Introduction to Algorithms and Data Structures

Lecture 4: More asymptotics: O, Ω and Θ

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Where we're heading . . .

Recall our runtime functions T_I, T_M for **InsertSort**, **MergeSort**. We've mentioned (and will prove in Lecture 5) that T_M grows slowly relative to T_I: $T_M = o(T_I)$.

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E.g. consider the following hierarchy of 'simple' functions:

$$\begin{array}{lllllllll} f_0(n) & = & 1 & f_1(n) & = & \lg n & f_2(n) & = & \sqrt{n} \\ f_3(n) & = & n & f_4(n) & = & n \lg n & f_5(n) & = & n^2 \\ f_6(n) & = & n^3 & f_7(n) & = & 2^n & f_8(n) & = & 2^{2^n} \quad \cdots \end{array}$$

Here $f_0 \in o(f_1)$, $f_1 \in o(f_2)$, . . .

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Here $f_0 \in o(f_1)$, $f_1 \in o(f_2)$, ...

Which of the above functions do T_I and T_M most closely 'resemble' in their essential growth rate?

The big guys: O, Ω, Θ

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Then say:

$$f \text{ is } \Theta(g) \iff f \text{ is } O(g) \text{ and } f \text{ is } \Omega(g).$$

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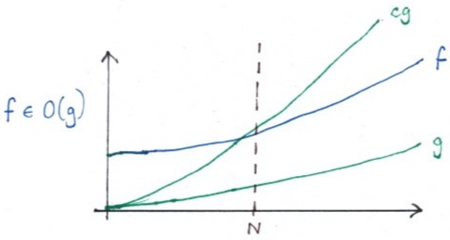

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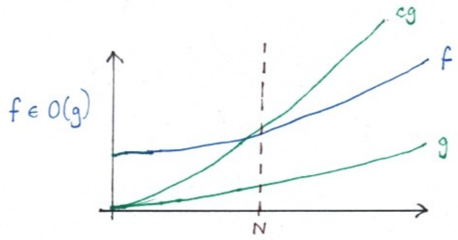

Write as f is $O(g)$, and call g an asymptotic upper bound for f.

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Intuition: $3n$ is the 'dominant' term; $\sqrt{n}$ is 'small change'.

Comparing o and O

We've defined:

$$f \text{ is } o(g) \quad \text{means} \quad \forall c > 0.\ \exists N.\ \forall n \geq N.\ f(n) < cg(n)$$
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Notation: Again, $O(g)$ is officially a set:

$$O(g) \ = \ \{f \ \mid \ \exists C \geq 0.\ \exists N.\ \forall n \geq N.\ f(n) \leq Cg(n)\}$$

But common to write e.g. $f = O(g)$ for $f \in O(g)$.

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So for all $n \geq 100$, we have $f(n) \leq 48n^2$.
In other words, $C = 48$, $N = 100$ will work.

A bit of freedom here . . .

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Advice: Make life easy for yourself!

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Some authors are less precise in distinguishing O and Θ (see CLRS, end of Chapter 3). But if Θ applies, it's fine only to mention O (or Ω) if that's the important bit.

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Moral: Do 'constant factors' matter? Depends where they occur!

Big O: final example

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Note that $\lg(n^7) = 7 \lg n$. So $C = 7$, $N = 1$ will do.

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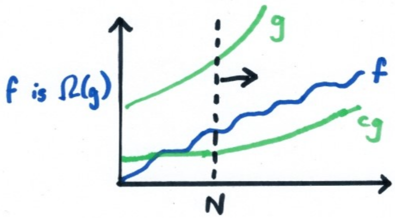

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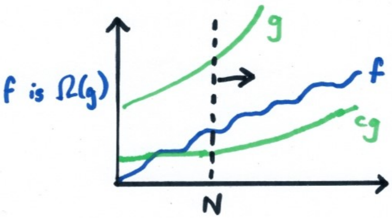

Not hard to show $f = \Omega(g) \iff g = O(f)$.

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Take $c = 1/2,\ N = 4$.

Then for all $n \geq N = 4$, we have $\sqrt{n} \leq n/2$, so

$$n - \sqrt{n} \ \geq\ n - n/2 \ =\ n/2 \ =\ cn$$

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Equivalently, $f \in \Theta(g)$ if and only if

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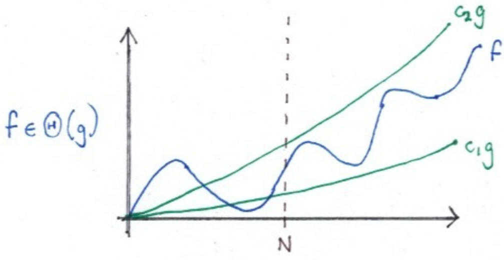

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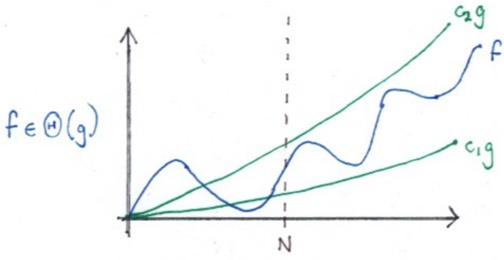

Note also that $f = \Theta(g) \iff g = \Theta(f)$.

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The dominant term is $3n^2$, the rest is small change.
So $f(n)$ will eventually be sandwiched between $2n^2$ and $4n^2$.
(Specifically, can take e.g. $c_1 = 2$, $c_2 = 4$, $N = 5$.)

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So $f(n)$ will eventually be sandwiched between $2n^2$ and $4n^2$.
(Specifically, can take e.g. $c_1 = 2$, $c_2 = 4$, $N = 5$.)

Example 2: $f(n) = 5 - 4/n$.

Examples of Θ

For each of the following functions f, identify some 'simple' g such that $f = \Theta(g)$.

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Example 2: $f(n) = 5 - 4/n$. Answer: $f(n) = \Theta(1)$.

That is, we're taking our 'g' to be the constant function $g(n) = 1$.
Then for any $n \geq 1$, we have

$$1.g(n) \ = \ 1 \ \leq \ 5 - 4/n \ \leq \ 5 \ = \ 5.g(n)$$

So taking $c_1 = 1$, $c_2 = 5$, $N = 1$ will work.

Harder example (optional slide)

Identify some simple g such that $f = \Theta(g)$.

Example 3: $f(n) = \Sigma_{i=1}^{n} 1/i$.
E.g. $f(4) = 1 + 1/2 + 1/3 + 1/4 = 2\frac{1}{12}$.

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[Moral: Can often use Θ-estimates when stuff is hard to compute exactly.]

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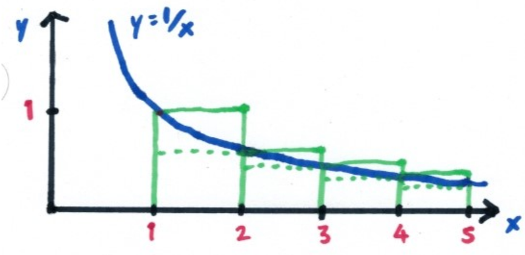
Idea: $f(n)$ is close to $\int_1^n (1/x)dx$, which is $\ln n$.

E.g. for $n = 4$:



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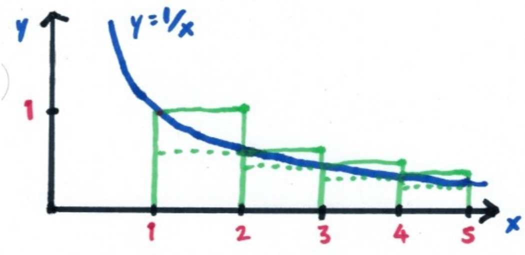

Actually, $\Theta(\ln n)$ is same as $\Theta(\lg n)$: see Tutorial Sheet 1.

Growth rates and algorithms

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Question: What do we expect all these T_J to have in common?

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Of the Gang of Five, we'll meet O and Θ most often.

Some common growth rates

Certain (types of) growth rates crop up frequently, and have names in common use.

- $\Theta(1)$: (within) constant time
- $\Theta(\lg n)$: logarithmic time
- $\Theta(n)$: linear time
- $\Theta(n \lg n)$: log-linear time
- $\Theta(n^2)$: quadratic time
- $\Theta(n^k)$ for some exponent k: polynomial time
- $\Theta(b^n)$ for some base b: exponential time

Reading (same as for Lecture 3):
Roughgarden Chapter 2
Kleinberg/Tardos Chapter 2, especially 2.2, 2.4
CLRS Chapter 3 (covers whole Gang of Five)
GGT Sections 3.3, 3.4.