

Introduction to Theoretical Computer Science

Lecture 11: NP-Completeness

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Hardness

Definition

A problem P_1 is *polynomially reducible* to P_2 , written $P_1 \leq_P P_2$, if there is a polynomially-bounded reduction from P_1 to P_2 .

Recall:

To prove that a problem P_2 is *hard*, show that there is an *easy* reduction from a known hard problem P_1 to P_2 .

Definition

A problem P is *NP-Hard* if, for *every* $A \in \mathbf{NP}$, $A \leq_P P$

- If a problem P_1 is **NP-hard** and $P_1 \leq_P P_2$ then P_2 is **NP-Hard**.
- To prove that a problem P_2 is **NP-hard**, show that there's a *polynomial* reduction from a known **NP-hard** P_1 to P_2 .

Completeness

Question

If any **NP**-hard problem is shown to be in **P**, what does that mean?

Definition

A problem is **NP-complete** if it is both **NP**-hard and in **NP**.

Do **NP**-Complete Problems Exist?

There are **many** such problems, including *HPP* and Timetabling. In fact, almost all **NP**-problems encountered in practice are either in **P**, or **NP**-complete.

Computers and Intractability - A guide to theory of NP-completeness, M.R. Garey and D.S. Johnson, Freeman 1979 lists a whole bunch.

The original **NP**-Complete problem

The *Cook-Levin theorem* states that a particular **NP** problem, *SAT*, is **NP**-complete. The theorem is usually proved for TMs; we shall do it later for RMs.

Why Cook-Levin?

The notion of **NP**-completeness, and the theorem, were due to Stephen Cook (and partly Richard Karp)—in the West. But as with many major mathematical results of the mid-20th century, they were discovered independently in the Soviet Union, by Leonid Levin. Since the fall of the Iron Curtain made Soviet maths more accessible, we try to attribute results to both discoverers.

SAT

SAT is a very significant problem about **boolean formulae**.

The SAT Problem

Given a boolean formula ϕ over a set of boolean variables X_i , is there an **assignment** of values to X_i which **satisfies** ϕ ?

(i.e. makes ϕ true?)

Equivalently, is ϕ non-contradictory?

- $(A \vee B) \wedge (\neg B \vee C) \wedge (A \vee C)$ is satisfiable, e.g. by making A and C true. $(A \wedge B) \wedge (\neg B \wedge C) \wedge (A \wedge C)$ is not satisfiable.
- The size of a SAT problem is the number of symbols in ϕ .
- SAT is obviously in **NP**: just nondeterministically “guess” an assignment and check it.
- It's also apparently exponential in reality: no obvious way to avoid checking all possible assignments (the truth table method).

The Proof

- **The SAT problem is in NP:** Nondeterministically guess an assignment and check it in polynomial time.
- **The SAT problem is NP-Hard:** Shown by reduction from **any NP problem** to SAT.

The Reduction

Suppose $(D, Q) \in \mathbf{NP}$. We shall construct a reduction $Q \leq_P \text{SAT}$. Given an instance $d \in D$, we shall construct a formula ϕ_d which can be satisfied if its variables describe the successful executions of an NRM checking Q . This machine can be polynomially bounded, so the size of ϕ_d will be polynomial in the size of d .

The Variables

Our NRM for Q , $M = (R_0, \dots, R_{m-1}, l_0, \dots, l_{n-1})$ runs for s steps (i.e. $p(|d|)$ where d is our input and p is our polynomial bound).

Name	Meaning	How Many
C_{tj}	Program counter at step t is on l_j .	$s \cdot n$
R_{tik}	k th bit of R_i at step t .	$(2s) \cdot m \cdot s$

Why $2s$?

How big can the registers get? Running s steps of $\text{ADD}(0, 0)$ will make R_0 double s times, if it starts at $2^{|d|}$ then we need $2^{|d|+s}$ capacity. Then w.l.o.g. 2^{2s} i.e. $2s$ bits is enough.

The Formula

$$C_{00} \wedge \rho_{\text{init}} \wedge \chi_{\text{one}} \wedge \bigwedge_t \chi_t \wedge \alpha$$

Name	Meaning	How Many
χ_{one}	Program counter is in one place.	$s \cdot n^2$
χ_t	Step $t + 1$ follows from step t .	$s^2 \cdot m$
ρ_{init}	Initial register values	$m \cdot n$
α	machine accepts	s

Details

Some formulae are easy:

Program counter is in one place

$$\chi_{\text{one}} \equiv \bigwedge_t \bigvee_j \left(C_{tj} \wedge \bigwedge_{j' \neq j} \neg C_{tj'} \right)$$

Some are more tedious:

Step $t + 1$ follows from step t

$\chi_t \equiv \varphi_t \wedge \rho_t$ where φ_t models **control flow** changes and ρ_t models **register** changes.

$\varphi_t \equiv \bigvee_j (C_{tj} \wedge v_{tj})$, where v_{tj} is:

- $C_{t+1,j+1}$ if l_j is INC, ADD, or SUB.
- $C_{t+1,j+1} \vee C_{t+1,j'}$ if l_j is MAYBE(j')
- $((\bigvee_k R_{tik}) \wedge C_{t+1,j+1}) \wedge ((\bigwedge_k \neg R_{tik}) \wedge C_{t+1,j'})$
if l_j is DECJZ(i, j')

Register changes

The formulae concerning registers are very tedious, but can easily be found. See your hardware course!

Exercise

Write a formula $\rho_{tj'j}^+$, which states that at step $t + 1$, R_j will have the sum of the values in R_j and $R_{j'}$ at step t .

Despite this being **very tedious**, these formulae are polynomial ($\mathcal{O}(s^4)$) !

3SAT

- ϕ is in *conjunctive normal form* if it is of the form $\bigwedge_i \bigvee_j P_{ij}$ where each P_{ij} is a *literal* (either a variable P or negation of one $\neg P$).
- ϕ is in *k-CNF* if each clause $\bigvee_j P_{ij}$ has at most k literals.

The Problem

3SAT is the problem of whether a satisfying assignment exists for a formula in *3-CNF*.

Reduction from SAT to 3SAT is difficult, because normally converting to 3-CNF is an exponential blowup. The *Tseitin encoding* is used instead to give a not-equivalent but *equisatisfiable* formula.

Note: For TM, the Cook-Levin theorem is shown directly for 3SAT.

Clique

The *CLIQUE* problem

Given a graph $G = (V, E)$ and a number k , a *k-clique* is a k -sized subset C of V , such that every vertex in C has an edge to every other. (C forms a complete subgraph.) Decide whether G has a k -clique.

Exercise: Why is *CLIQUE* \in NP?

Reducing from 3SAT, we have a formula

$$\varphi = \bigwedge_{1 \leq i \leq k} (x_{i1} \vee x_{i2} \vee x_{i3})$$

The Graph

Each x_{ij} is a vertex. Connect x_{ij} to $x_{i'j'}$ iff: $i \neq i'$ and $x_{i'j'}$ is not the negation of x_{ij} .

i.e. we connect literals in different clauses so long as they are not inconsistent.

Why does this work?

Since the vertices in one clause are disconnected, finding a k -clique amounts to finding one literal for each clause, such that they are all consistent — and so represent a satisfying assignment. Conversely, any satisfying assignment generates a k -clique.

P vs. NP

As previously mentioned, we **don't know** if **P** and **NP** are really distinct classes.

Find a polynomial time algorithm for any **NP**-hard problem and you can win yourself one million US dollars from the Clay Institute. (Also hire bodyguards because most web/banking security depends on such problems being hard.)

Many complexity theory results start with “if **P** \neq **NP**...”

Example (**NP**-Intermediacy)

A problem is **NP**-Intermediate if it is in **NP** but not in **P** nor **NP**-complete.

If **P** \neq **NP**, then **graph isomorphism** is such a problem

(and there aren't many others).

NP in Practice

As far as we know, **NP** problems are just hard: need exponential search, so $\mathcal{O}(p(n) \cdot 2^n)$. So how do we solve them in practice?

- Randomized algorithms are often useful. Allow algorithms to toss a coin. Surprisingly one can get randomized algorithms that solve e.g. 3SAT in time $\mathcal{O}(p(n) \cdot \frac{4}{3}^n)$.

(Why is this useful? $2^{100} \approx 10^{31}$, while $1.33^{100} \approx 10^{12}$)

Catch: (really) small probability of error!

- In many special classes (e.g. sparse graphs, or almost-complete graphs), heuristics lead to fast results. See <http://satcompetition.org/> for the state of the art.

Next time...

We'll be looking at the **boundaries** of the class **NP**, and what lays beyond. Specifically, the classes of **coNP** and **PSPACE**, as well as the *polynomial hierarchy*, analogous to the arithmetic hierarchy we've already seen, but contained entirely within **PSPACE** decidable problems.