Introduction to Algorithms and Data Structures

Lecture 3: Asymptotics: o and ω

John Longley

School of Informatics
University of Edinburgh

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This lecture: In what sense is MergeSort 'fundamentally faster' than InsertSort? o and ω.

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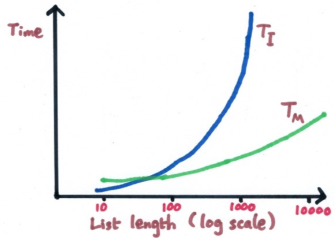
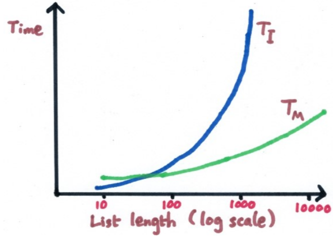
Could then plot a graph (schematic only):



Comparing T_I and T_M



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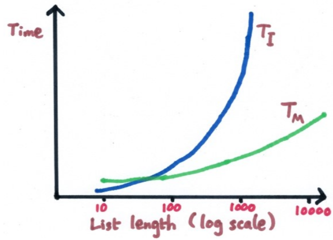

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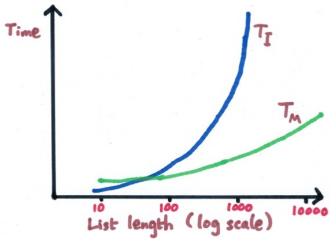

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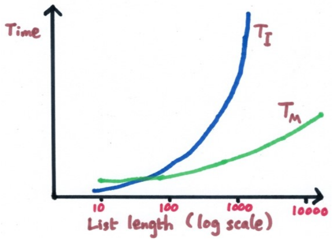

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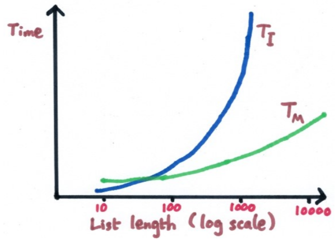

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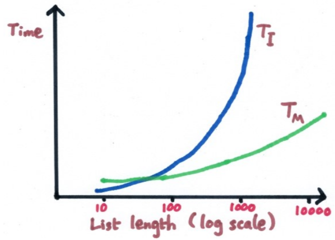

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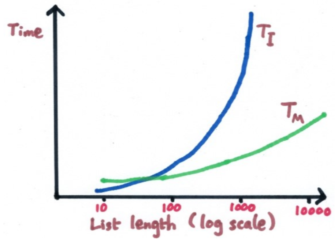

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But doesn't capture the essential difference ...

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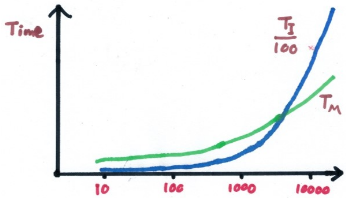

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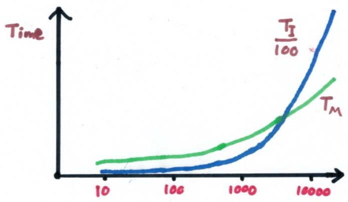

In symbols: $\exists N. \forall n \geq N.\ T_M(n) < 0.01\, T_I(n)$.
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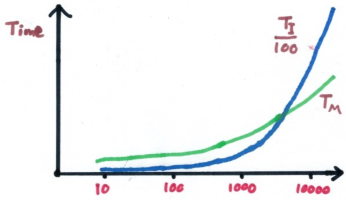

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Question: What if we replaced 0.01 by 0.0001? Or by 0.000001?

Growth rates and 'little o'

Intuition (will justify later): For *any* scale-down factor c, however close to zero, $cT_I(n)$ will eventually break out and overtake T_M:

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Equivalent to saying $g(n)/f(n) \to \infty$ as $n \to \infty$ (if $f : \mathbb{N} \to \mathbb{R}_{>0}$).

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We can take $c = 1/7$. It's then true for *any* $n \geq 0$ that

$$n + 1000000 \ > \ n \ \geq \ 6n/7 \ = \ c.6n$$

So it's clear that $\forall N. \exists n \geq N.\ n + 1000000 \geq c.6n$ (given N, can just take $n = N$).

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Common convention: Write '$o(g)$' to mean 'some (unspecified) function in the set $o(g)$'. E.g.

$$f \;=\; o(g)\,, \qquad\qquad f(n) = 3n^2 + o(n)$$

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So, ‘f is $o(g)$’ technically means $f \in o(g)$.

Common convention: Write ‘$o(g)$’ to mean ‘some (unspecified) function in the set $o(g)$’. E.g.

$$f \;=\; o(g)\,, \qquad\qquad f(n) = 3n^2 + o(n)$$

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(Exercise if you like maths: Prove this from the definition of o.)

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(where every step can be rigorously justified).

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General advice: Sketch graphs to understand what's going on!

And finally: ω

ω is dual to o. Recall that $f = o(g)$ means:

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(Compare: $x > y$ if and only if $y < x$.)

We'll tend to use o more than ω.

Next time: O, Ω, Θ.
(Most presentations start with these!)

Reading for Lectures 3 and 4:
Roughgarden Chapter 2
Kleinberg/Tardos Chapter 2, especially 2.2, 2.4
CLRS Chapter 3 (covers whole Gang of Five)
GGT Sections 3.3, 3.4.