Introduction to Theoretical Computer Science

Lecture 12: The Polynomial Hierarchy, Alternation and **PSPACE**

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The Class **CoNP**

trick **Question**

Is the class **NP** closed under complement?

We don't know.

Question

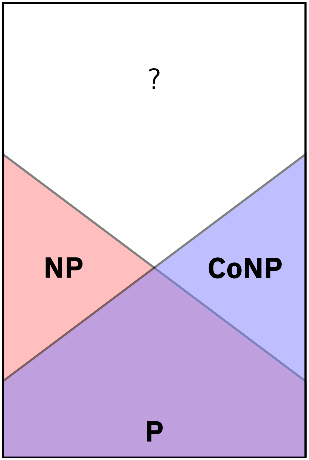
Why can't we just flip the answer, like we do for **P**?

Nondeterministic machines accept if just one path accepts. To flip the answer of an NRM, we'd have to accept an answer if all paths reject. This is no longer an (angelic) NRM (resp. NTM).

Question

What is **NP** $\cap$ **CoNP**? is it **P**? (we don't know)

What we have now



Sigmas

We shall introduce notation to describe polynomial problems.

Sigma

The set $\Sigma_1^{\mathbf{P}}$ describes all problems that can be phrased as $\{y \mid \exists^{\mathbf{P}} x \in \mathbb{N}.\ R(x, y)\}$, where R is a **P**-decidable predicate and $\exists^{\mathbf{P}} x \ldots$ indicates that x is of size polynomial in the size of y.

- If a problem $Q \in \Sigma_1^{\mathbf{P}}$ then Q is in **NP**. **Why**?
  (we can "guess" an x and polynomially test $R(x, y)$)
- If a problem Q is in **NP** then $P \in \Sigma_1^{\mathbf{P}}$. **Why?**

Certificates

We can say that x is a *certificate* showing which "guesses" can made by our NRM giving an accepting run.

So, **NP** $= \Sigma_1^{\mathbf{P}}$.

Pis

Pi

The set $\Pi_1^{\mathbf{P}}$ describes all problems that can be phrased as $\{y \mid \forall^{\mathbf{P}} x \in \mathbb{N}.\ R(x, y)\}$, where R is a **P**-decidable predicate and $\forall^{\mathbf{P}} x \ldots$ indicates that x is of size polynomial in the size of y.

$$\begin{array}{rcl} \overline{\Sigma_1^{\mathbf{P}}} & = & \overline{\{x \mid \exists^{\mathbf{P}} y.\ R(x, y)\}} \\ & = & \{x \mid \neg\exists^{\mathbf{P}} y.\ R(x, y)\} \\ & = & \{x \mid \forall^{\mathbf{P}} y.\ \neg R(x, y)\} \\ & = & \Pi_1^{\mathbf{P}} \end{array}$$

As $\Sigma_1^{\mathbf{P}}$ is **NP**, $\Pi_1^{\mathbf{P}}$ is **CoNP**.

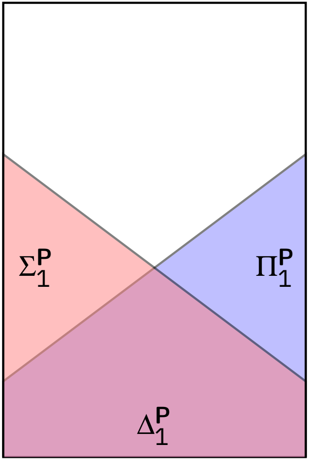
Deltas

Delta

The set $\Delta_1^{\mathbf{P}}$ describes the set **P**

This is different from definitions in the arithmetic hierarchy where Δ_1 describes the intersection of Σ_1 and Π_1.

Relabeling



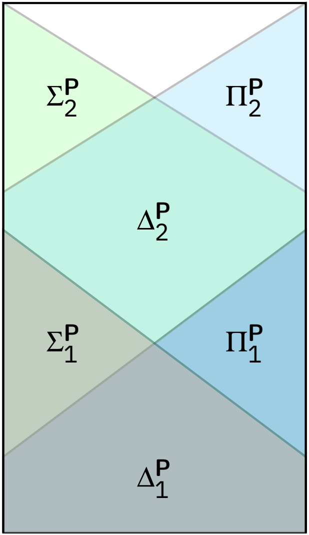
Moving Higher

Definitions

- $\Sigma_2^{\mathbf{P}}$ is all problems of form $\{x \mid \exists^{\mathbf{P}} y. \forall^{\mathbf{P}} z.\ R(x, y, z)\}$.
- $\Pi_2^{\mathbf{P}}$ is all problems of form $\{x \mid \forall^{\mathbf{P}} y. \exists^{\mathbf{P}} z.\ R(x, y, z)\}$.
- $\Delta_2^{\mathbf{P}} = \mathbf{P}^{NP}$ (deterministic polynomial time with NP oracle.

Note that $\Sigma_1^{\mathbf{P}}, \Pi_1^{\mathbf{P}}, \Delta_1^{\mathbf{P}}$ are all $\subseteq \Delta_2^{\mathbf{P}}$ (and therefore $\subseteq \Sigma_2^{\mathbf{P}}$ and $\subseteq \Pi_2^{\mathbf{P}}$).
Why?
(our R can simply “ignore” one of the parameters)

The Polynomial Hierarchy



An equivalent characterisation

We can define in terms of oracles:

- $\Delta_2^{\mathbf{P}}$ is all problems that are decidable in polynomial time by some deterministic TM/RM with an $\mathcal{O}(1)$ oracle for some complete problem in $\Sigma_1^{\mathbf{P}}$, i.e. it is **P** with an $\mathcal{O}(1)$ oracle for **NP**.
- $\Sigma_2^{\mathbf{P}}$ allows the TM/RM to be nondeterministic, i.e. it is **NP** with an $\mathcal{O}(1)$ oracle for **NP**.
- $\Pi_2^{\mathbf{P}}$ is **CoNP** with an oracle for **NP**.

Building up

In general, for any $n > 1$:

- $\Delta_n^{\mathbf{P}}$ is all problems that are decidable by some deterministic, polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem $\in \Sigma_{n-1}^{\mathbf{P}}$.
- $\Sigma_n^{\mathbf{P}}$ are all problems that are decidable by some nondeterministic, polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem $\in \Sigma_{n-1}^{\mathbf{P}}$.
- $\Pi_n^{\mathbf{P}}$ are all problems decidable by some *co-nondeterministic*, polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem $\in \Sigma_{n-1}^{\mathbf{P}}$.

Co-nondeterminism

Could also be called *demonic* nondeterminism. Like our normal (angelic) nondeterminism but only accepts if *all* paths accept.

Definition with oracles

X^Y means decision problems solvable in X with the help of an $\mathcal{O}(1)$ oracle for problems in Y.

Definitions

- $\Delta_0^{\mathbf{P}} := \Sigma_0^{\mathbf{P}} := \Pi_0^{\mathbf{P}} := \mathbf{P}$
- $\Sigma_{i+1}^{\mathbf{P}} := NP^{\Sigma_i^{\mathbf{P}}}$
- $\Pi_{i+1}^{\mathbf{P}} := coNP^{\Sigma_i^{\mathbf{P}}}$
- $\Delta_{i+1}^{\mathbf{P}} := \mathbf{P}^{\Sigma_i^{\mathbf{P}}}$

Alternation

Alternation

Equivalently $\Sigma_n^{\mathbf{P}}$ are all problems that can be phrased as some alternation of (**P**-bounded) quantifiers, starting with $\exists^{\mathbf{P}}$:

$$\{w \mid \exists^{\mathbf{P}} x_1 . \forall^{\mathbf{P}} x_2 . \exists^{\mathbf{P}} x_3 . \forall^{\mathbf{P}} x_4 . \ldots x_n .\ R(w, x_1, \ldots, x_n)\}$$

$\Pi_n^{\mathbf{P}}$ starts instead with $\forall^{\mathbf{P}}$:

$$\{w \mid \forall^{\mathbf{P}} x_1 . \exists^{\mathbf{P}} x_2 . \forall^{\mathbf{P}} x_3 . \exists^{\mathbf{P}} x_4 . \ldots x_n .\ R(w, x_1, \ldots, x_n)\}$$

Alternating Machines

Alternating Machines combine the acceptance modes of both angelic and demonic nondeterministic machines.

Alternating Register Machines

Consider NRMs where instead of just a MAYBE instruction we have a $\text{MAYBE}^{\forall}$ instruction and a $\text{MAYBE}^{\exists}$ instruction.

- $\text{MAYBE}^{\exists}$ is a nondeterministic choice where we accept if one branch accepts.
- $\text{MAYBE}^{\forall}$ is a nondeterministic choice where we accept only when both branches accept.

Alternating Turing Machines are defined by labelling states with either $\forall$ or $\exists$.

Alternating Machines and the Polynomial Hierarchy

- The class $\Sigma_n^{\mathbf{P}}$ could equivalently be defined as the class of problems decided in polynomial time by an alternating machine that initially uses $\exists$-nondeterminism, and every path in the machine swaps quantifiers (i.e. to $\forall$ or back to $\exists$) at most $n - 1$ times.
- $\Pi_n^{\mathbf{P}}$ is the same, except that we start with $\forall$ instead.

The class **AP**

AP is the class of all problems decidable by an alternating machine in polynomial time, without any restriction on swapping quantifiers.
AP is known to be equal to **PSPACE** (more on this in a moment)

A Fragile House of Cards

Warning

The polynomial hierarchy could collapse at any point.
(i.e. all of the classes in the PH could be equal)

We don't know that **P** $\neq$ **PSPACE**, and the entire polynomial hierarchy is contained inside **AP** which $=$ **PSPACE**.

Wait, what's **PSPACE**?

An RM/TM is *$f(n)$-space-bounded* if it may use only $f(\mathit{inputsize})$ space.
For TMs, space means cells on tape; for RMs, #bits in registers.
PSPACE is the class of problems solvable by polynomially-space-bounded machines.

The following inclusions follow from the definitions.
P $\subseteq$ **NP** $\subseteq$ **PSPACE** $\subseteq$ **EXPTIME**.
At least one of these inclusions must be strict, because we know that **P** $\subset$ **EXPTIME**. But we don't know which one.
The common conjecture is that they are all strict.