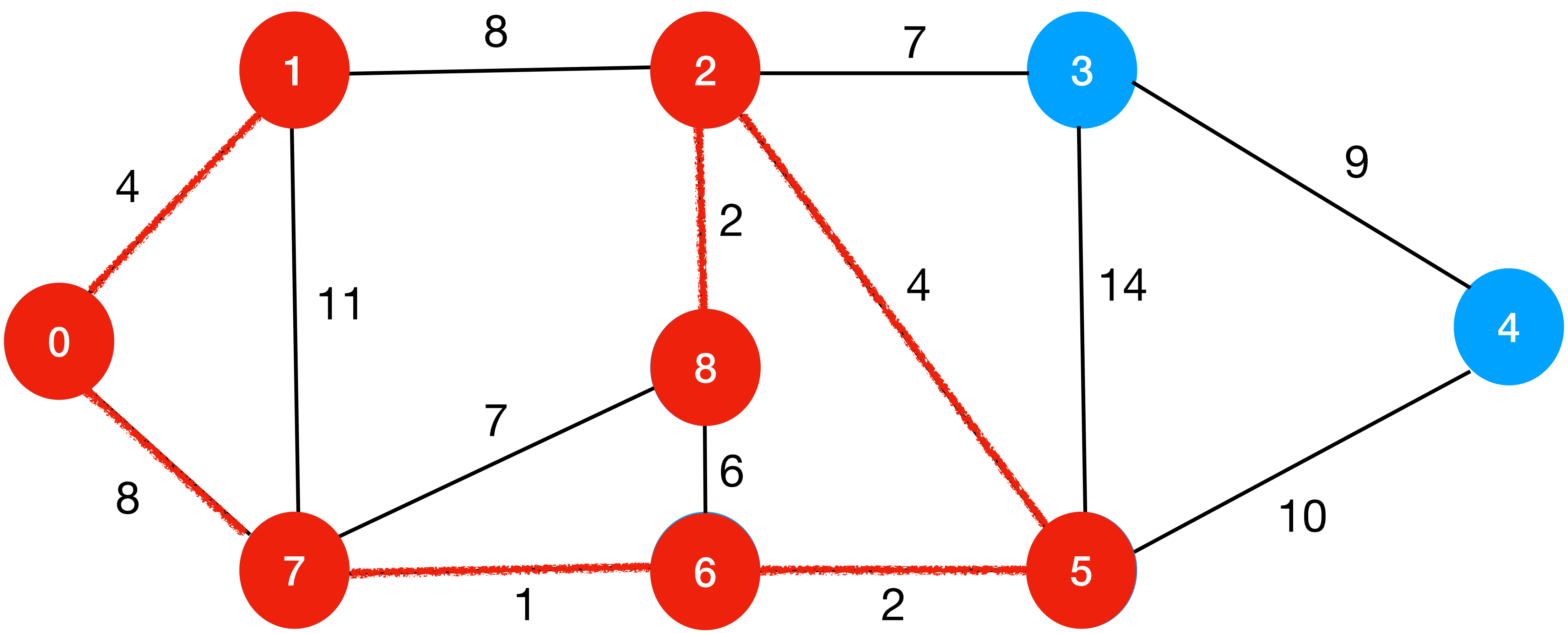

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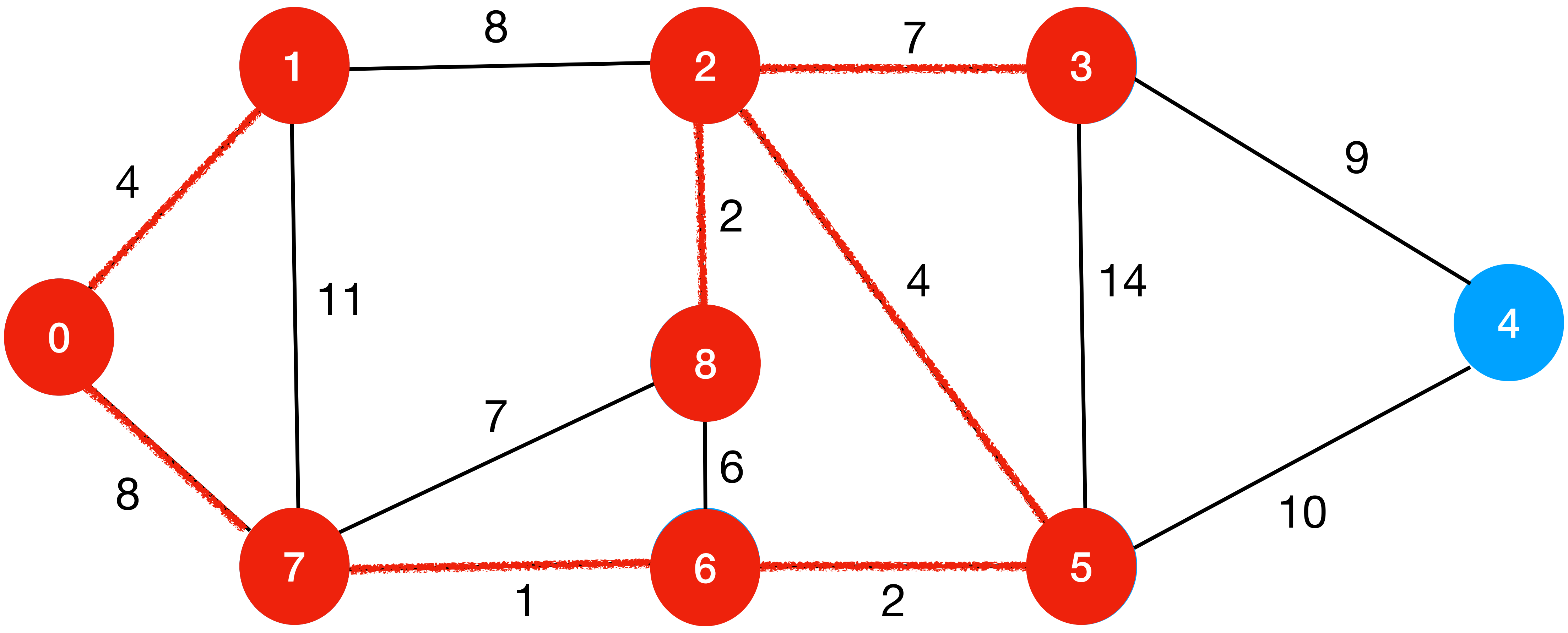

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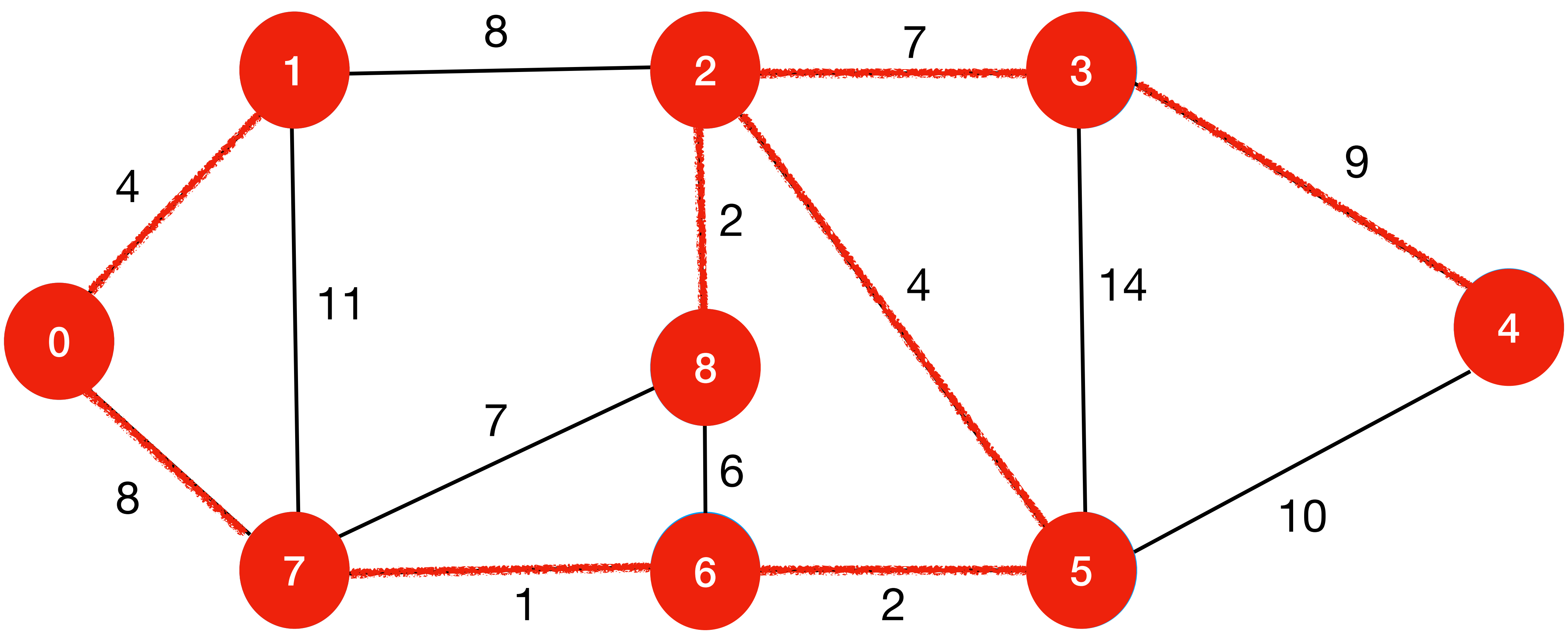

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$$\Theta(n^2)$$

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For Min-Priority Queues, the elements with the smallest values are those with the highest priority.

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We will not cover this here; it was covered in IADS last year.

e.g. see KT Chapter 2.5, CLRS Chapter 6.5. (but you would have to also read 6.1 - 6.3).

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Running time: $O(m \log n)$

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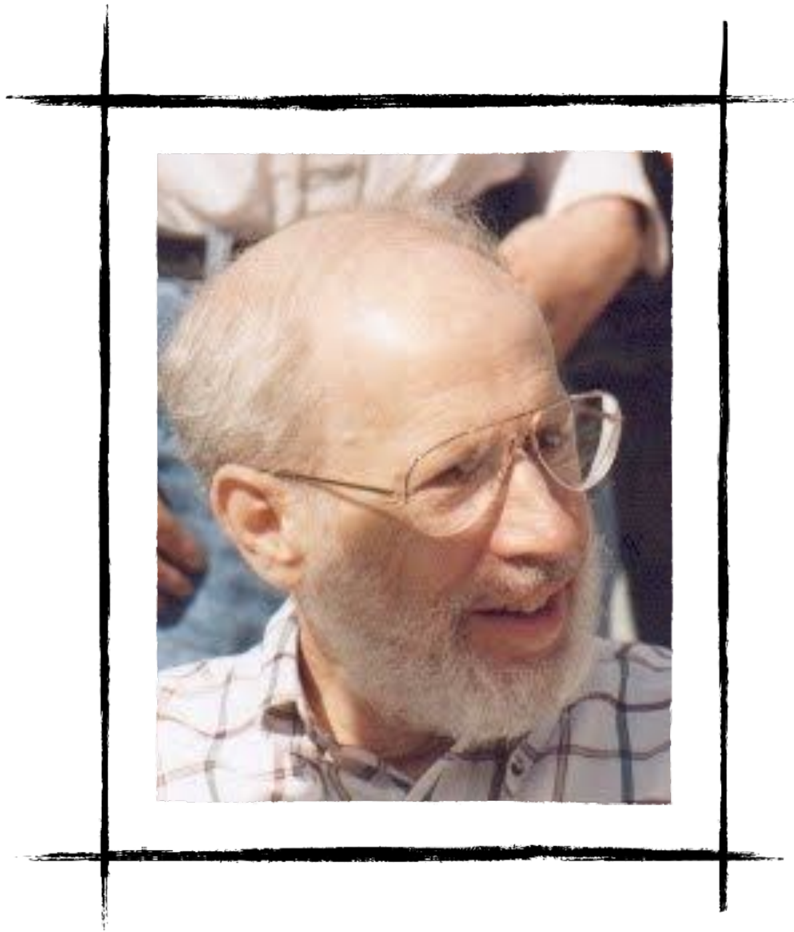

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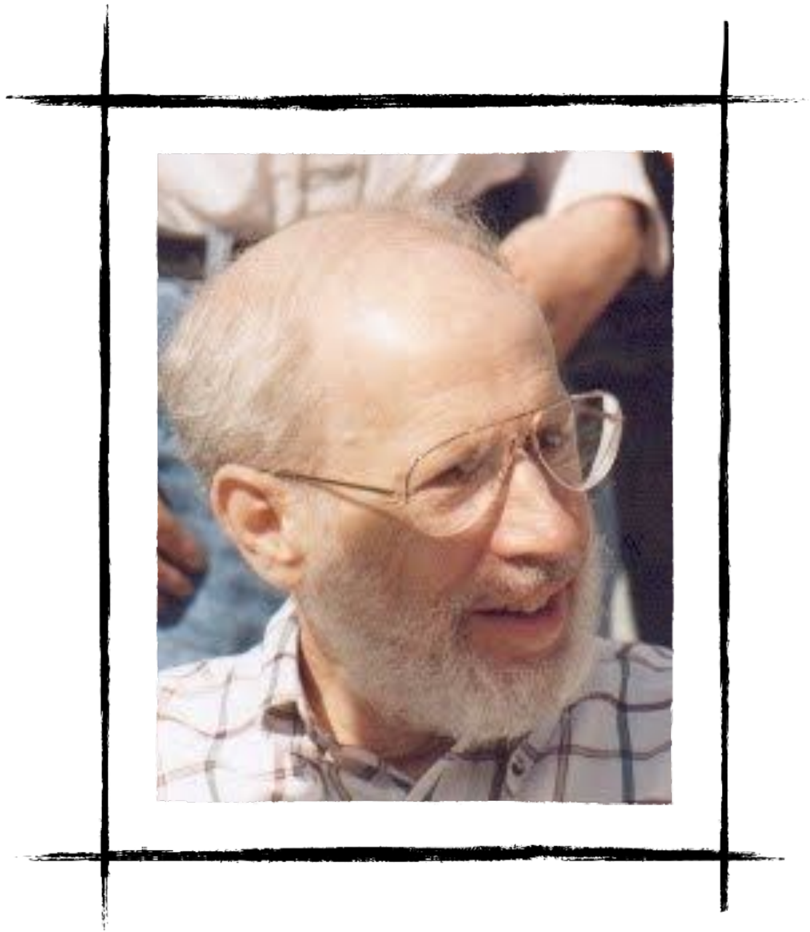

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What is the tricky part here?

Do we always add the new edge e to T?

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How can we find the connected component of v?

Graph Traversal (Search)

We would like to go over all the possible nodes of an (undirected) graph.

There are different ways of doing that.

Two systematic ways:

Depth-First Search

Breadth-First Search

KT Chapter 3.2.

CLRS Chapter 20.2, 20.3

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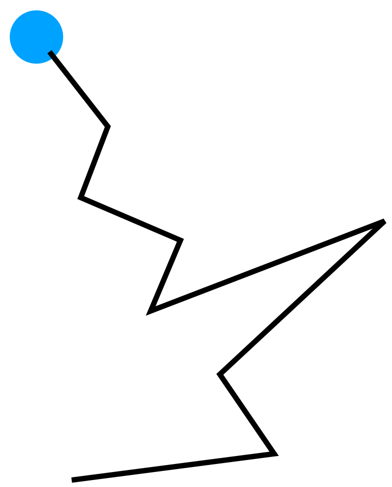

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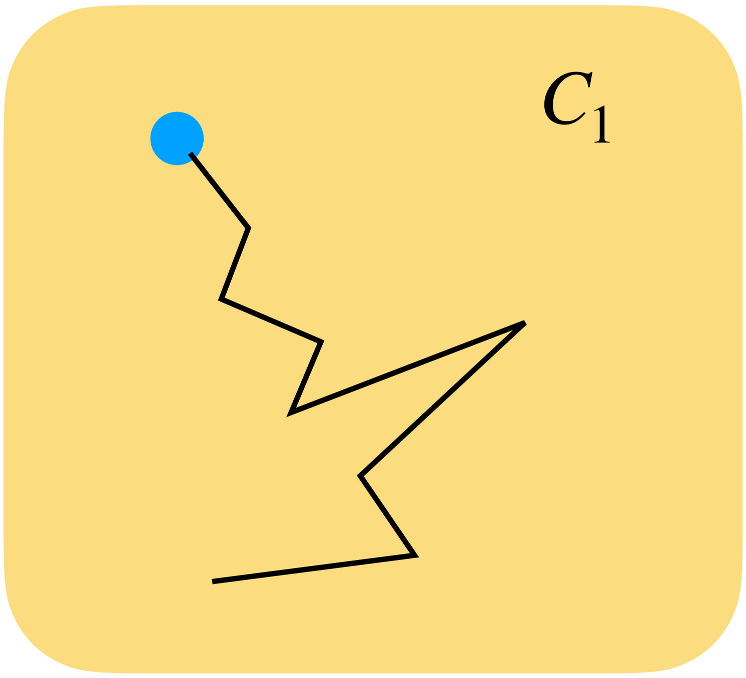

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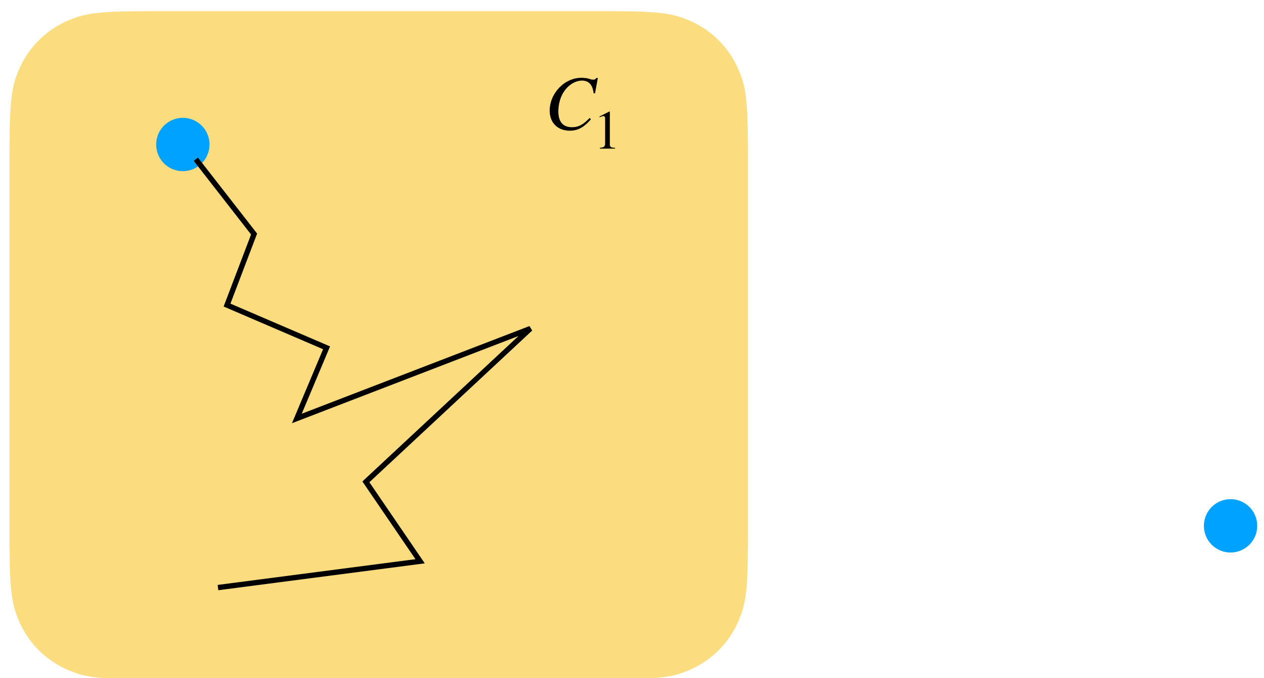

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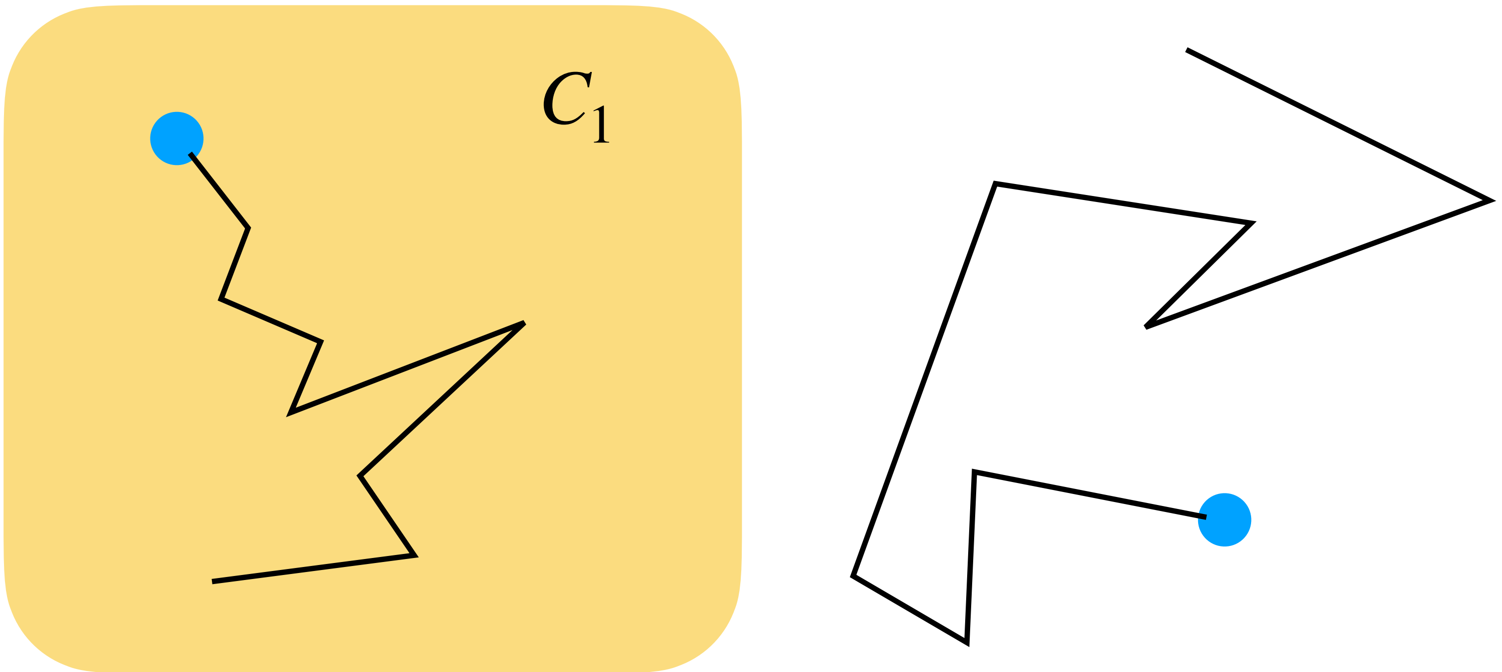

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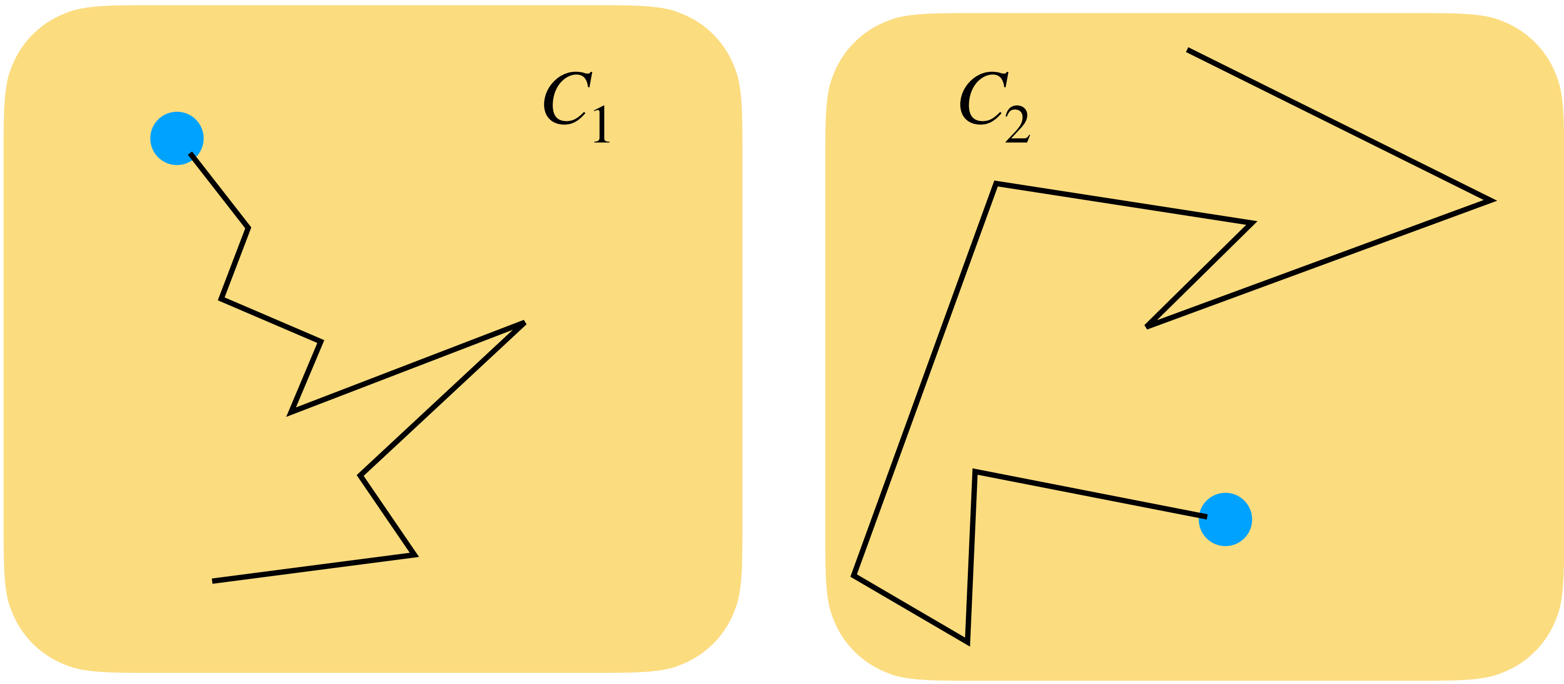

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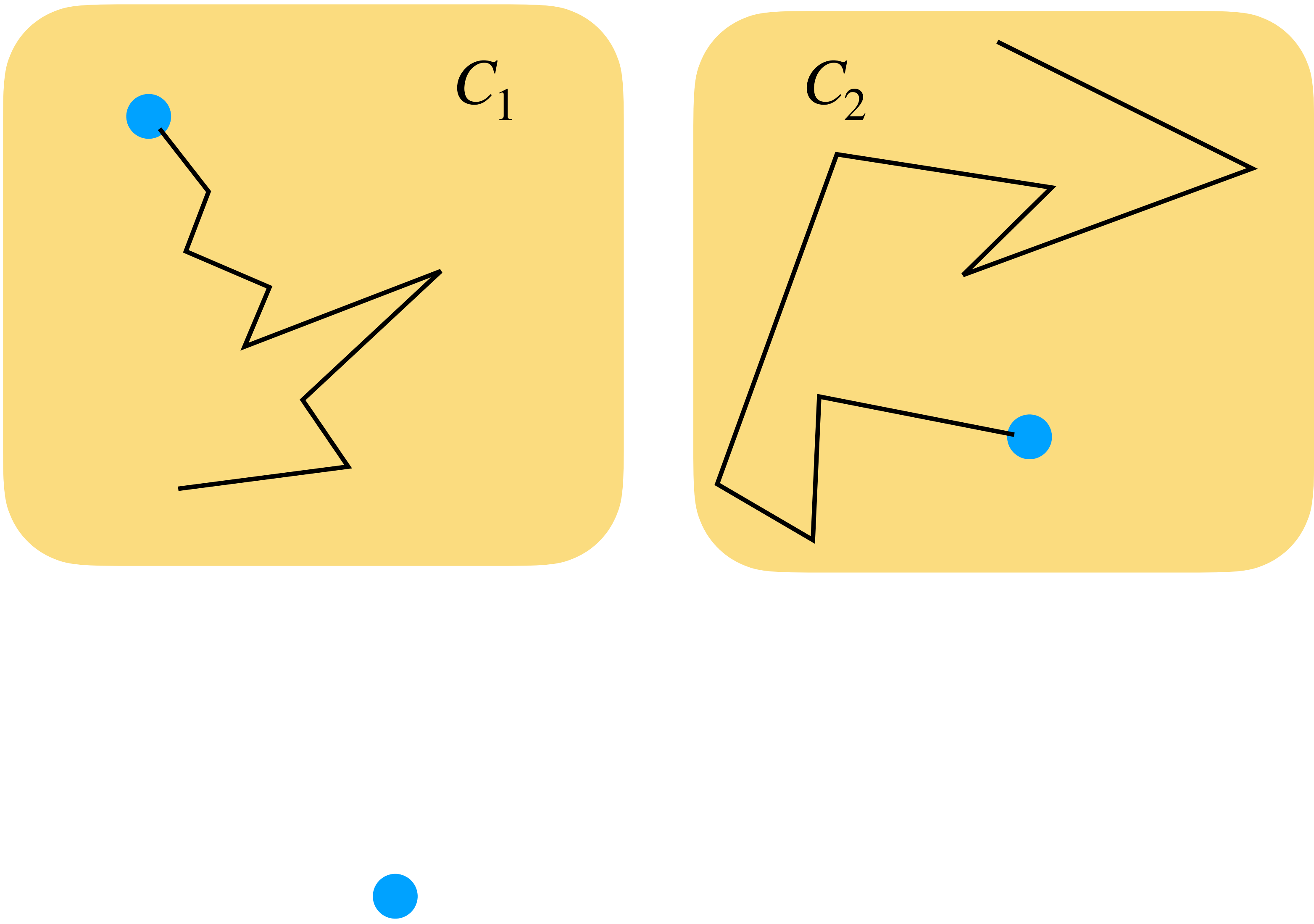

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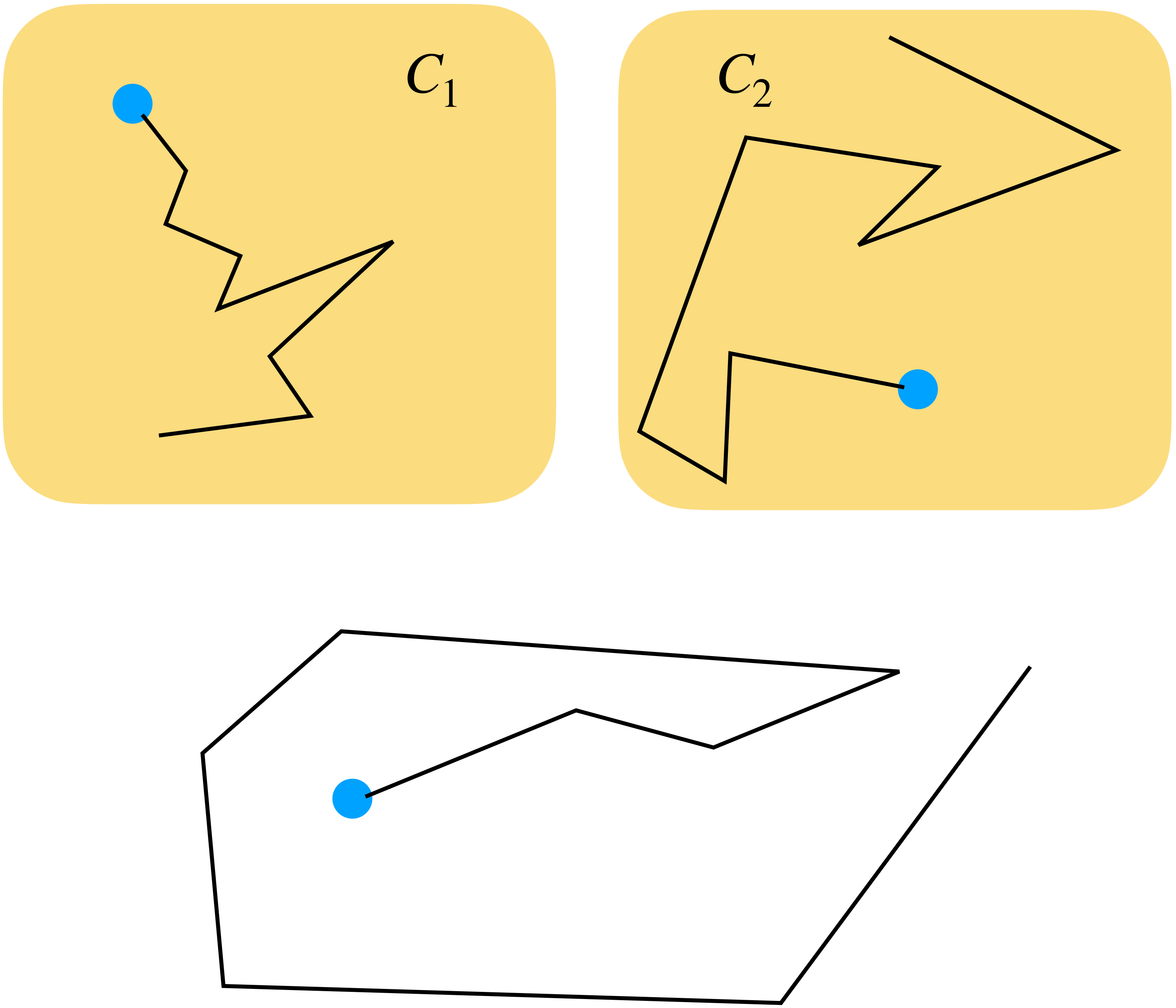

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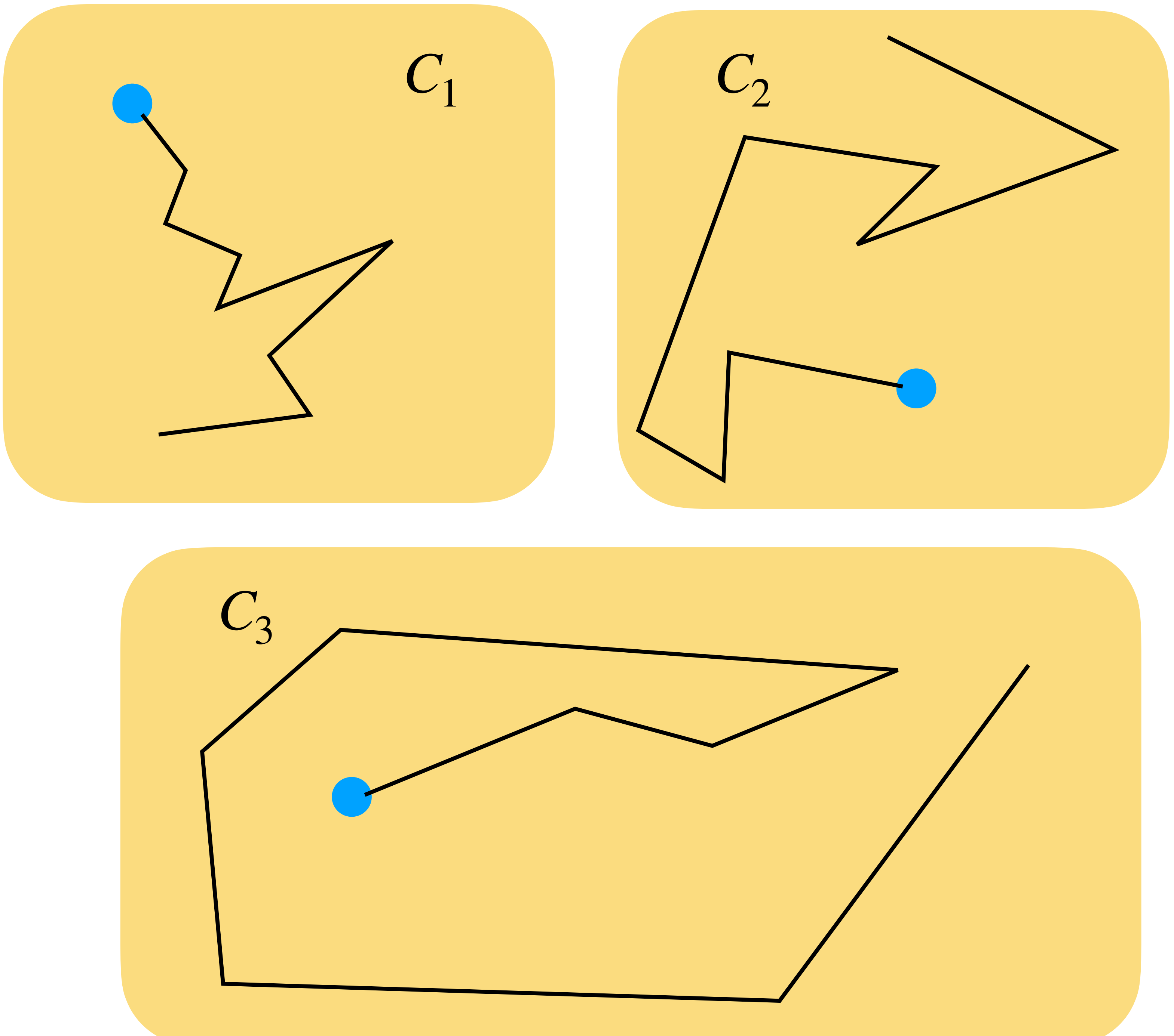

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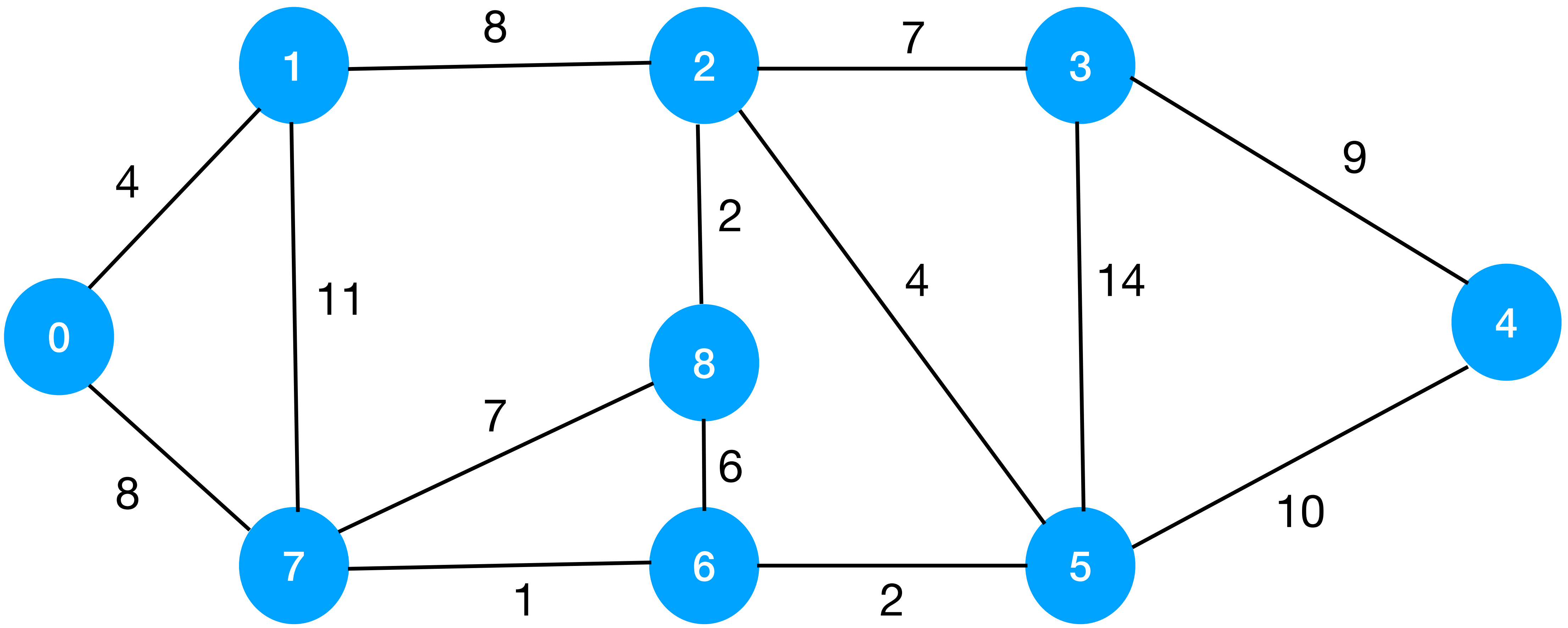
So, for a candidate edge $e = \{w, v\}$ we can check in $O(1)$ time whether the endpoints are in the same connected component or not.

But on which graph?

Example



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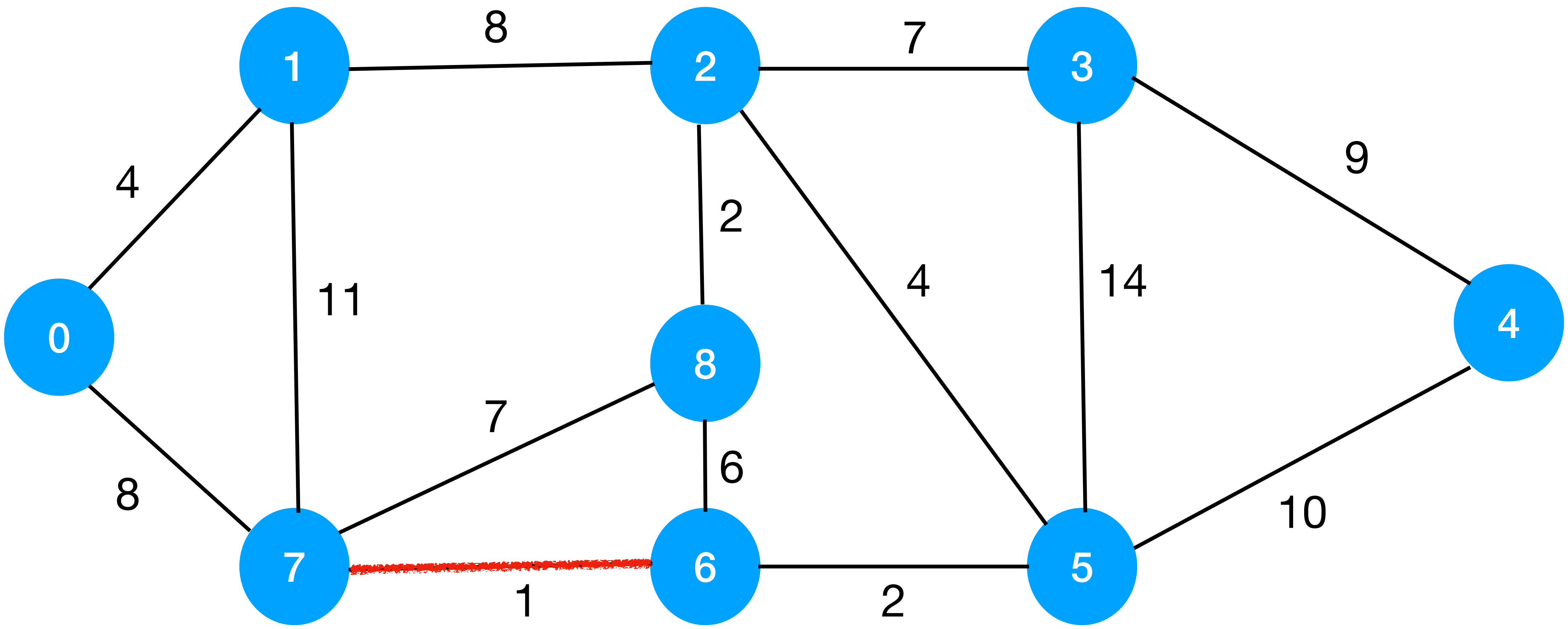

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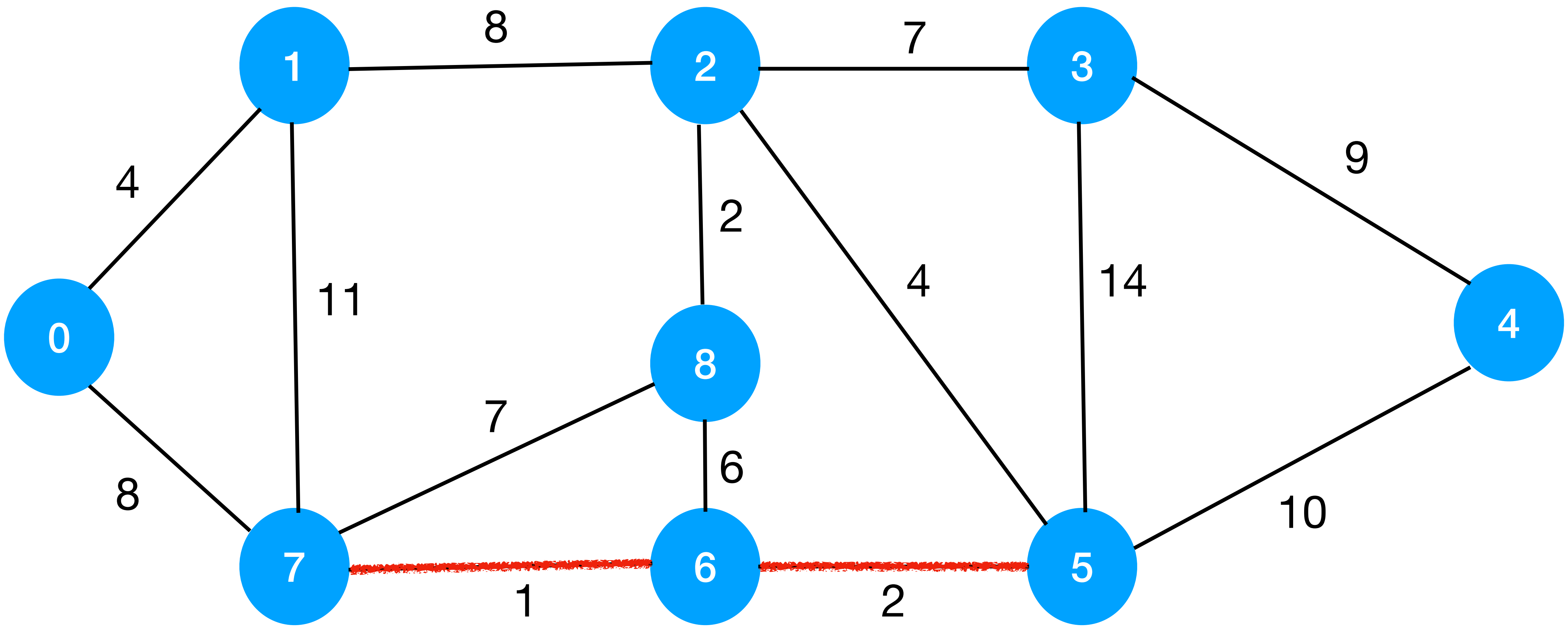

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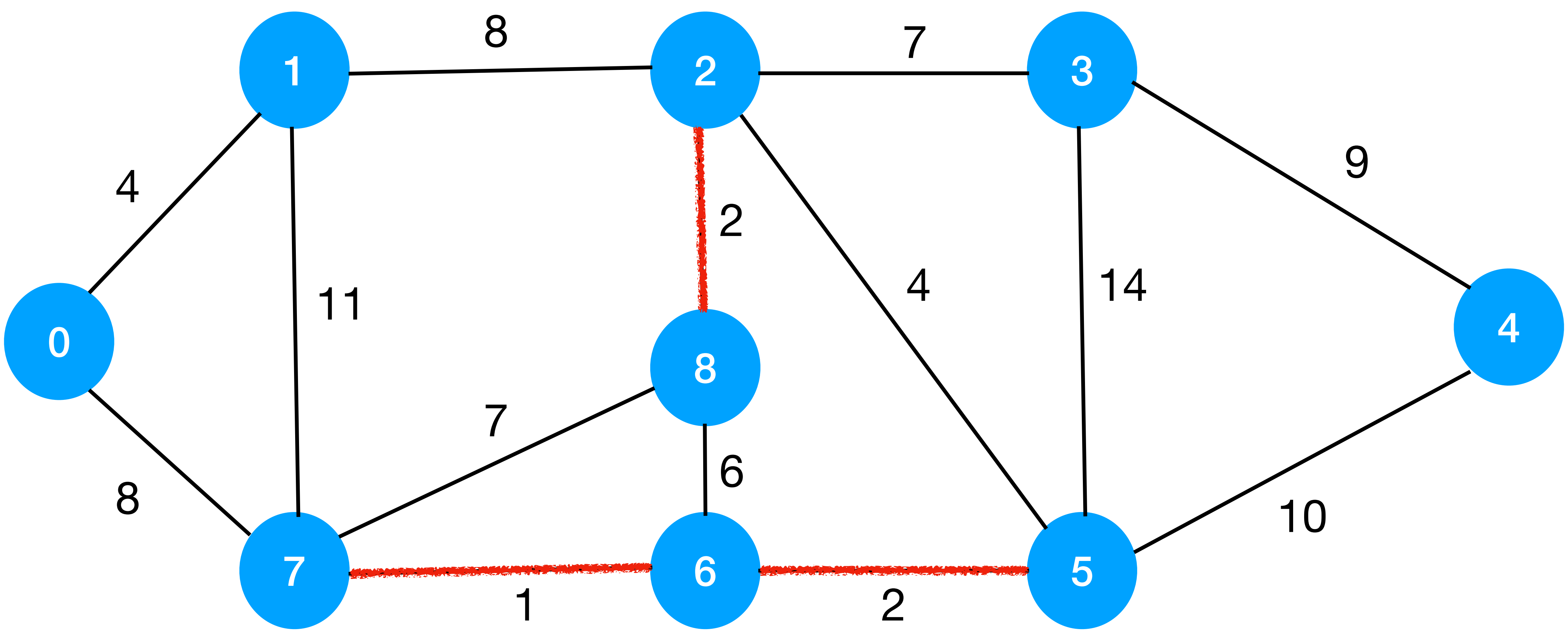

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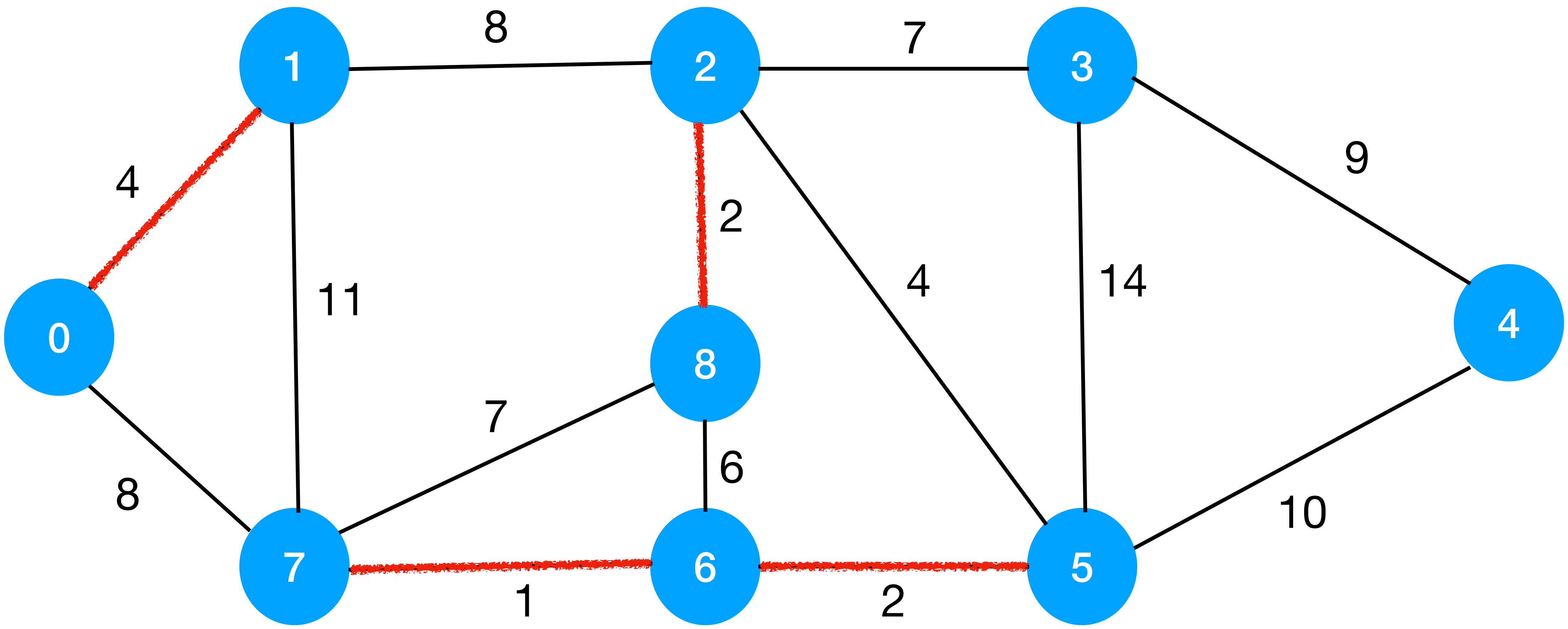

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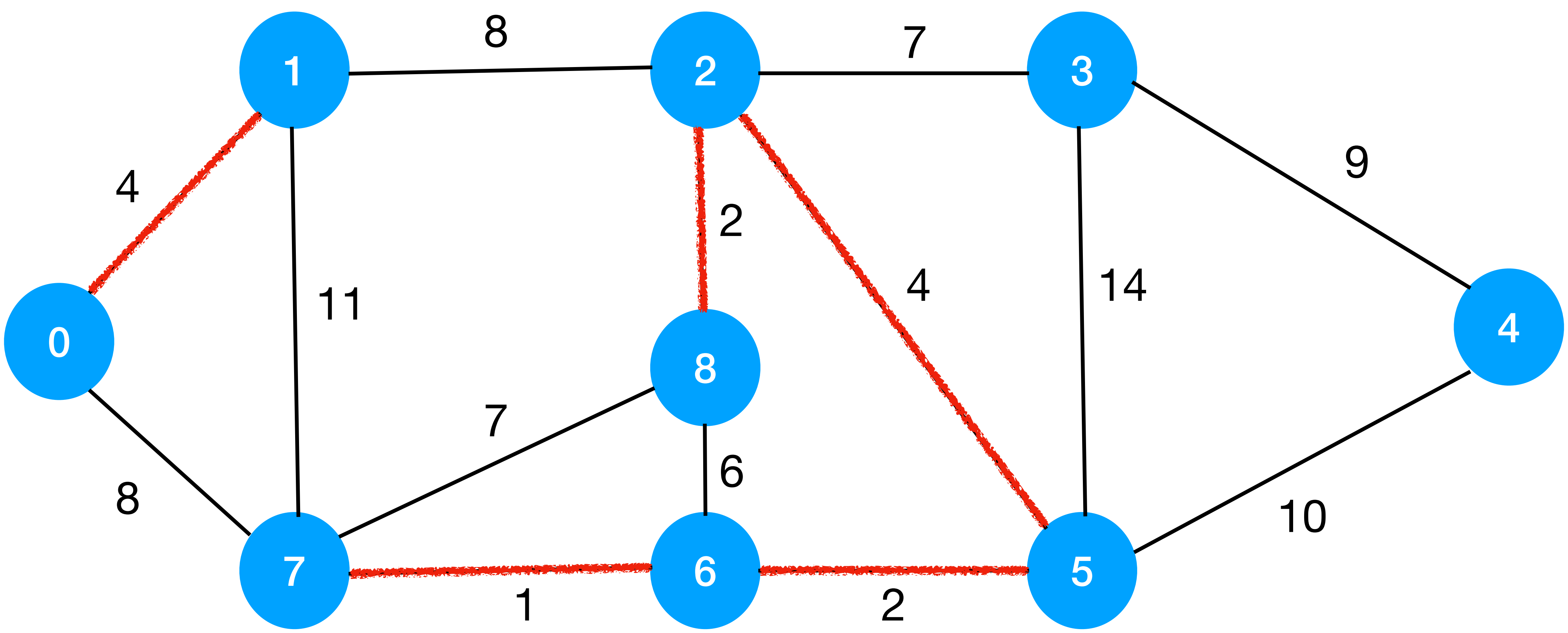

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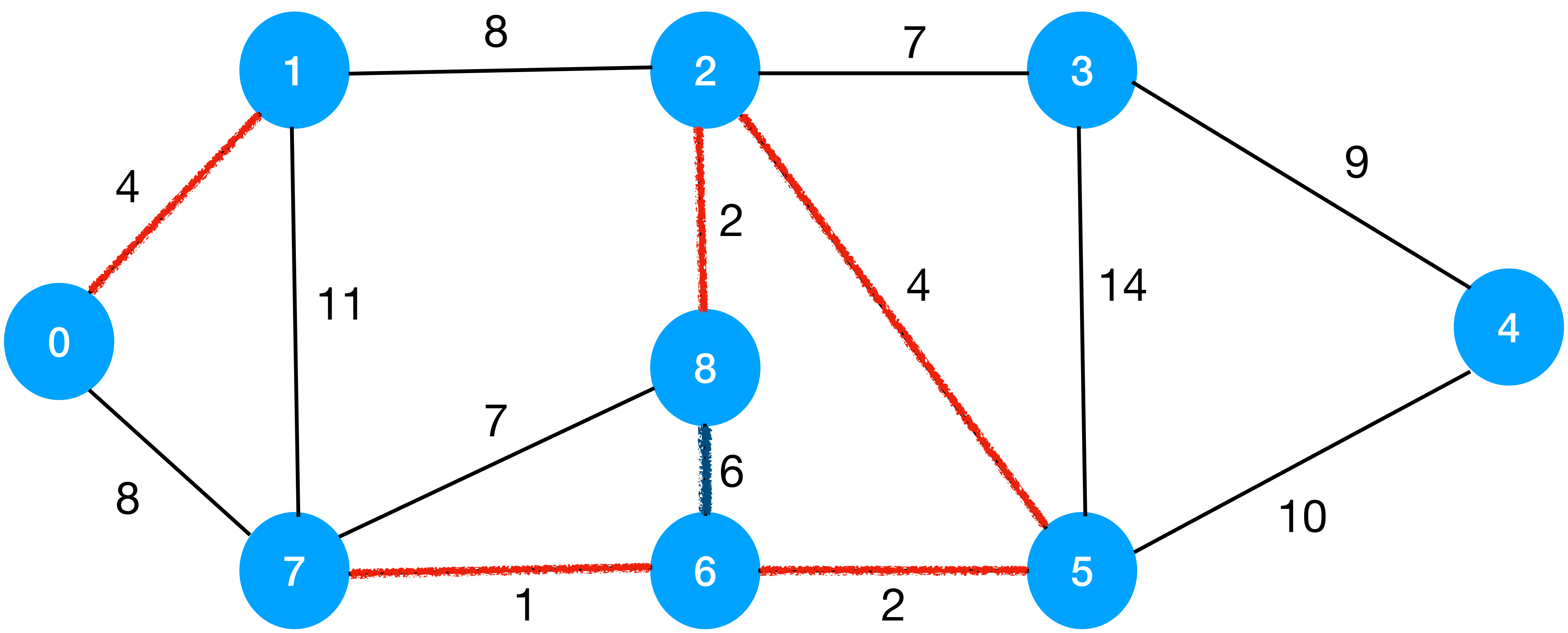

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We actually don't have to compute the connected component for every edge! We can compute all the connected components in $O(m+n)$ time.

So, for a candidate edge $e = \{w, v\}$ we can check in $O(1)$ time whether the endpoints are in the same connected component or not.

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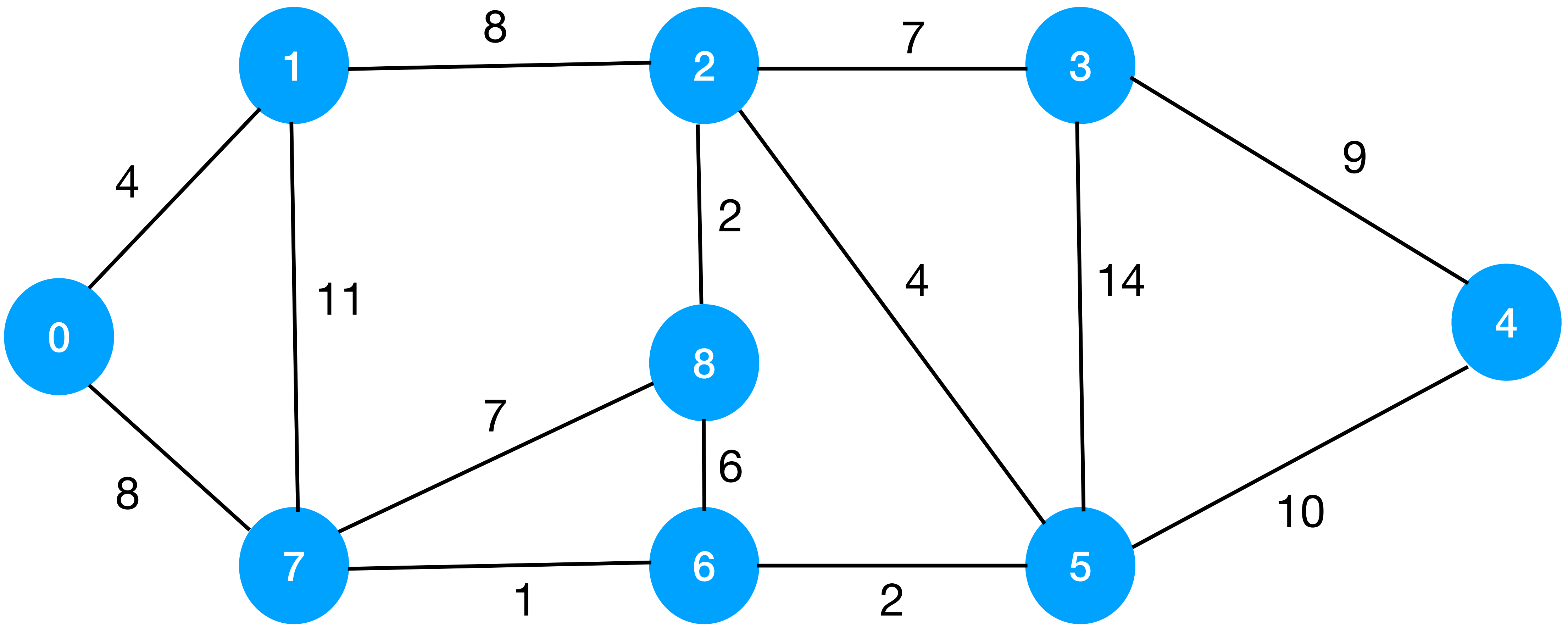

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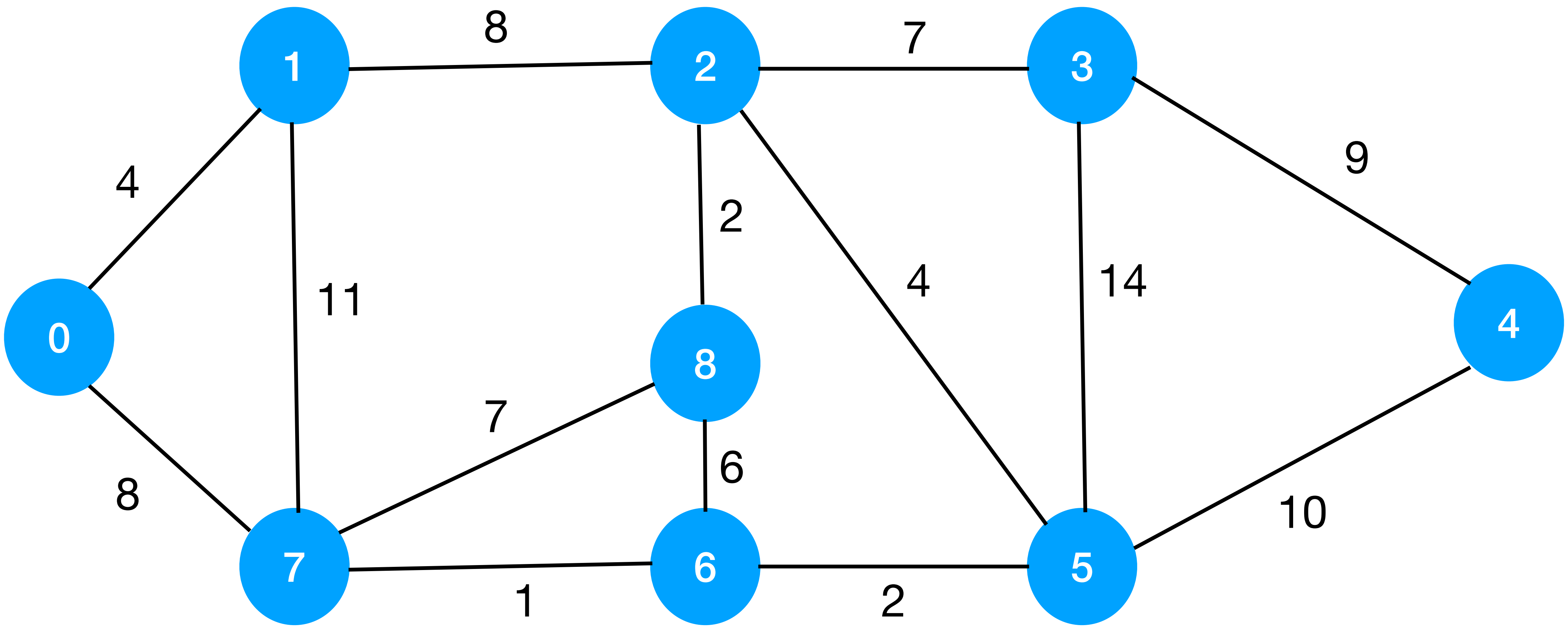

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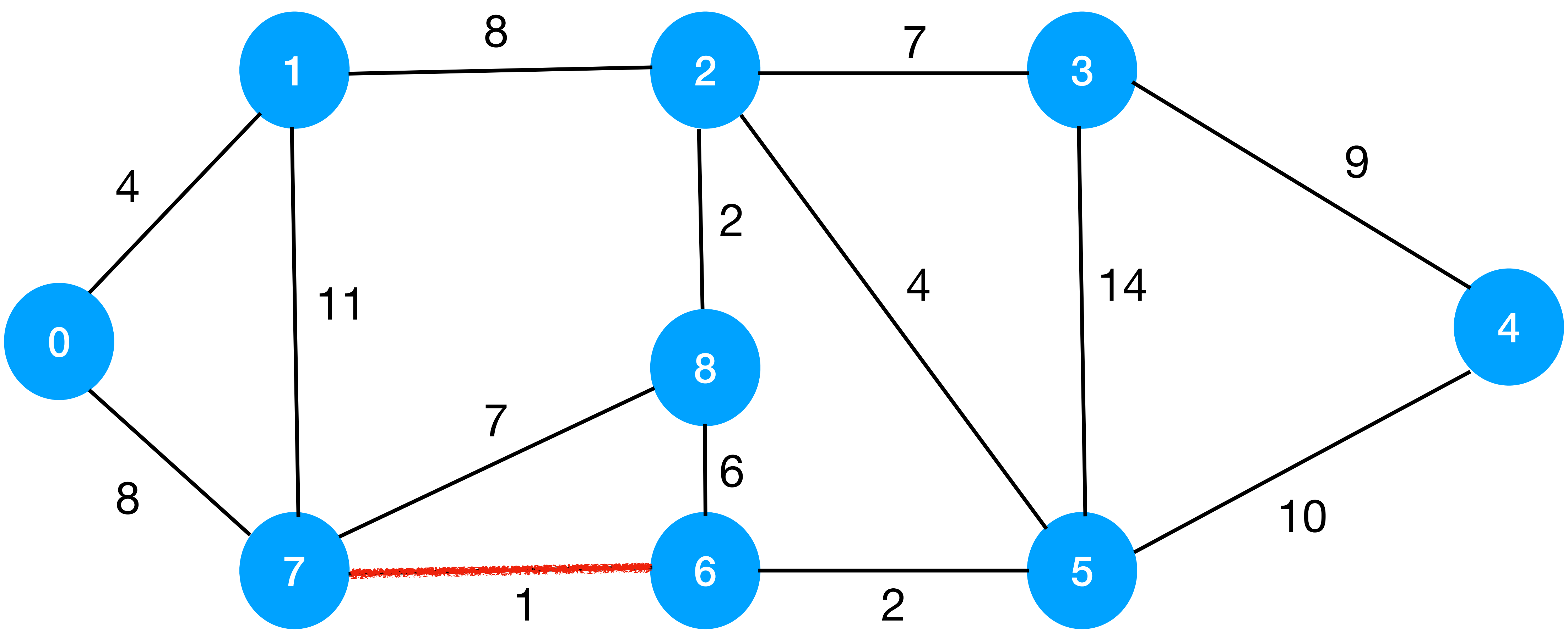

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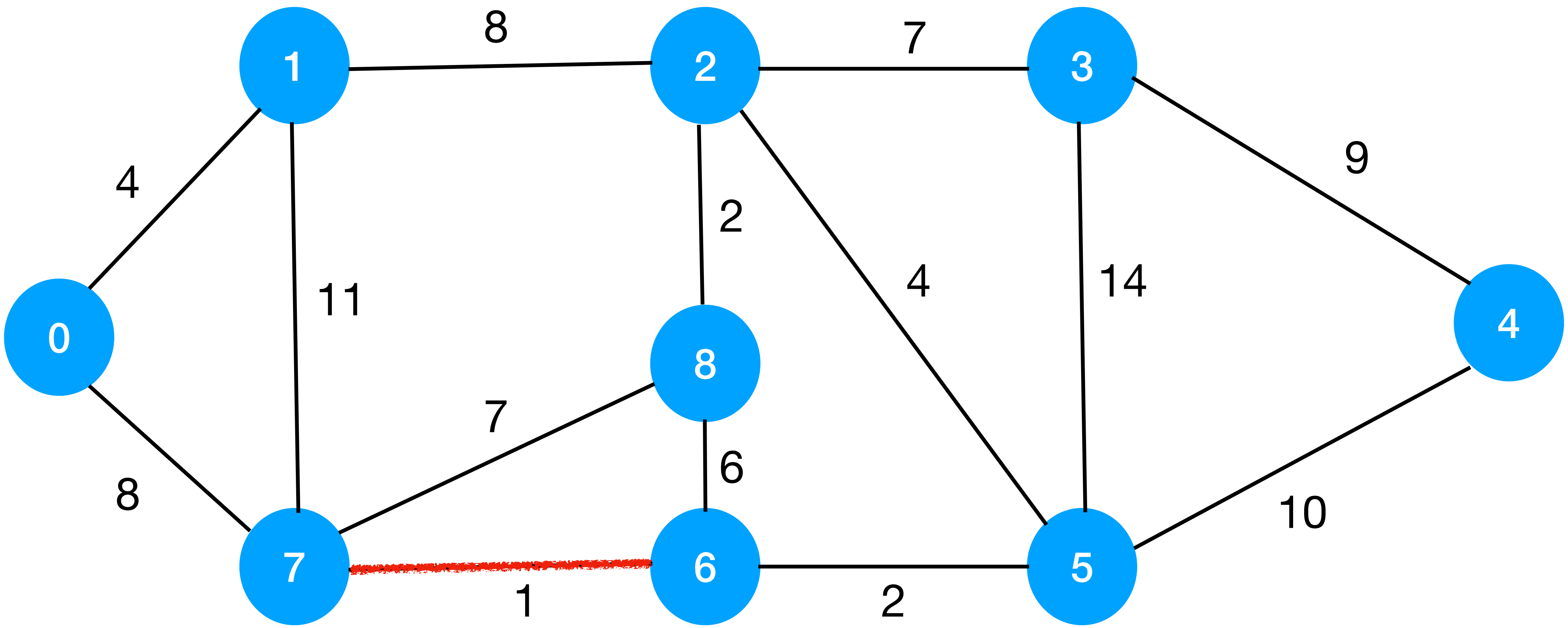

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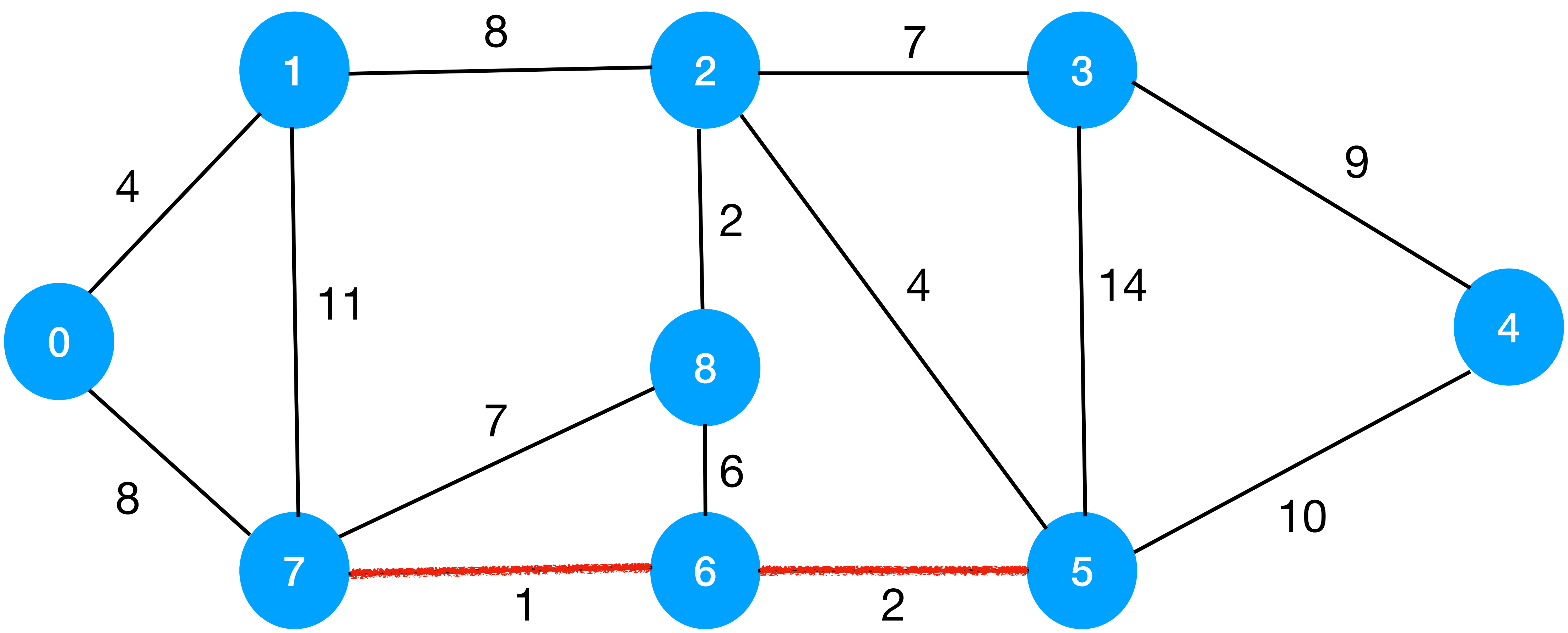

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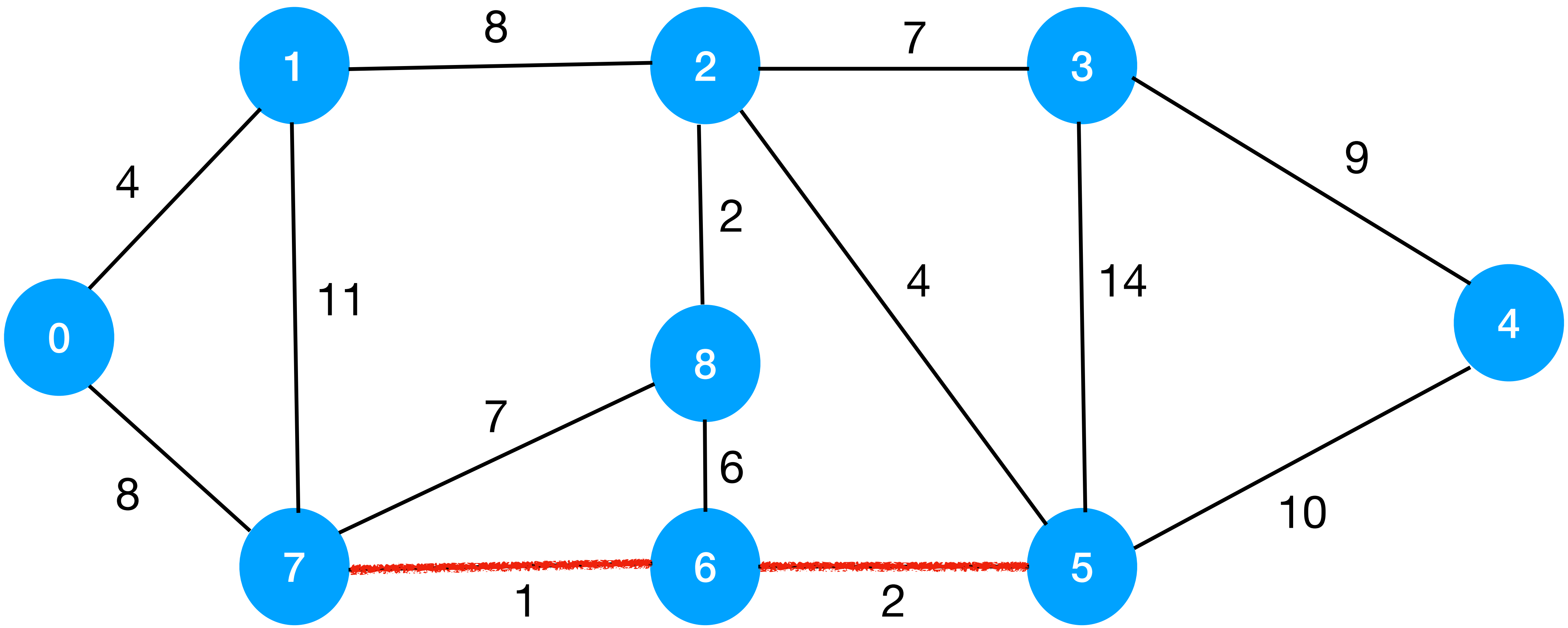

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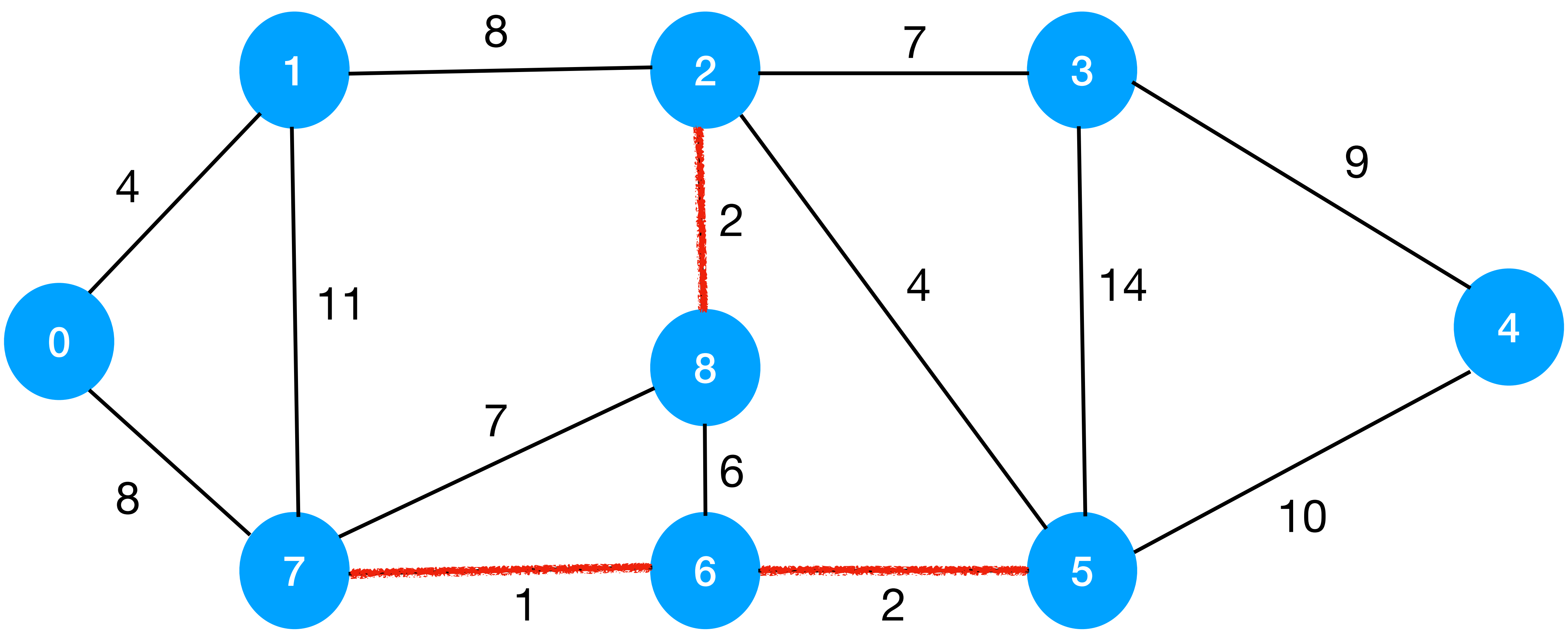

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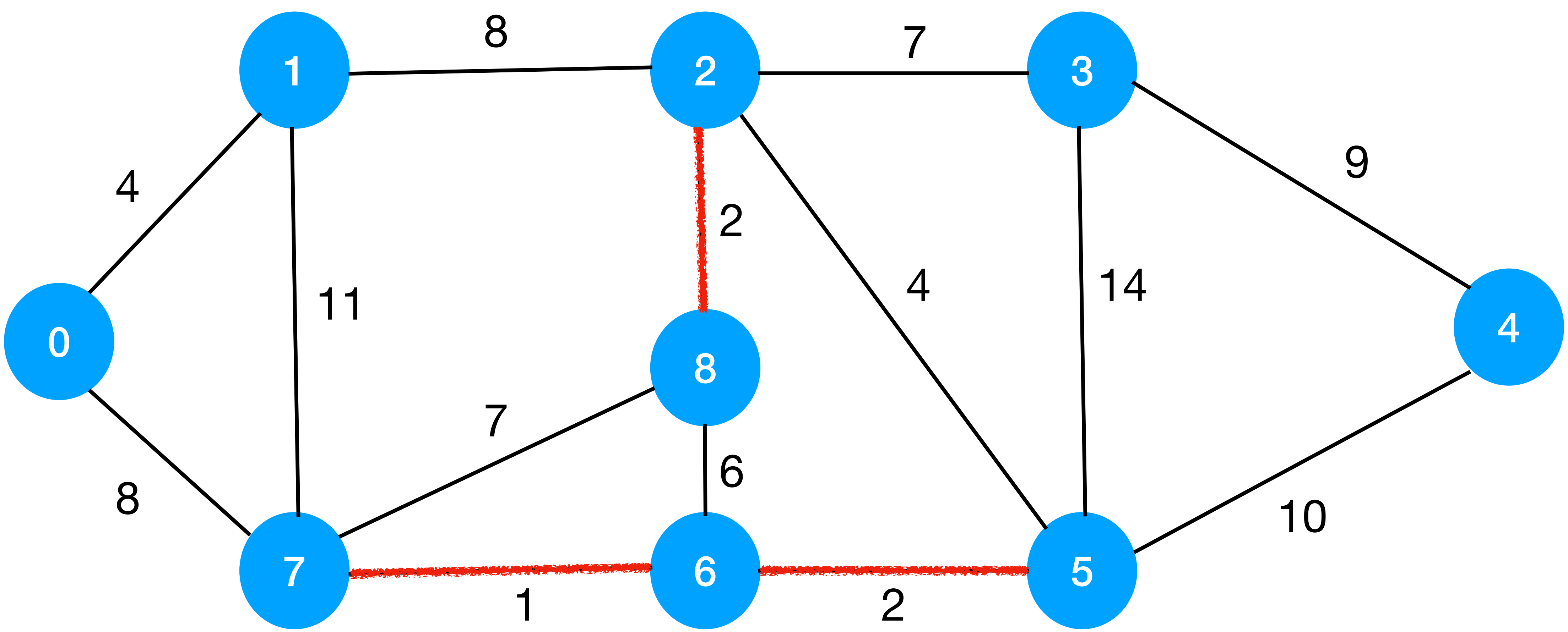

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We need to implement it using actual data structures.

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Intuition: There can only be a few sets of very large size, so all the other Union(A, B) operations should be pretty cheap.

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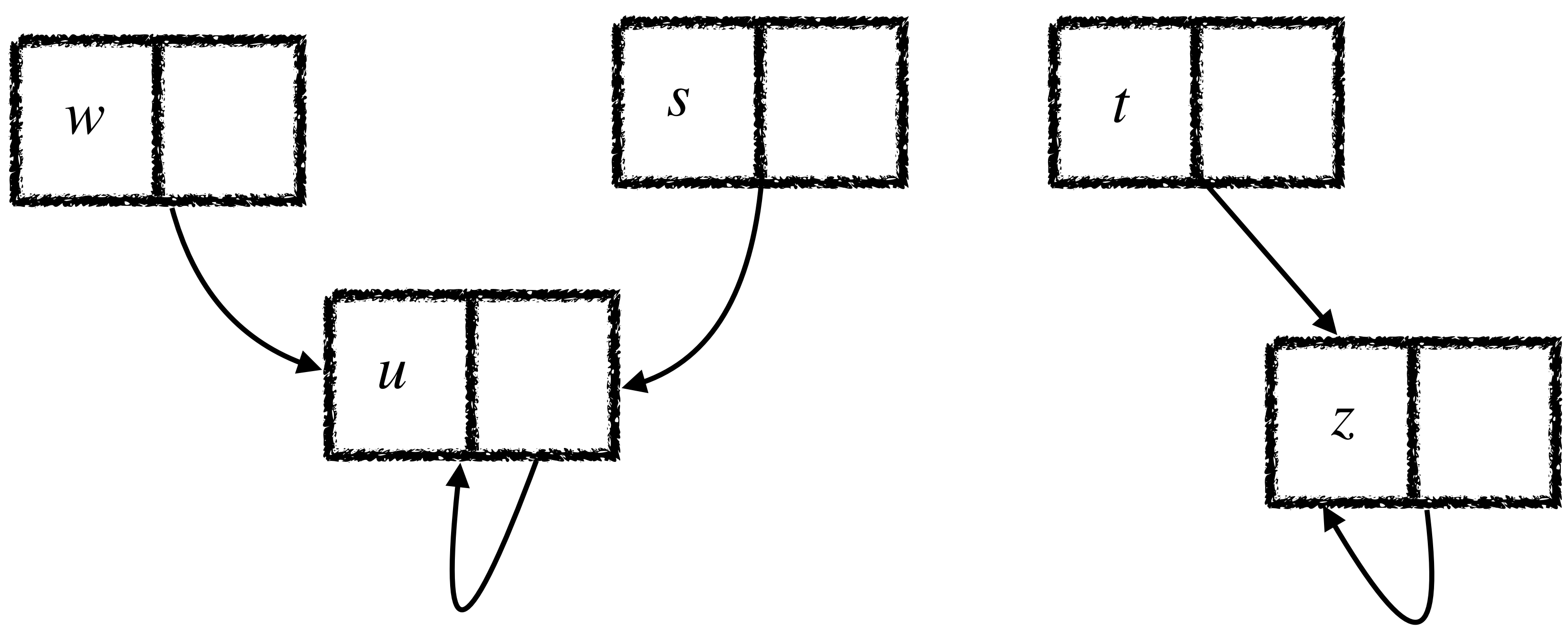
Naming: Name a set S by the name of one of its elements v.

Pointers: Every element v points to some element u (possibly the same).



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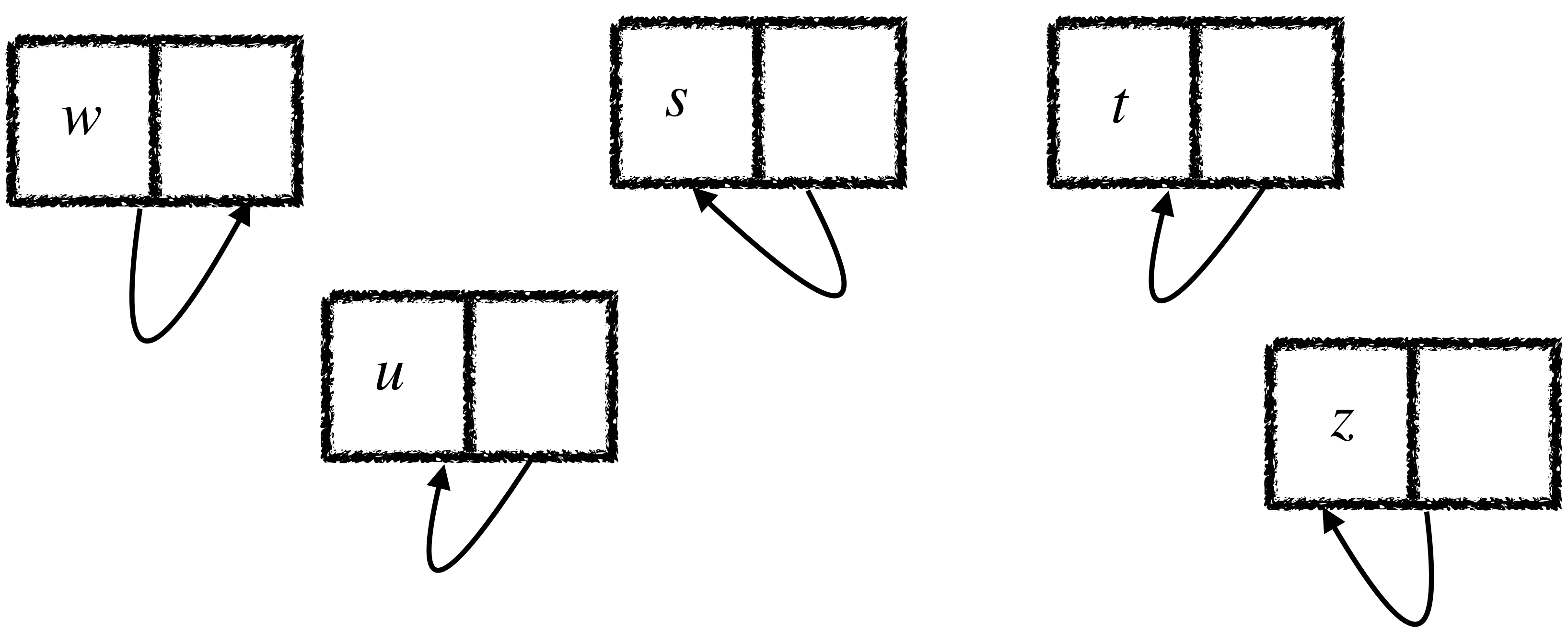

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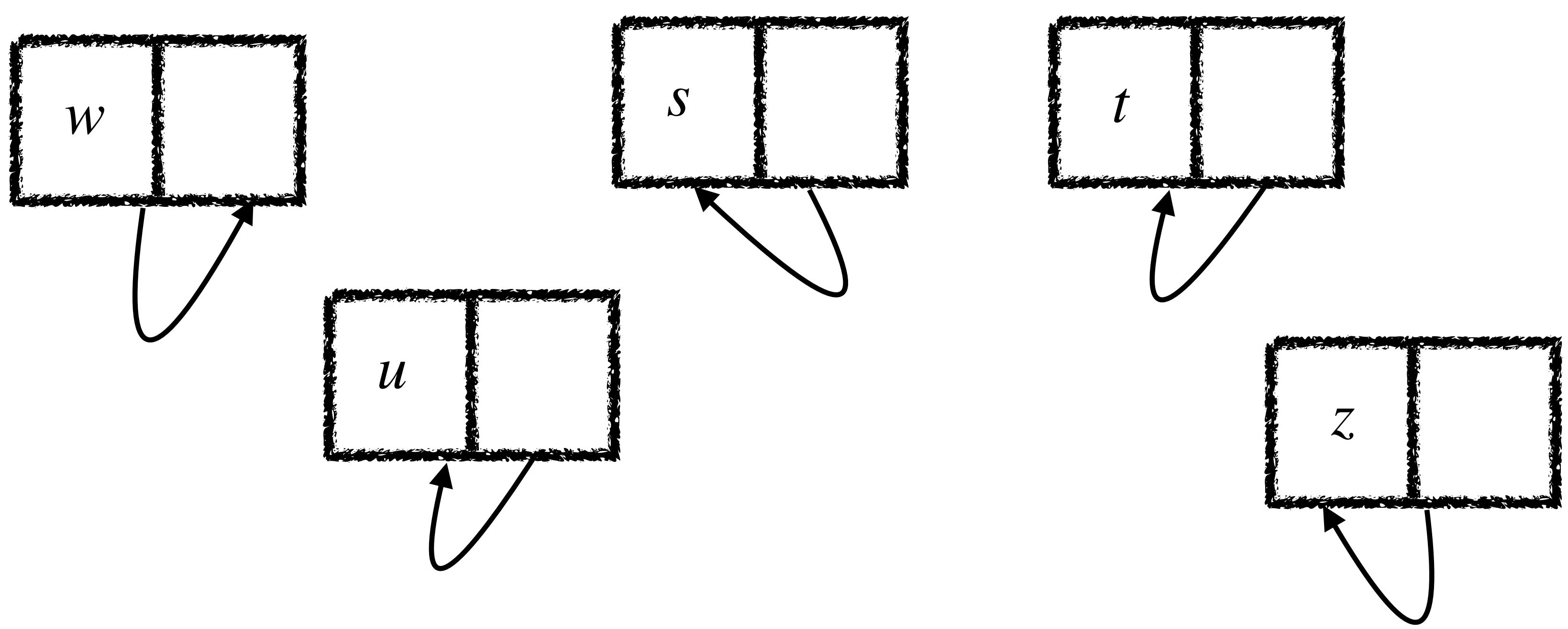
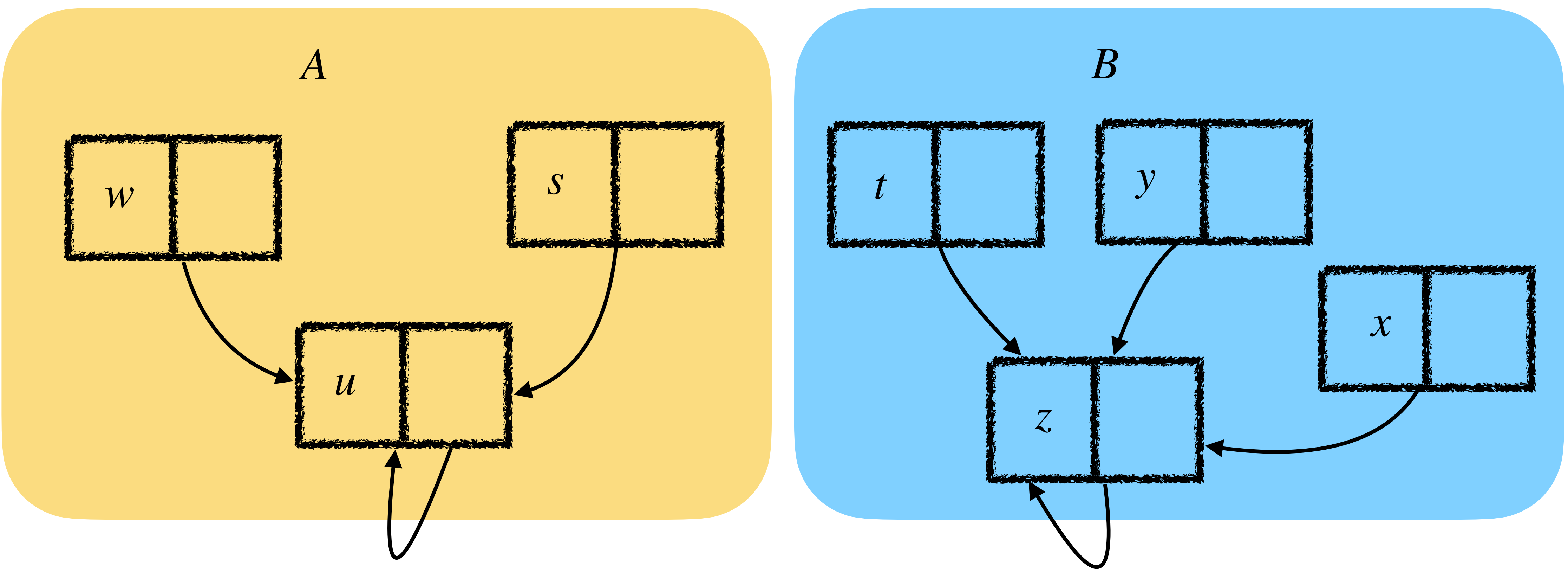
$O(n)$ time



A Better Implementation

Union(A, B): Redirect the pointer of the smallest set to the largest set.



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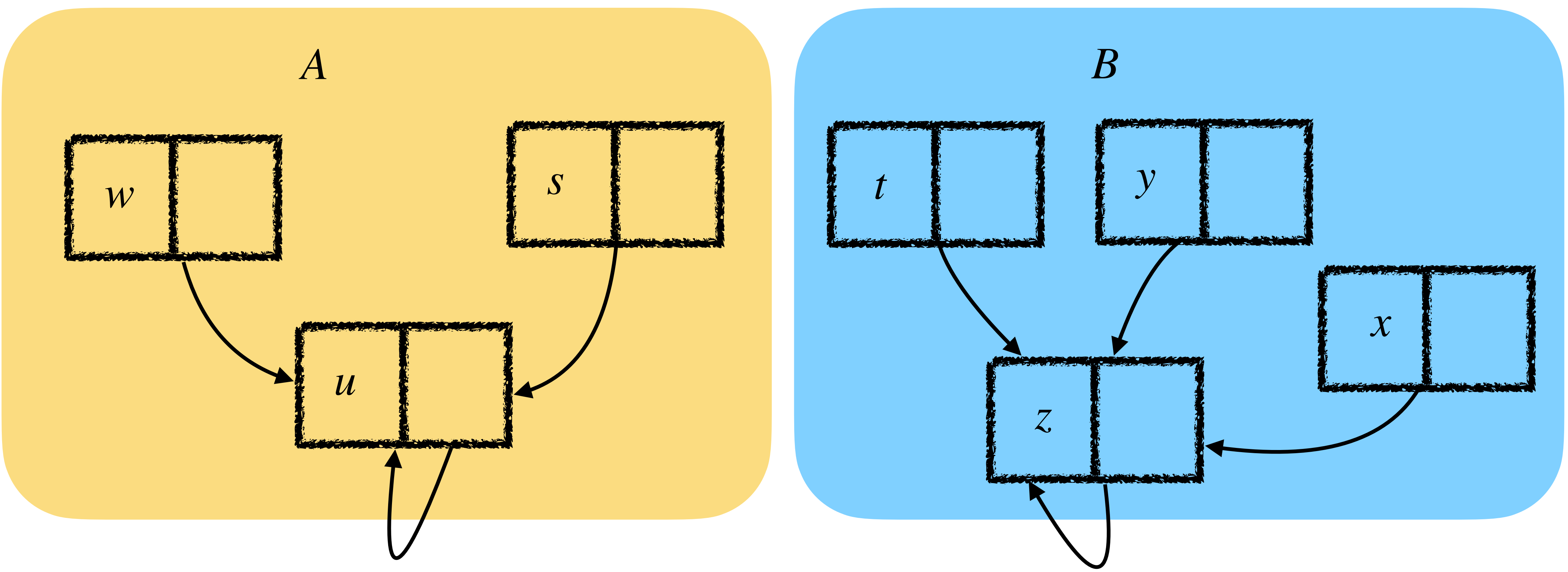

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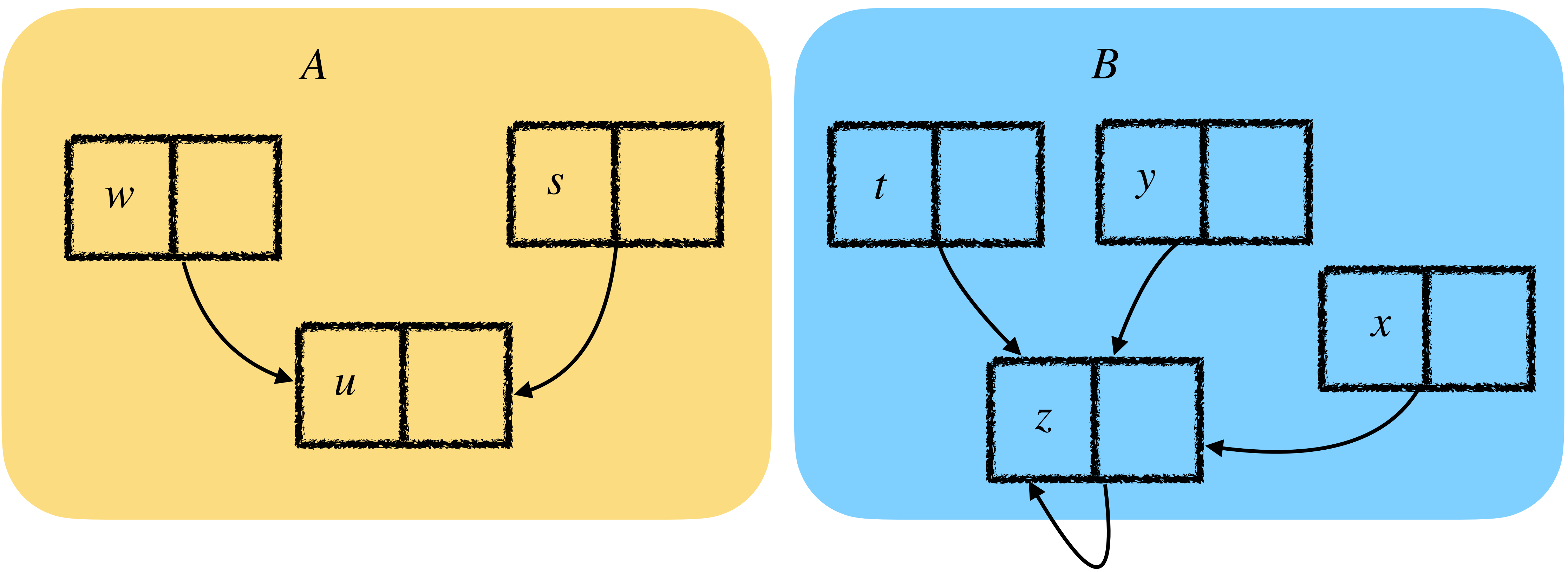

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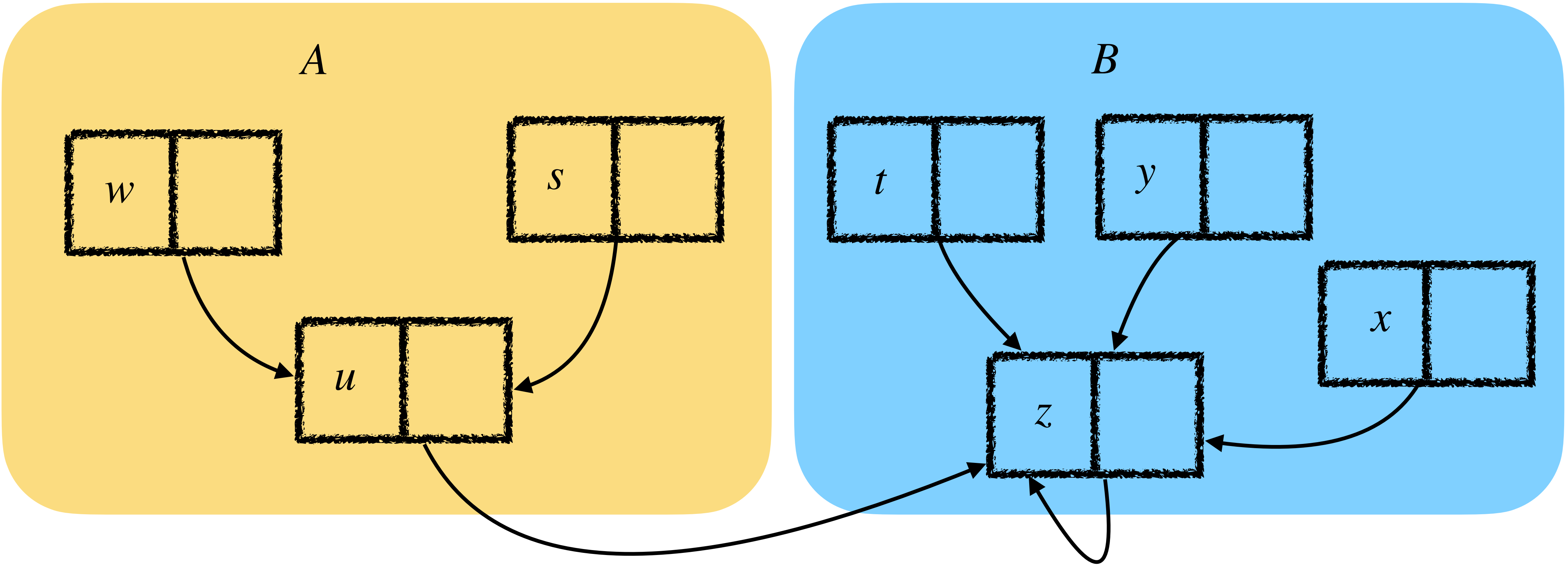

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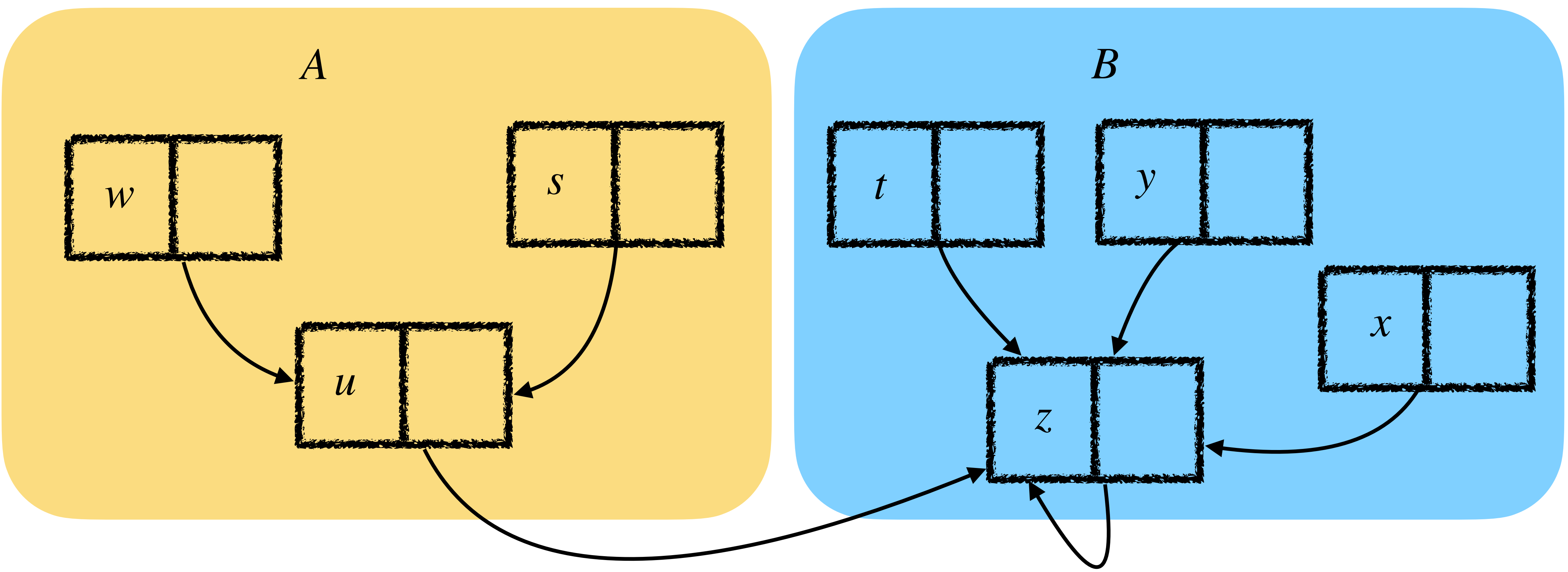

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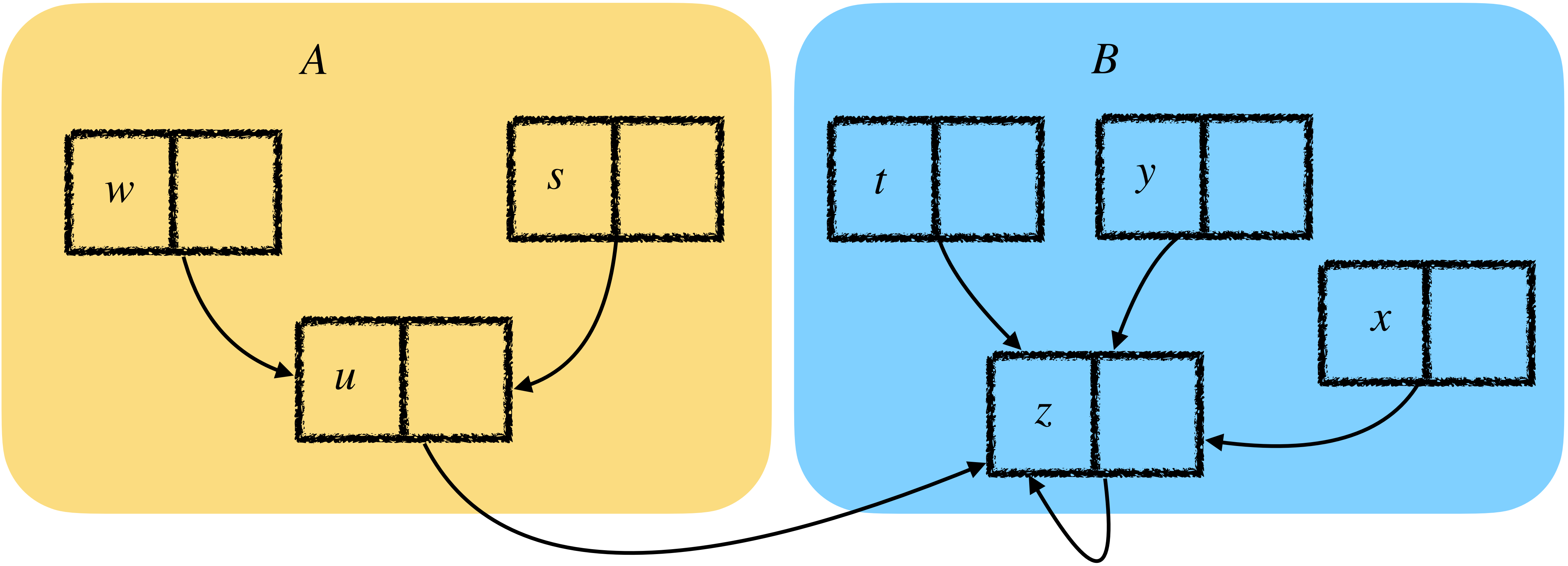
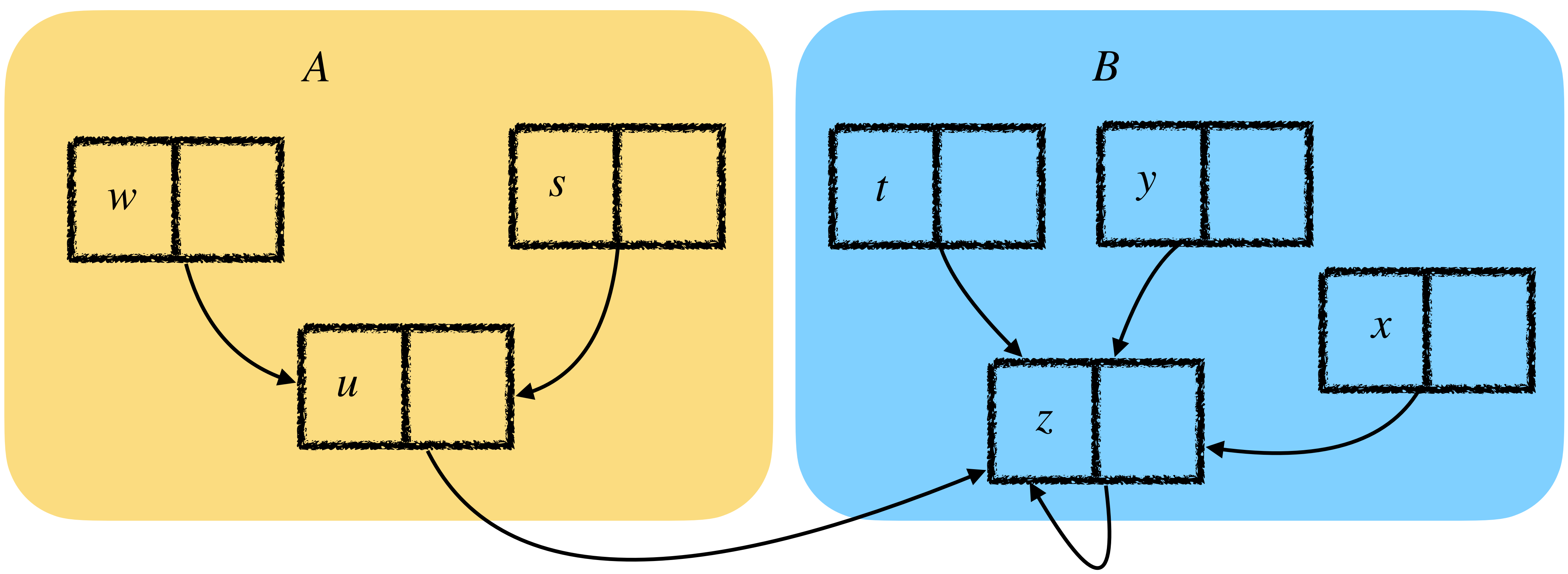


How much time needed for **Union**(A, B)? $O(1)$ time

A Better Implementation

Find(u): Follow the arrows to find the name of the set.



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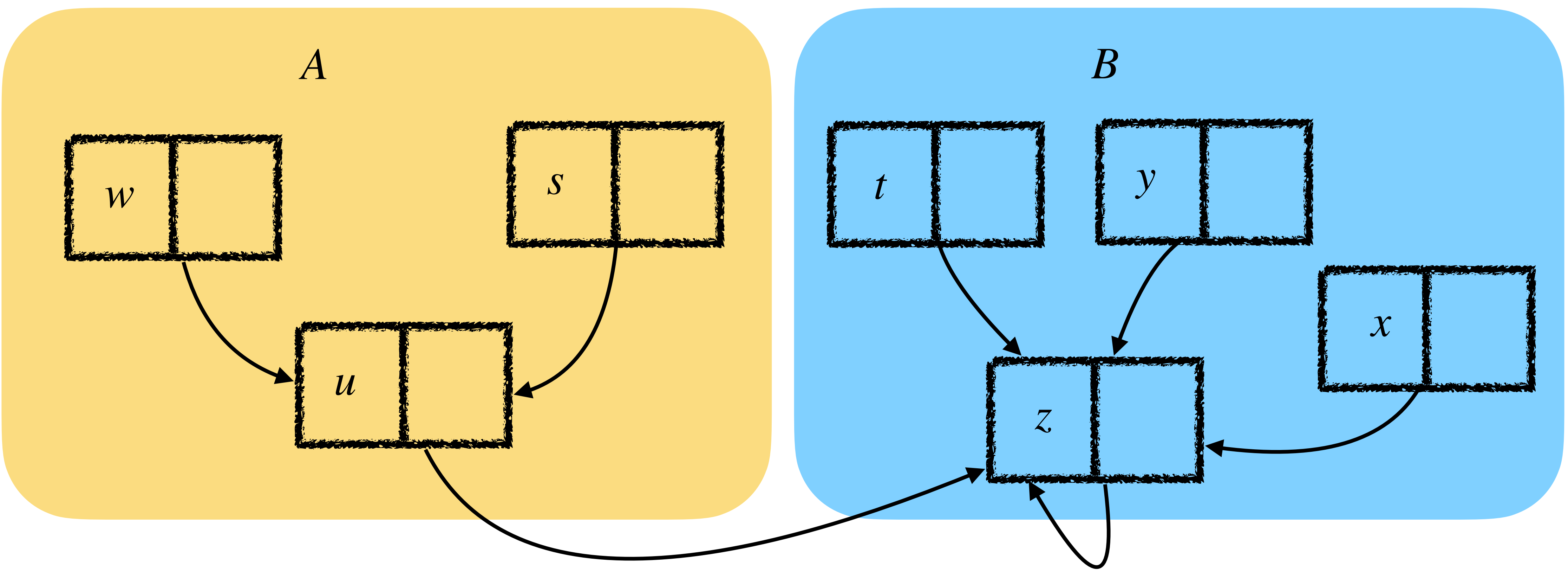

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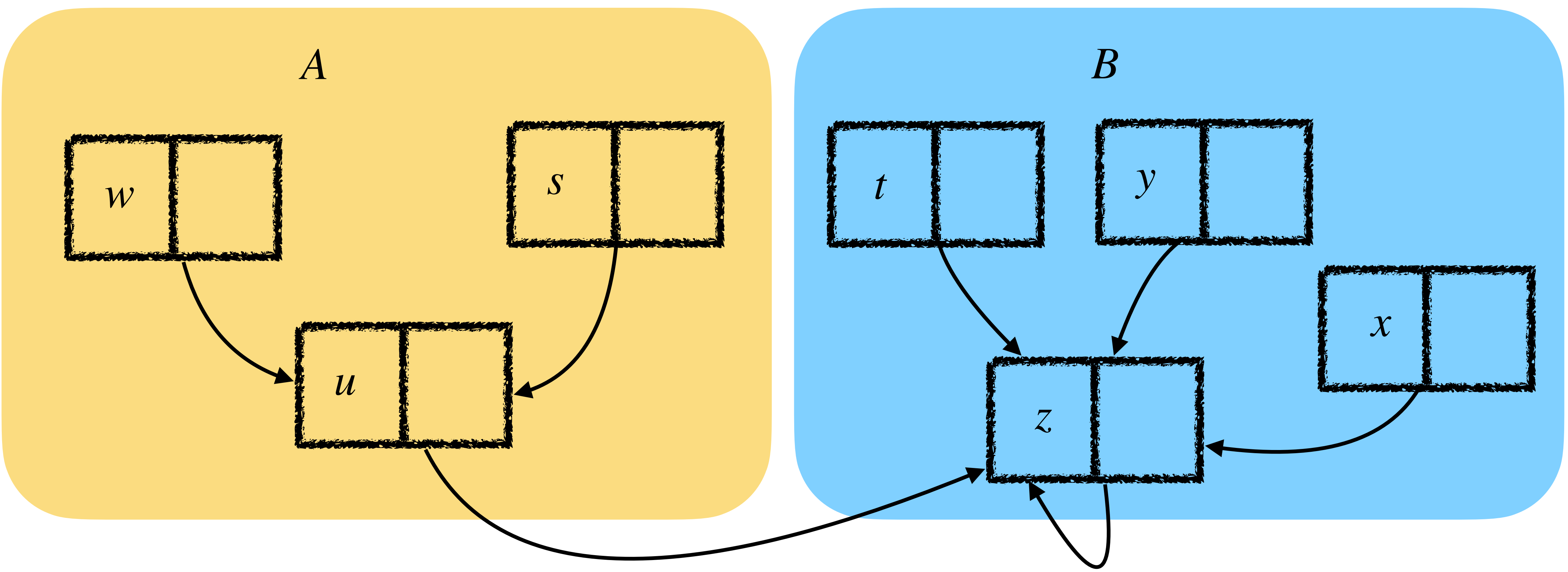

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So, at most how many name changes? $O(\log n)$ changes

Union-Find Operations

MakeUnionFind(S) creates a new Union-Find data structure where every element in S is a singleton set, i.e., $\{v_1\}, \{v_2,\}, \ldots \{v_k\}$ for $S = \{v_1, v_2, \ldots, v_k\}$

Find(u) returns the name of the set containing element u.

Union(A, B) changes the Union-Find data structure by merging the sets A and B into a single set.

Suffices to show $T\left(\text{Find}(u)\right)$, and $T\left(\text{Union}(A, B)\right)$ are $O(\log n)$ and $T\left(\text{MakeUnionFind}(v)\right)$ is $O(m \log n)$

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An even better implementation?

The pointer-based implementation can be made even better, using a similar argument as before, bounding the running time of a sequence of Find(u) operations rather than a single operation.

Details only if you are very interested: KT pp 197-199.

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Data structures is a very big chapter in itself and an active area of research.