Algorithms and Data Structures

Degeneracy, Geometry, and Duality

What about this dictionary?

Maximise $\zeta = 3 - 0.5x_1 + 2x_2 - 1.5w_1$

subject to

$$
\begin{aligned}
x_3 &= 1 - 0.5x_1 - 0.5w_1 \\
w_2 &= x_1 - x_2 + w_1
\end{aligned}
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$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

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entering variable

$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

What about this dictionary?

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\begin{array}{llrrrr}
\textbf{Maximise} & \zeta = & 3 & -0.5\,x_1 & +2\,x_2 & -1.5\,w_1 \\
\textbf{subject to} & x_3 = & 1 & -0.5\,x_1 & & -0.5\,w_1 \\
 & w_2 = & & x_1 & -\,x_2 & +\,w_1
\end{array}
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$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

leaving variable

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$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

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entering variable

We can increase the value of some nonbasic variable, here x_2

We should not violate any constraints though!

We don't want any of the slack variables to become negative.

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x_2 cannot be increased! Are we stuck?

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Equivalently: In a basic feasible solution, one of the basic variables is 0.

Degeneracy not necessarily and issue

entering variable

Maximise $\zeta = 5 + x_3 - x_1$

subject to

$$\begin{aligned} x_2 &= 5 + 2x_3 - 3x_1 \\ x_4 &= 7 - 4x_1 \\ x_5 &= x_1 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The LP is unbounded!

We can increase the value of some nonbasic variable, here x_3

We should not violate any constraints though!

We don't want any of the slack variables to become negative.

This does not happen regardless of how much we increase x_3.

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subject to

$$x_3 = 1 \quad -0.5\,x_1 \qquad\qquad -0.5\,w_1$$

$$w_2 = \qquad x_1 \quad - \quad x_2 \quad + \quad w_1$$

leaving variable

entering variable

$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

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"Degenerate pivots are quite common and usually harmless."

Let's not give up

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Maximise $\zeta = 3 - 0.5\,x_1 + 2\,x_2 - 1.5\,w_1$

subject to

$$x_3 = 1 - 0.5\,x_1 - 0.5\,w_1$$

$$\boxed{w_2} = x_1 - \boxed{x_2} + w_1$$

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$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

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$$x_3 = 1 - 0.5\,x_1 - 0.5\,w_1$$
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Increase the variable as much as we can (as before): here 0 increase

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Actually pivot!

We can increase the value of some nonbasic variable, here x_2

We should not violate any constraints though!

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Let's not give up

Maximise $\zeta = 3 - 0.5x_1 + 2x_2 - 1.5w_1$

subject to $x_3 = 1 - 0.5x_1 - 0.5w_1$

leaving variable $w_2 = x_1 - x_2 + w_1$ entering variable

$x_1, x_2, x_3, w_1, w_2 \geq 0$

Actually pivot!

$(x_1, x_2, x_3, w_1, w_2) = (0,0,1,0,0)$

We can increase the value of some nonbasic variable, here x_2

We should not violate any constraints though!

We don't want any of the slack variables to become negative.

Increase the variable as much as we can (as before): here 0 increase

The new dictionary

Maximise $\zeta = 3 + 1.5\,x_1 + 2\,w_2 + 0.5\,w_1$

subject to

$$x_3 = 1 - 0.5\,x_1 - 0.5\,w_1$$

$$x_2 = x_1 - w_2 + w_1$$

$$x_1, x_2, x_3, w_1, w_2 \geq 0$$

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$(x_1, x_2, x_3, w_1, w_2) = (0,0,1,0,0)$

We can now increase x_1 to $x_1 = 2$

The new dictionary

Maximise $\zeta = 3 + 1.5\, x_1 + 2\, w_2 + 0.5\, w_1$

entering variable

subject to $x_3 = 1 - 0.5\, x_1 - 0.5\, w_1$

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$x_2 = x_1 - w_2 + w_1$

$x_1, x_2, x_3, w_1, w_2 \geq 0$

$(x_1, x_2, x_3, w_1, w_2) = (0,0,1,0,0)$

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The pivot is not degenerate!

The new dictionary

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$$(x_1, x_2, x_3, w_1, w_2) = (0,0,1,0,0)$$

We can now increase x_1 to $x_1 = 2$

The pivot is not degenerate!

It will actually lead to a final dictionary, and an optimal solution.

Pictorially

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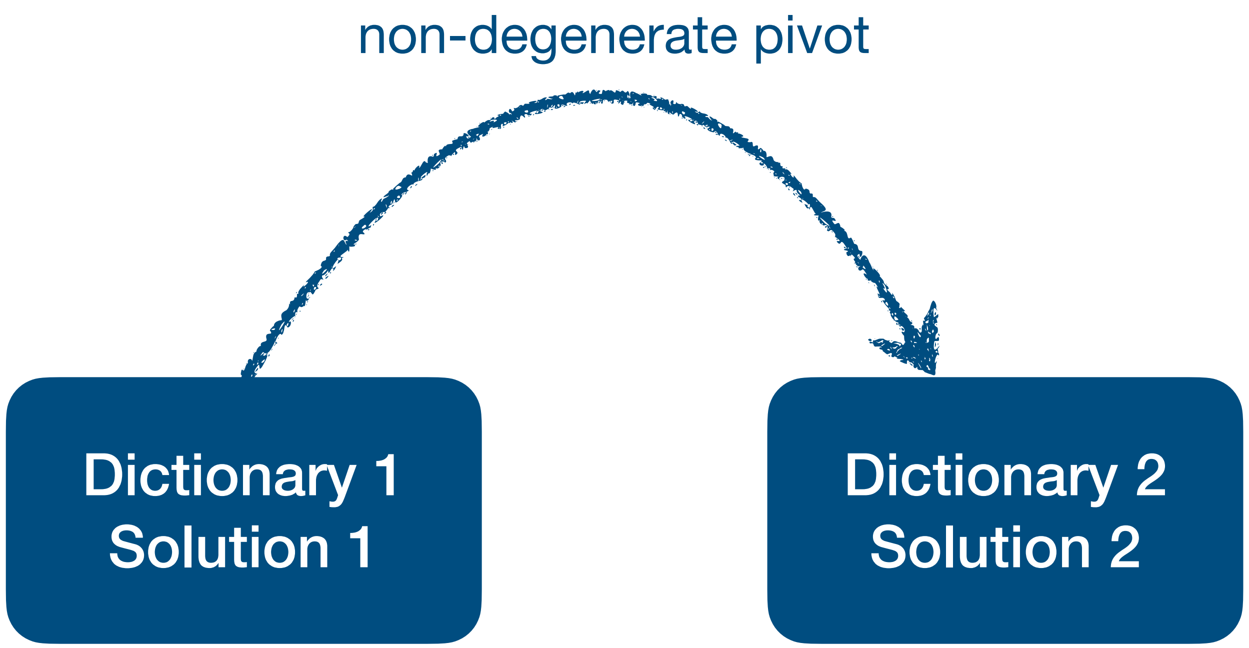
Dictionary 1
Solution 1

Pictorially



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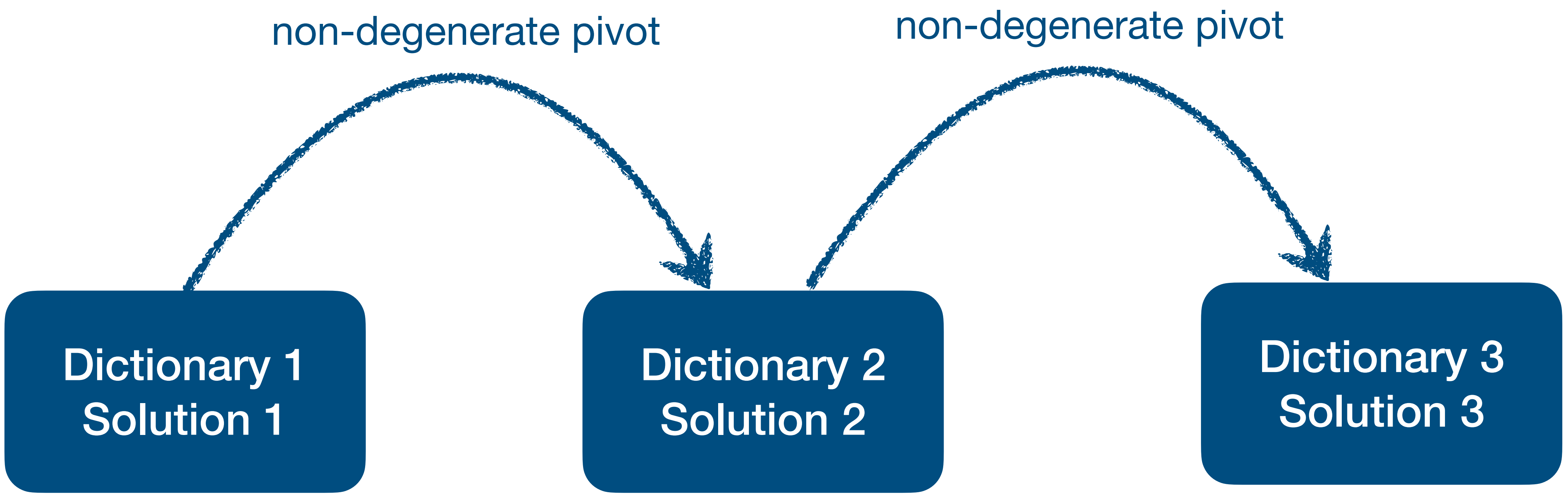

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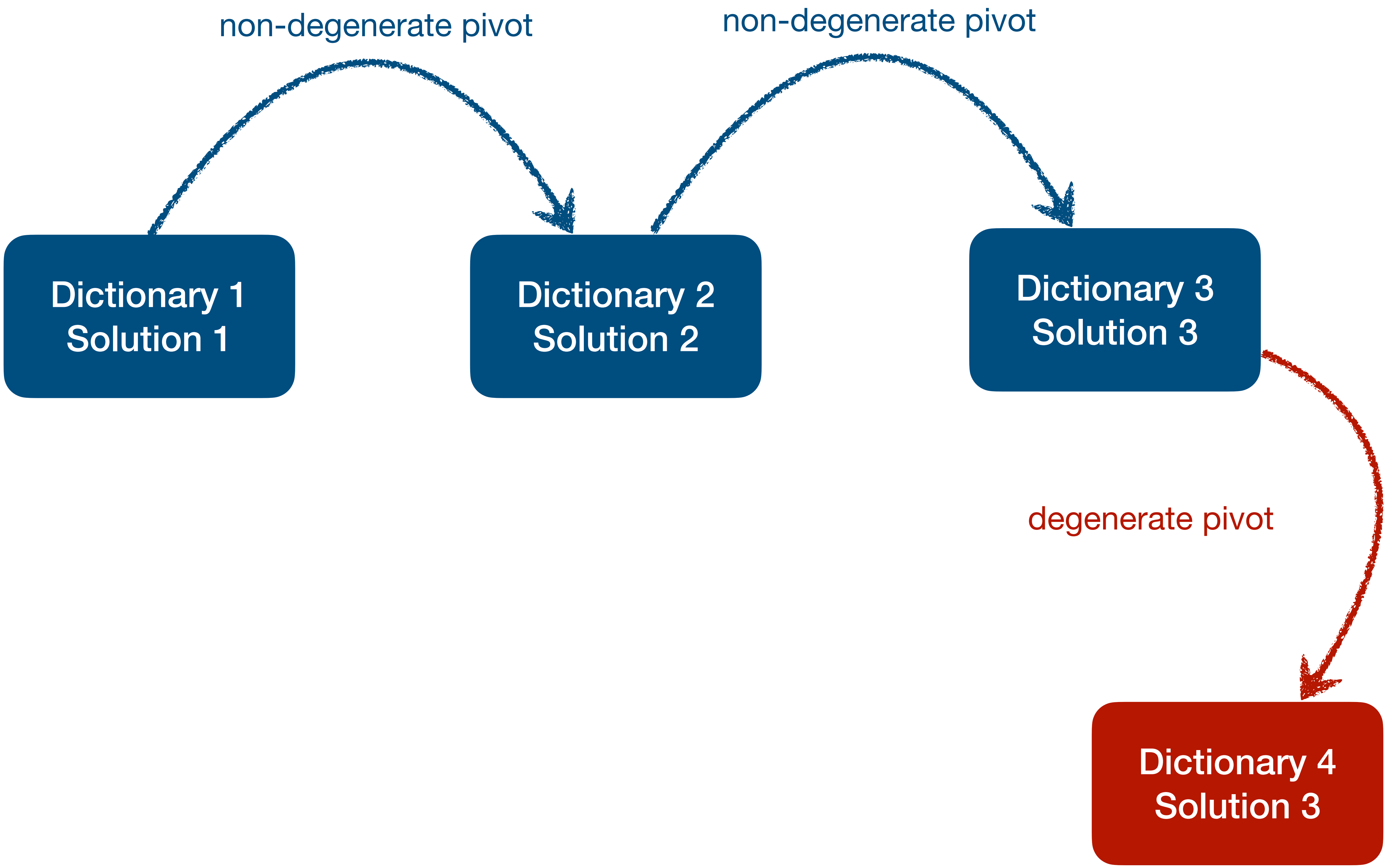

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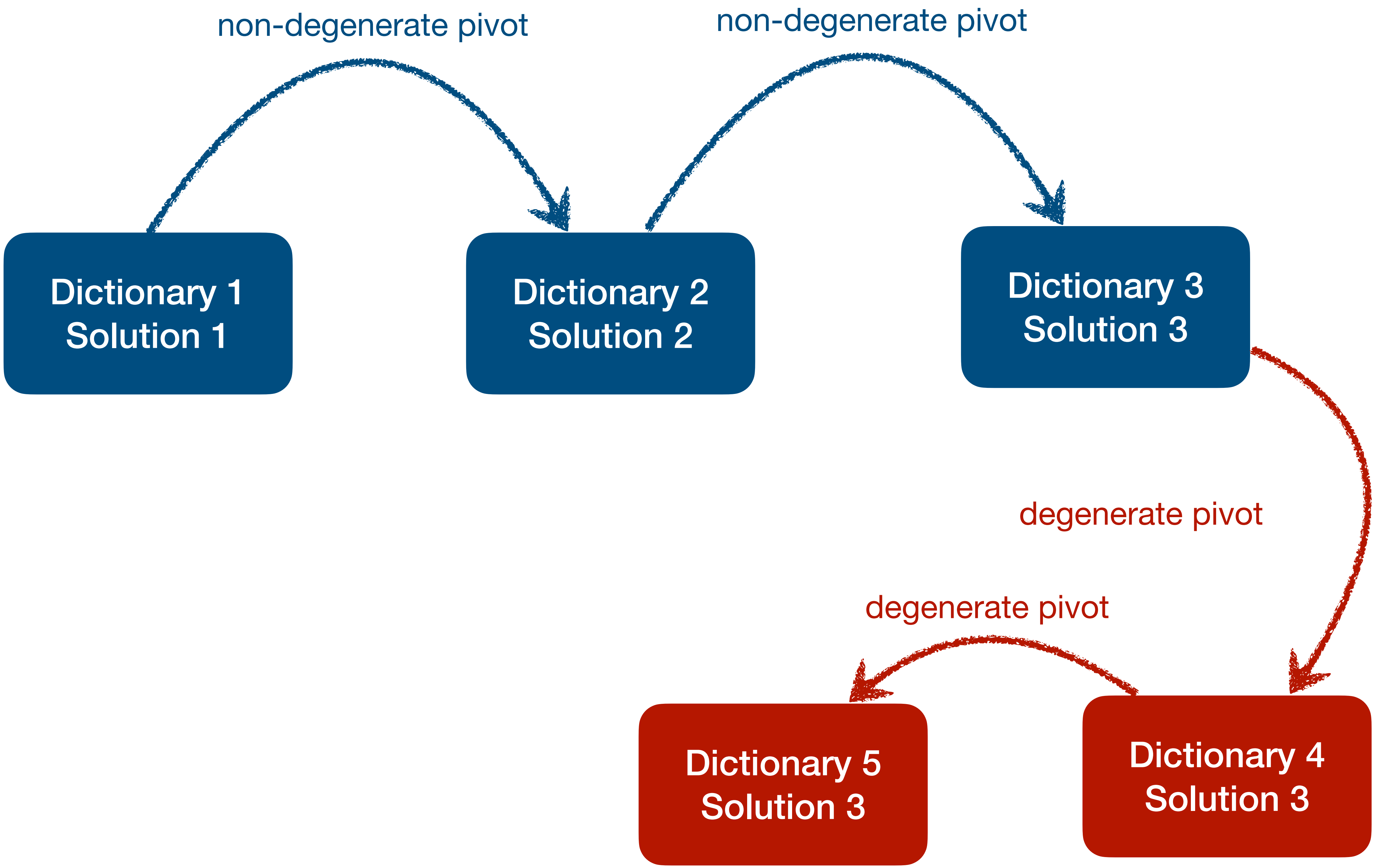

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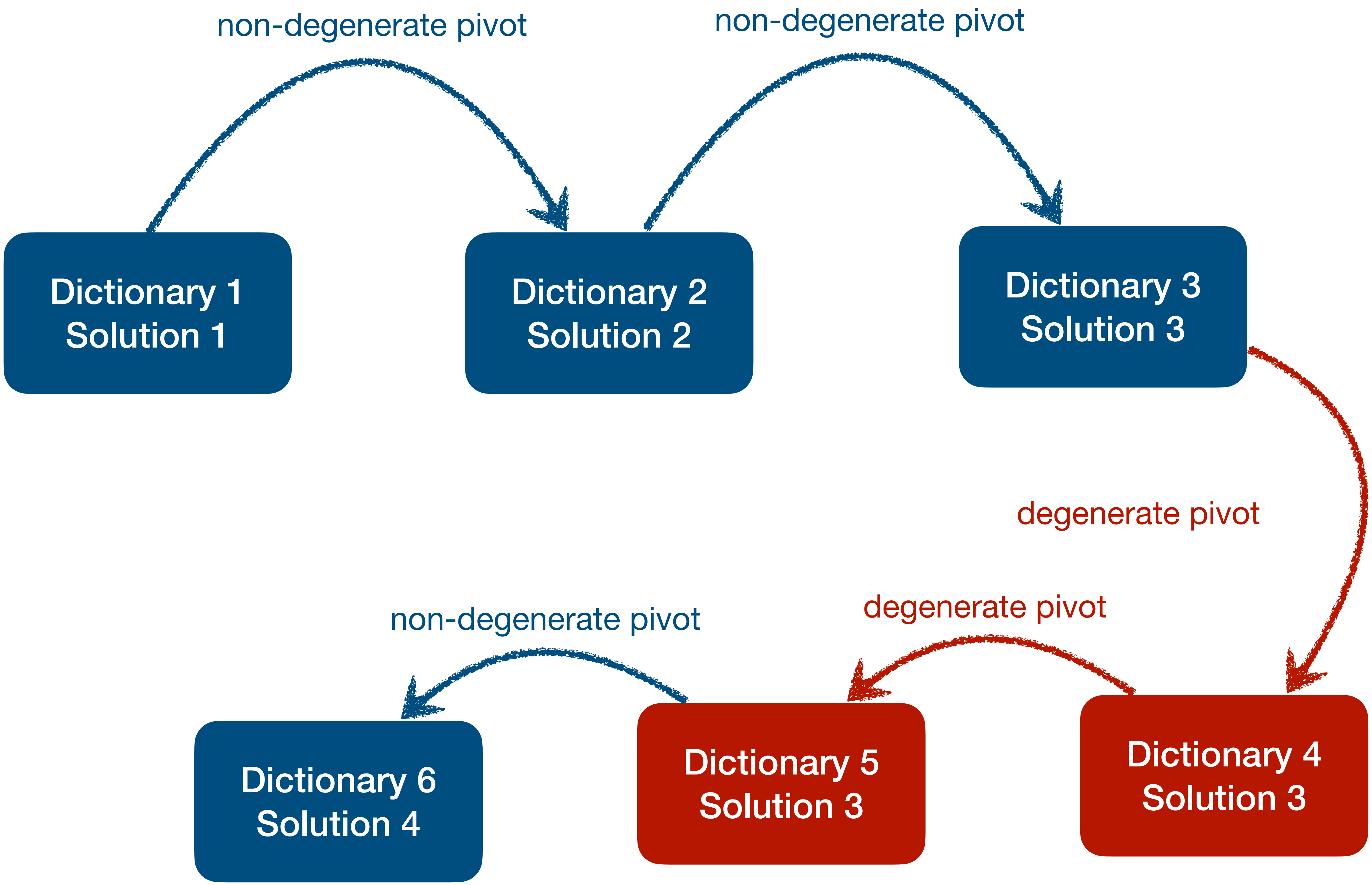

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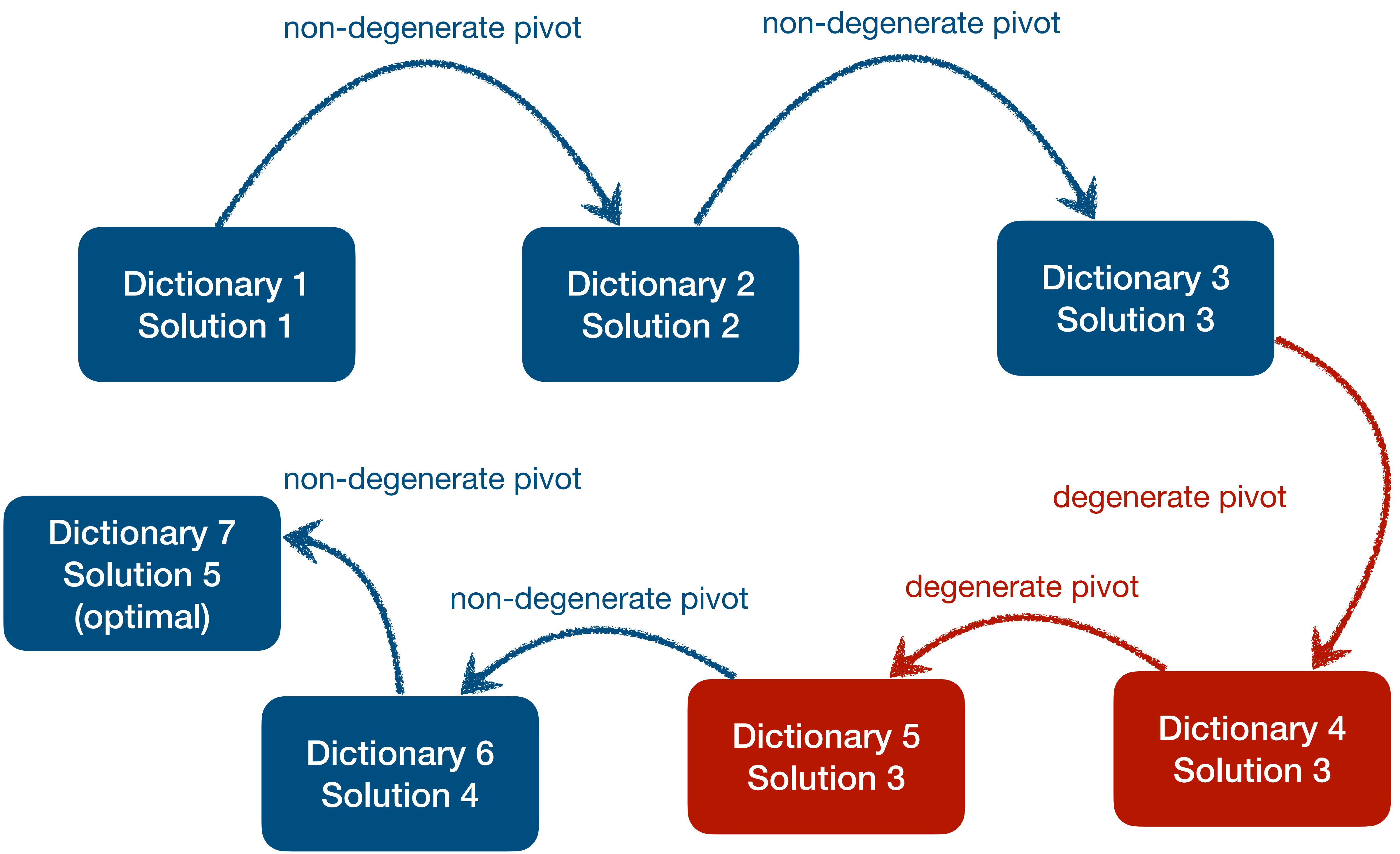

Could this happen though?

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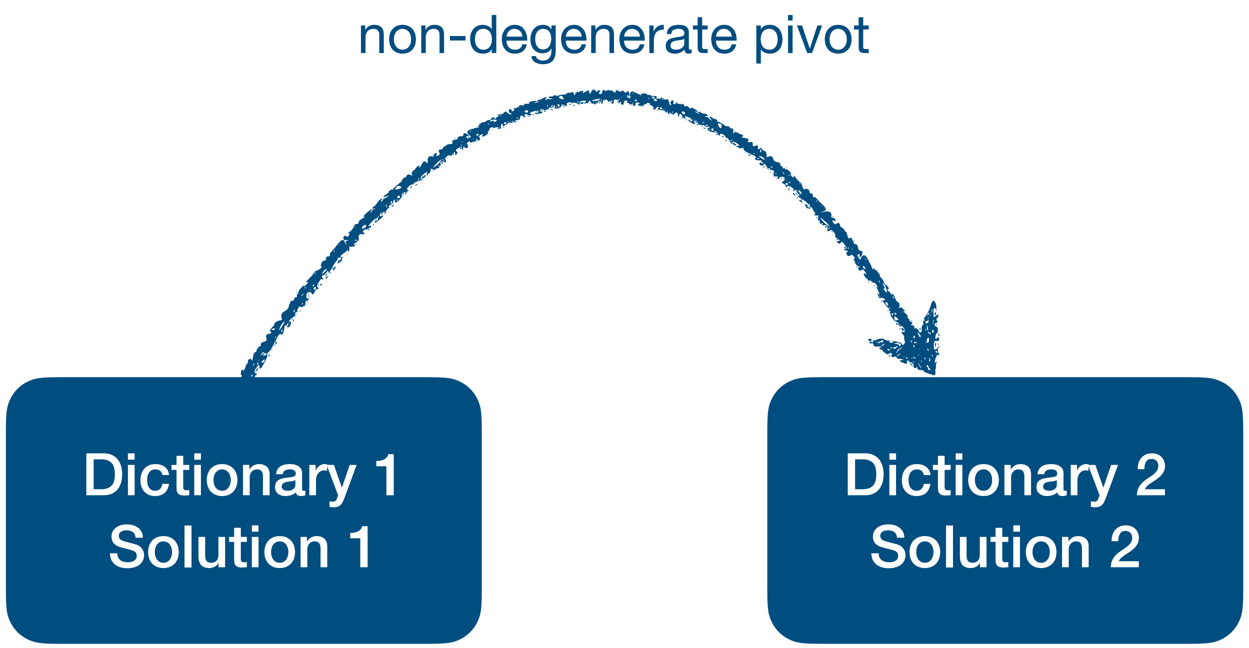

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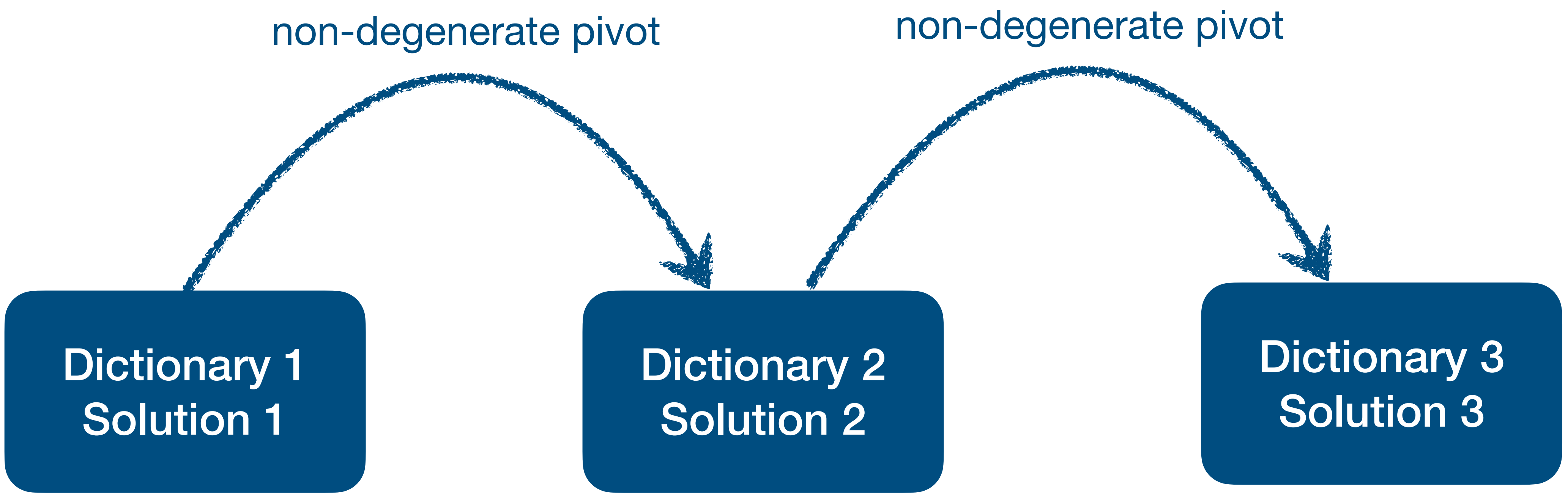

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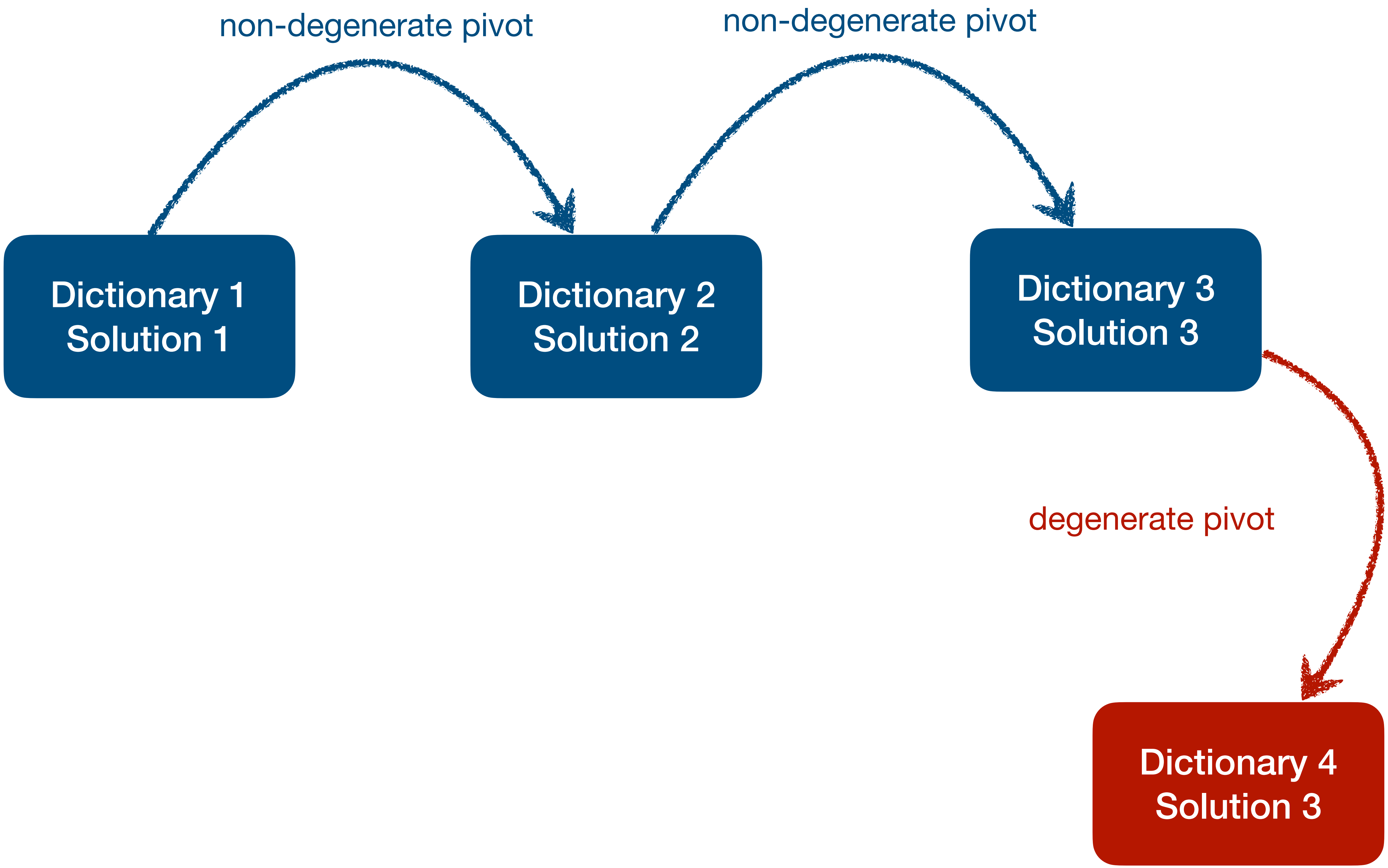

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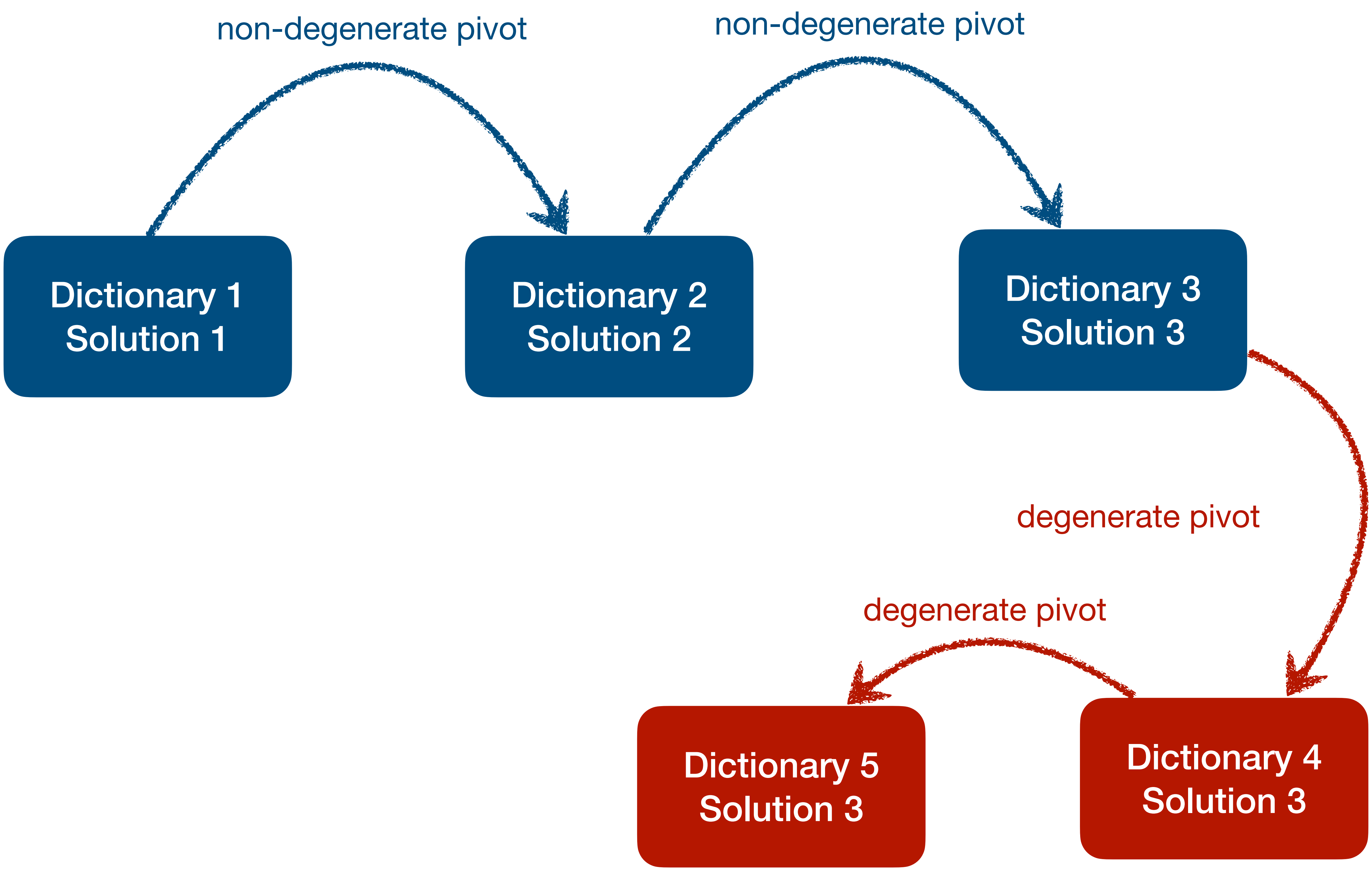

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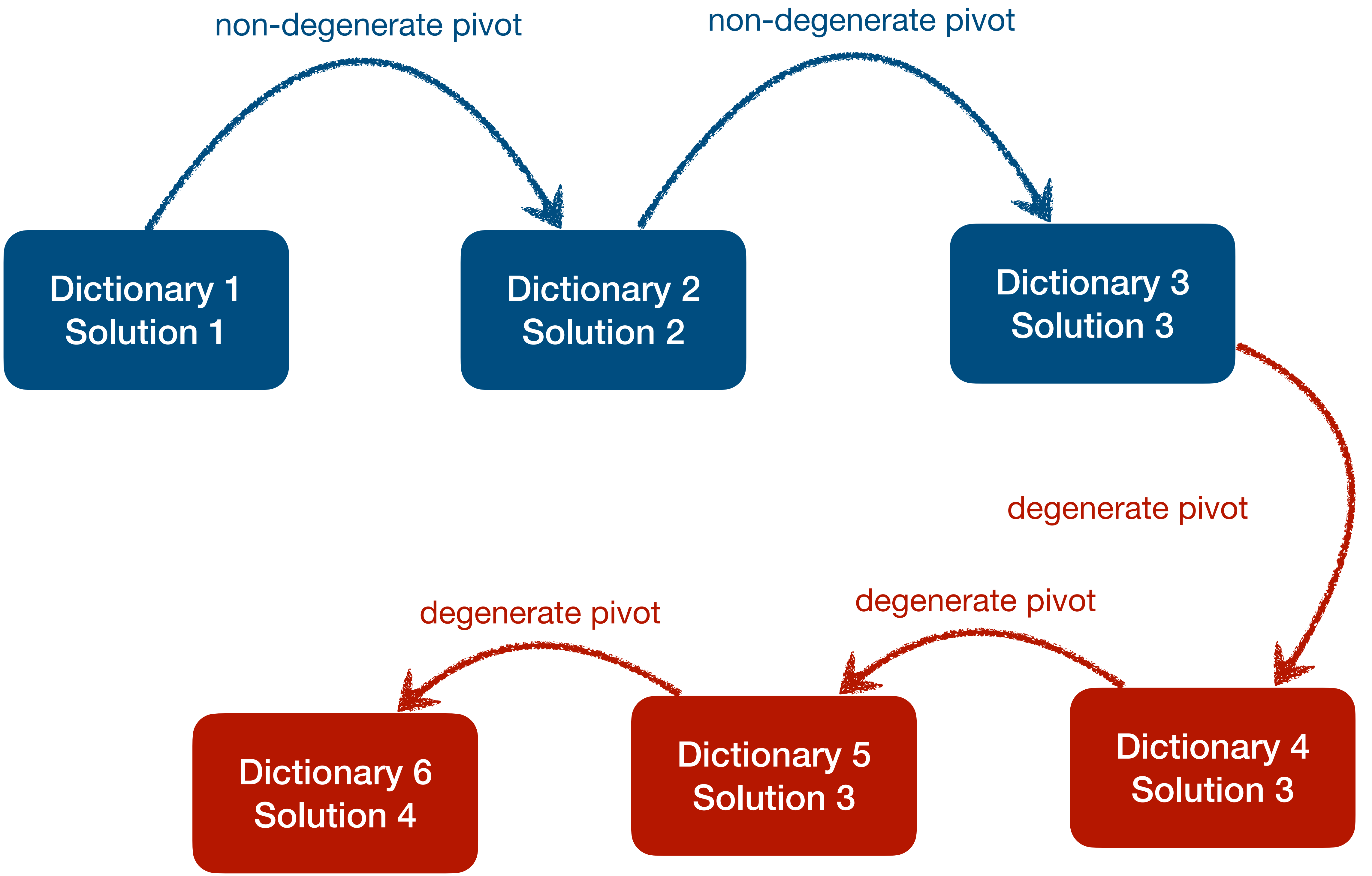

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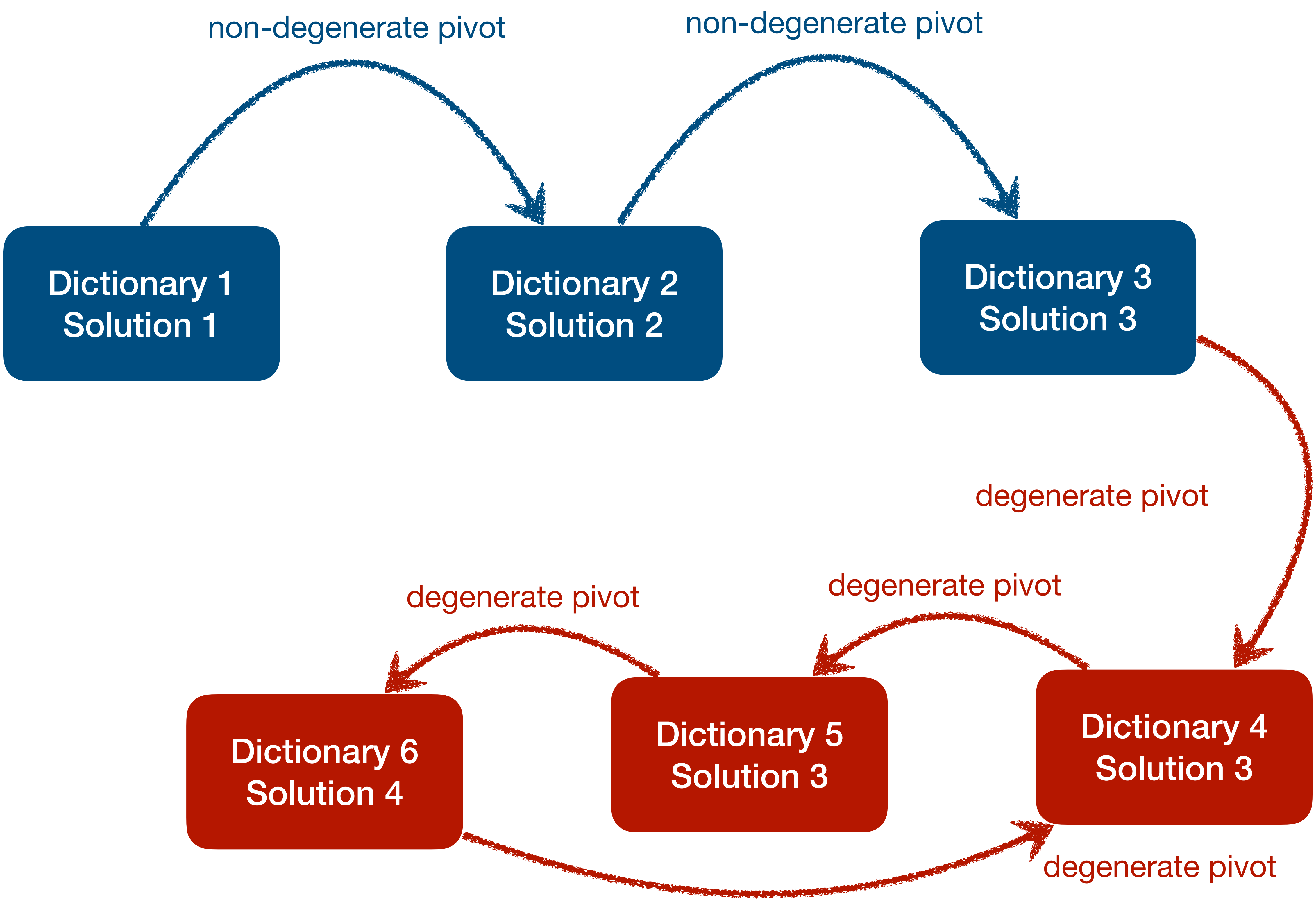

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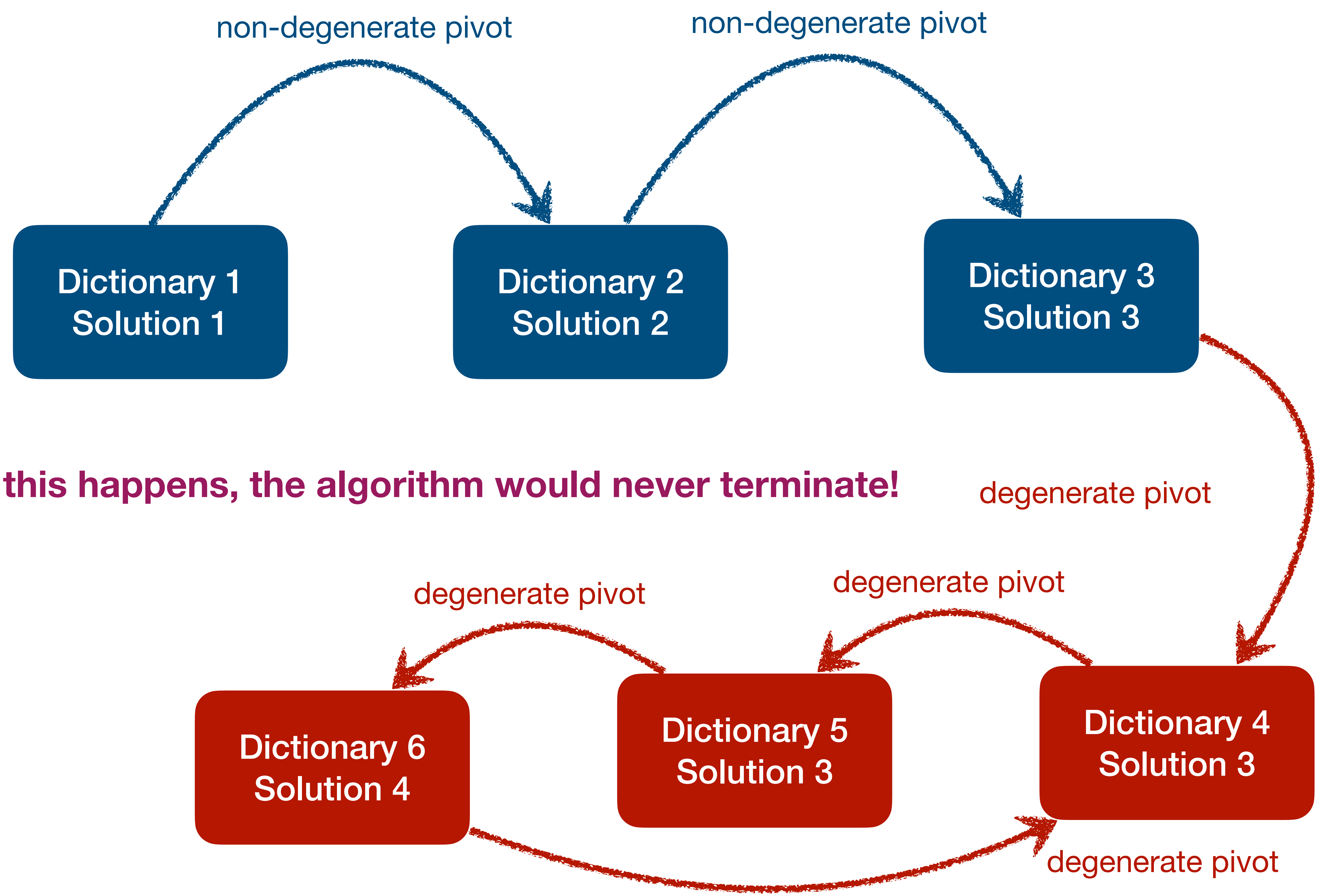

If this happens, the algorithm would never terminate!

Cycling

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In theory: Cycling can happen.

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In practice: Cycling rarely happens.

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But non-degenerate pivots are quite common.

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Can we avoid cycling in theory too?

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Can we avoid cycling in theory too?

Bland's rule: For both the entering variable and the leaving variable, choose the one with the smallest index.

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Theorem: If the simplex method does not cycle, it terminates.

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Proof: A dictionary is determined by which variables are basic and which are non-basic.

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There only $\binom{n+m}{m} = \frac{(n+m)!}{n!m!}$ possibilities.

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Geometry

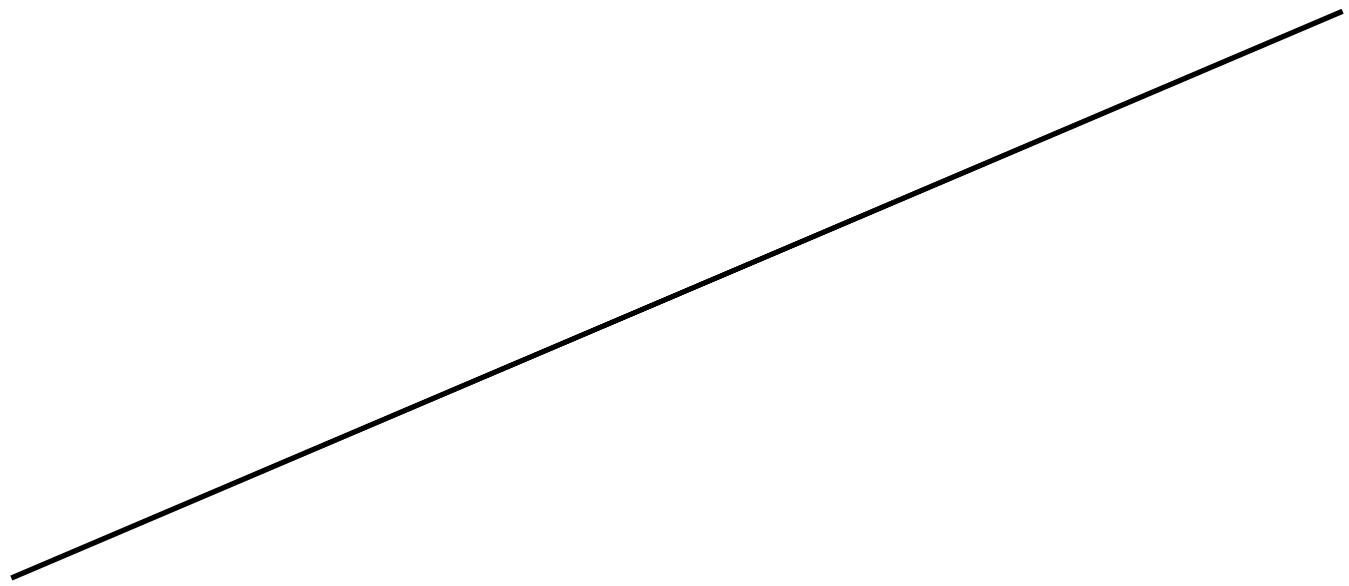
Every constraint corresponds to a hyperplane, which defines a halfspace.

The intersection of those halfspaces is the feasible region, which is a polyhedron (or polytope).

The part of each hyperplane that intersects with the feasible region is called a facet.

Geometry

Geometry



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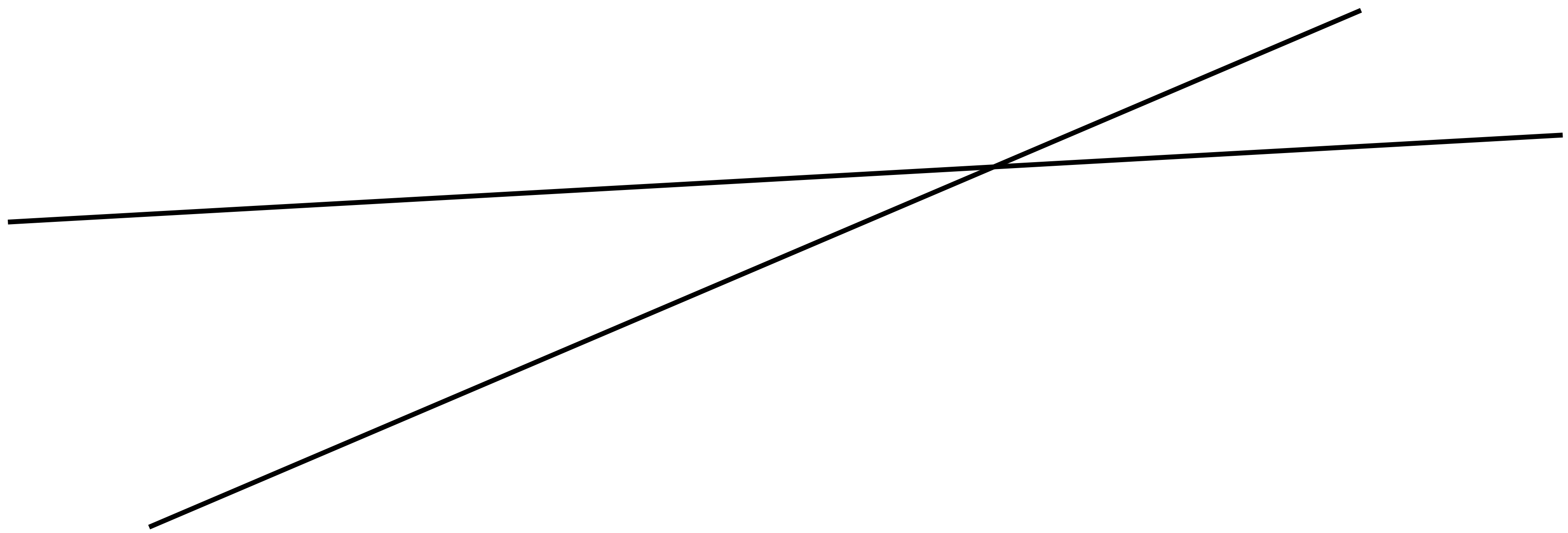

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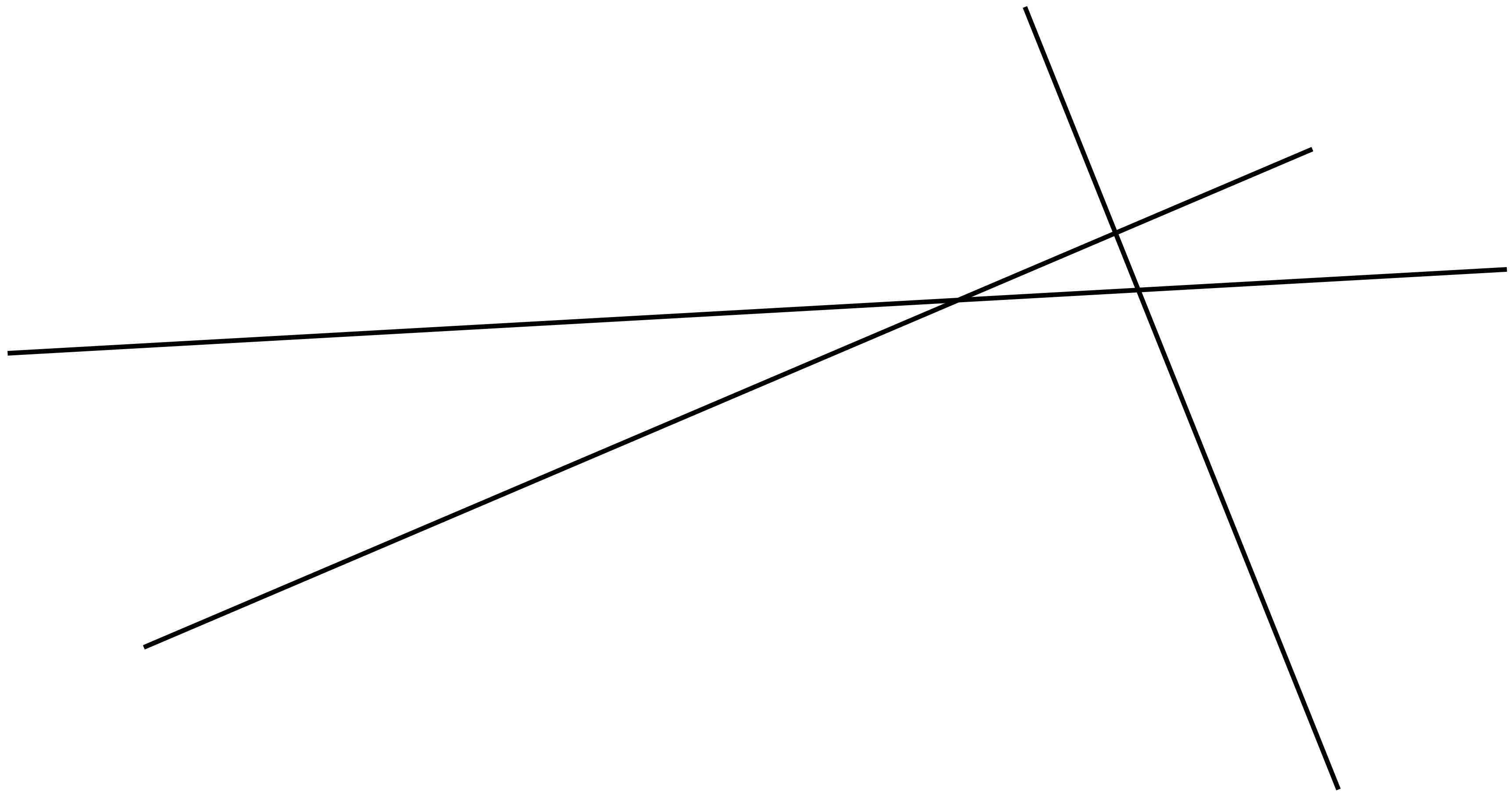

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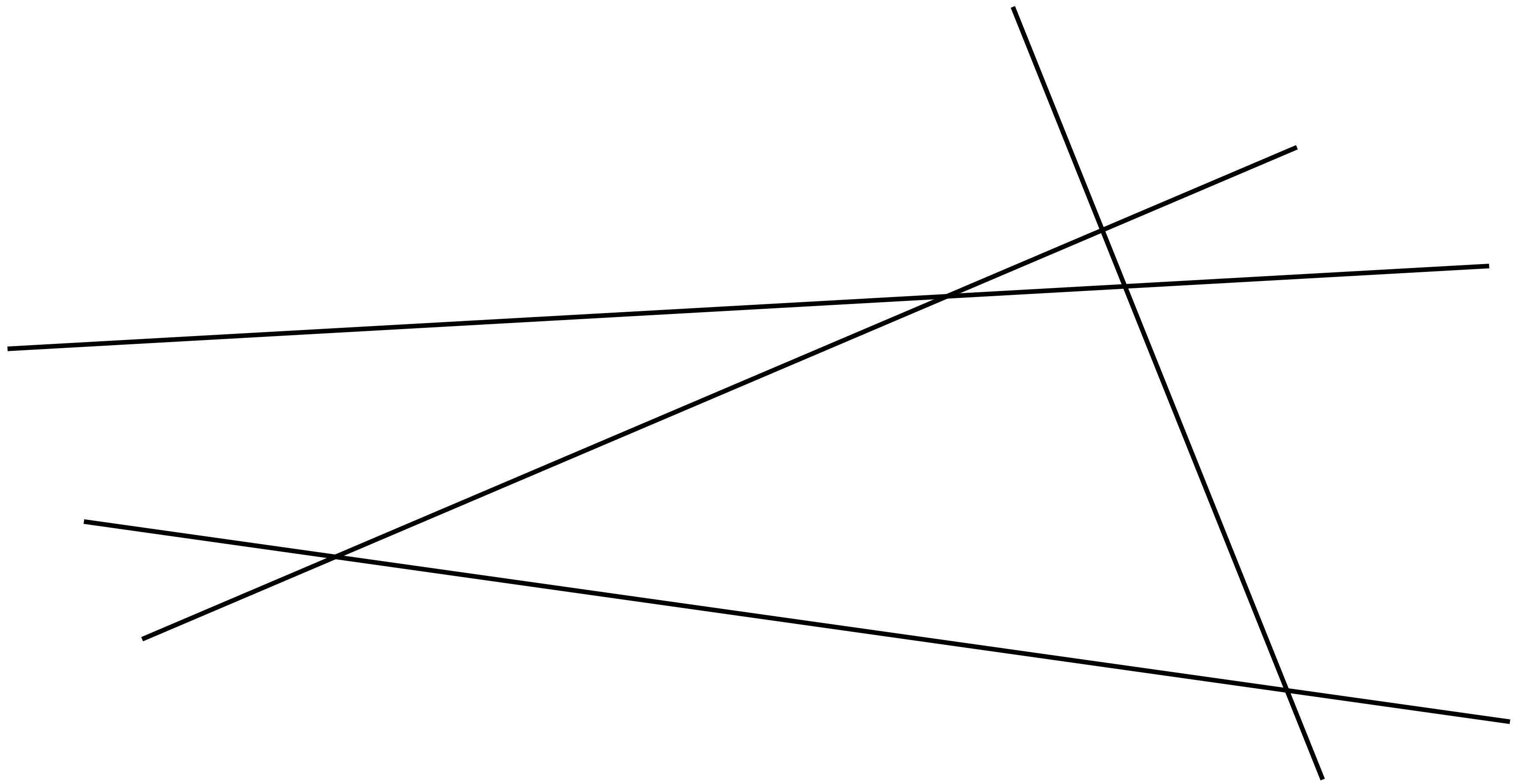

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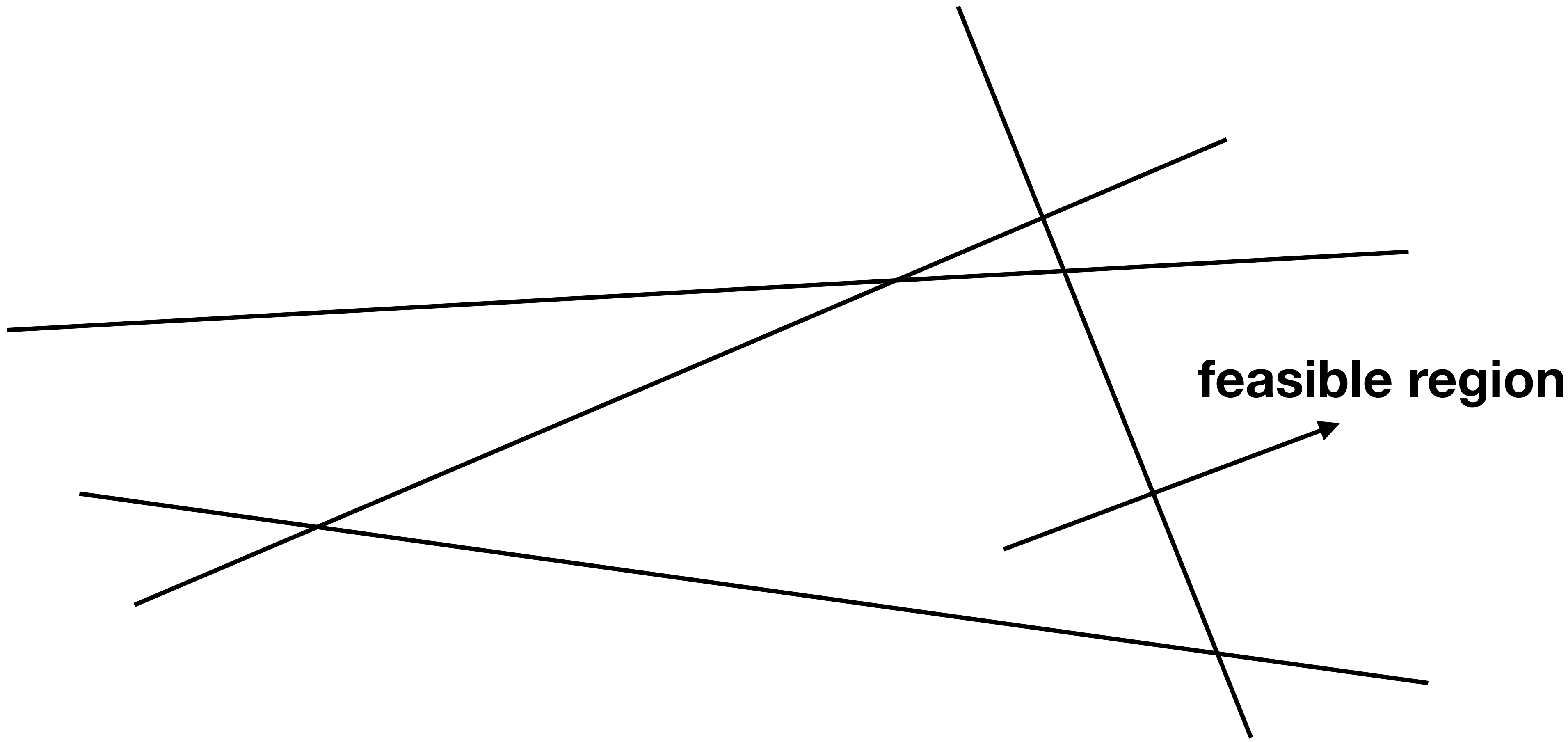

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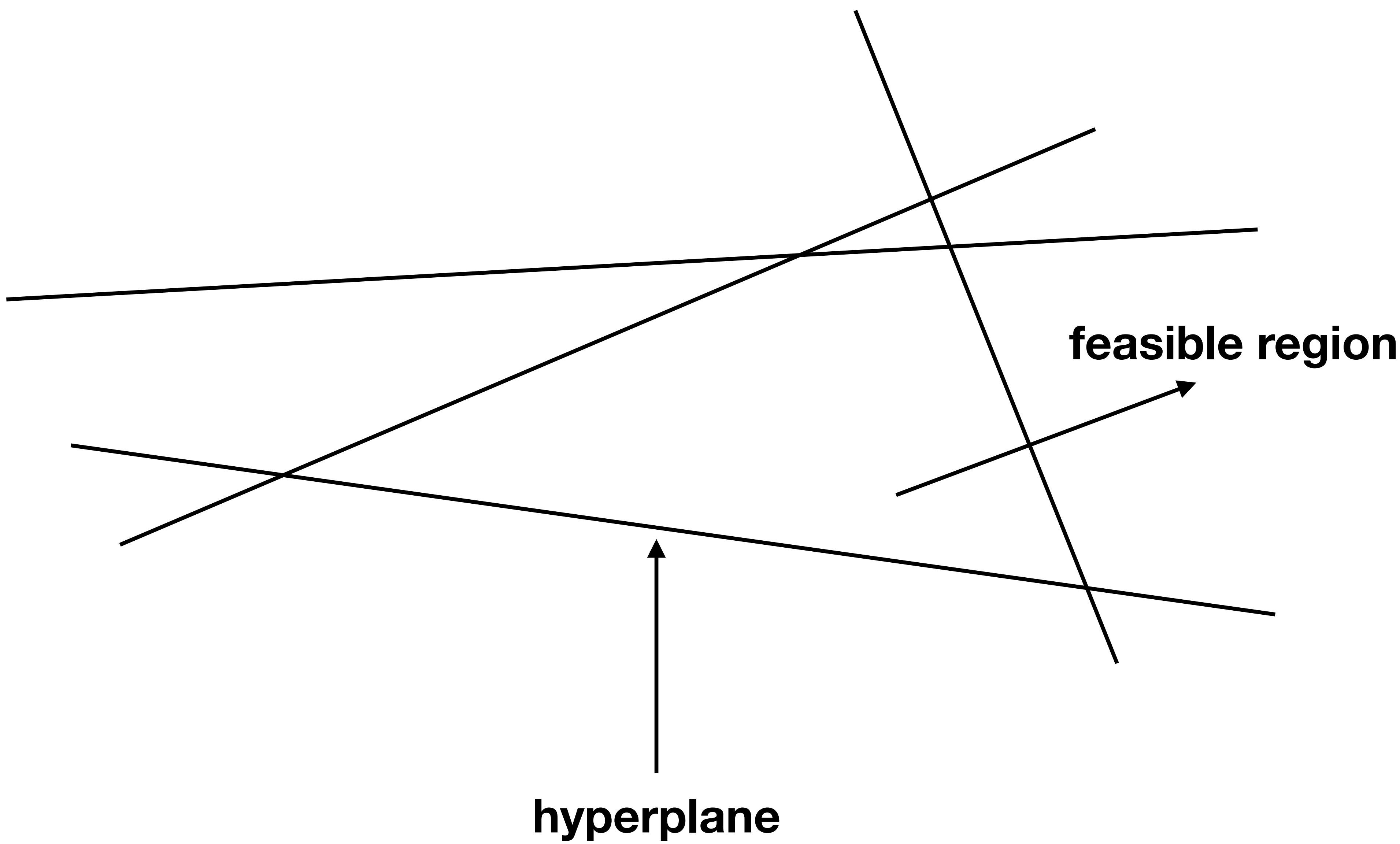

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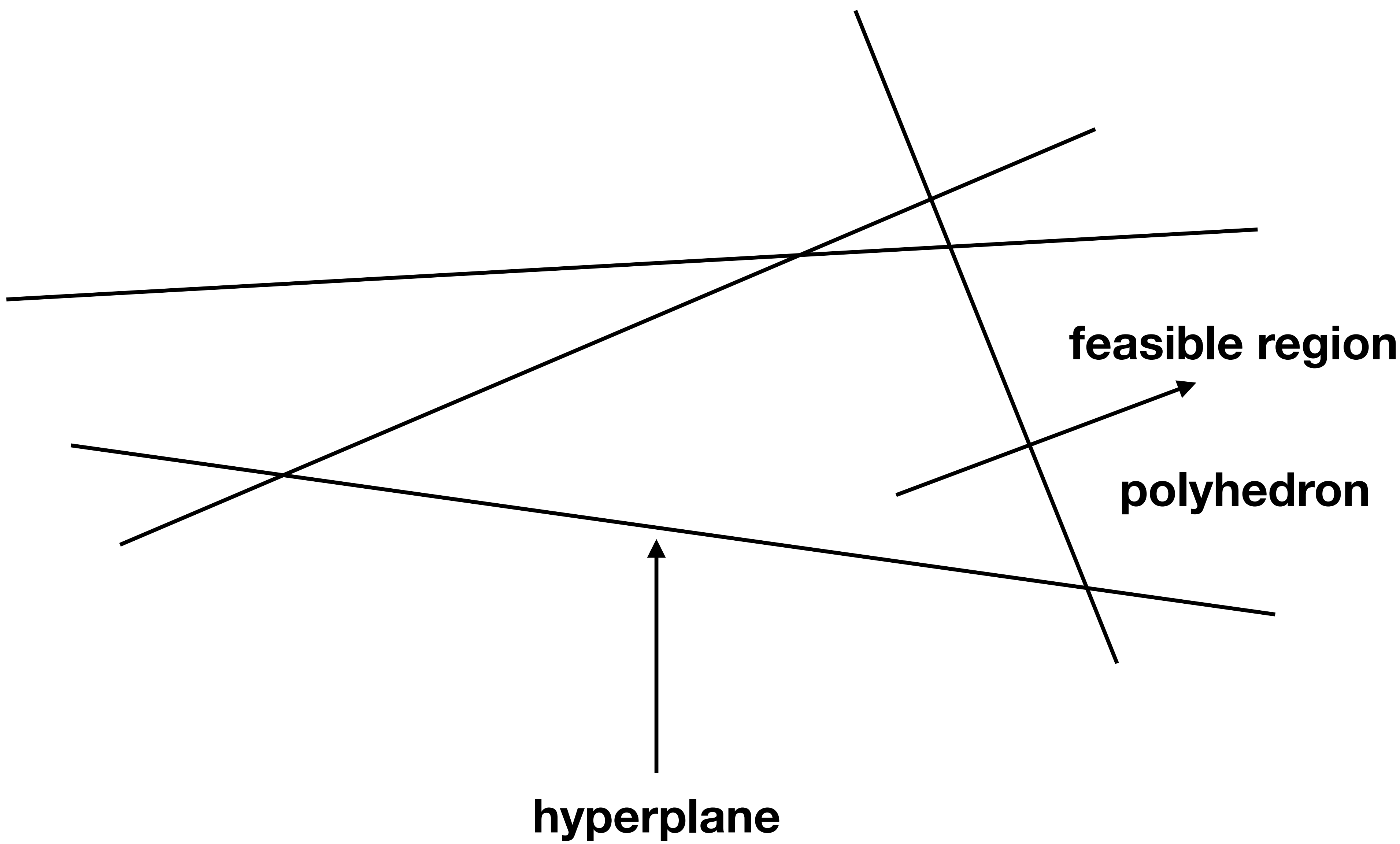

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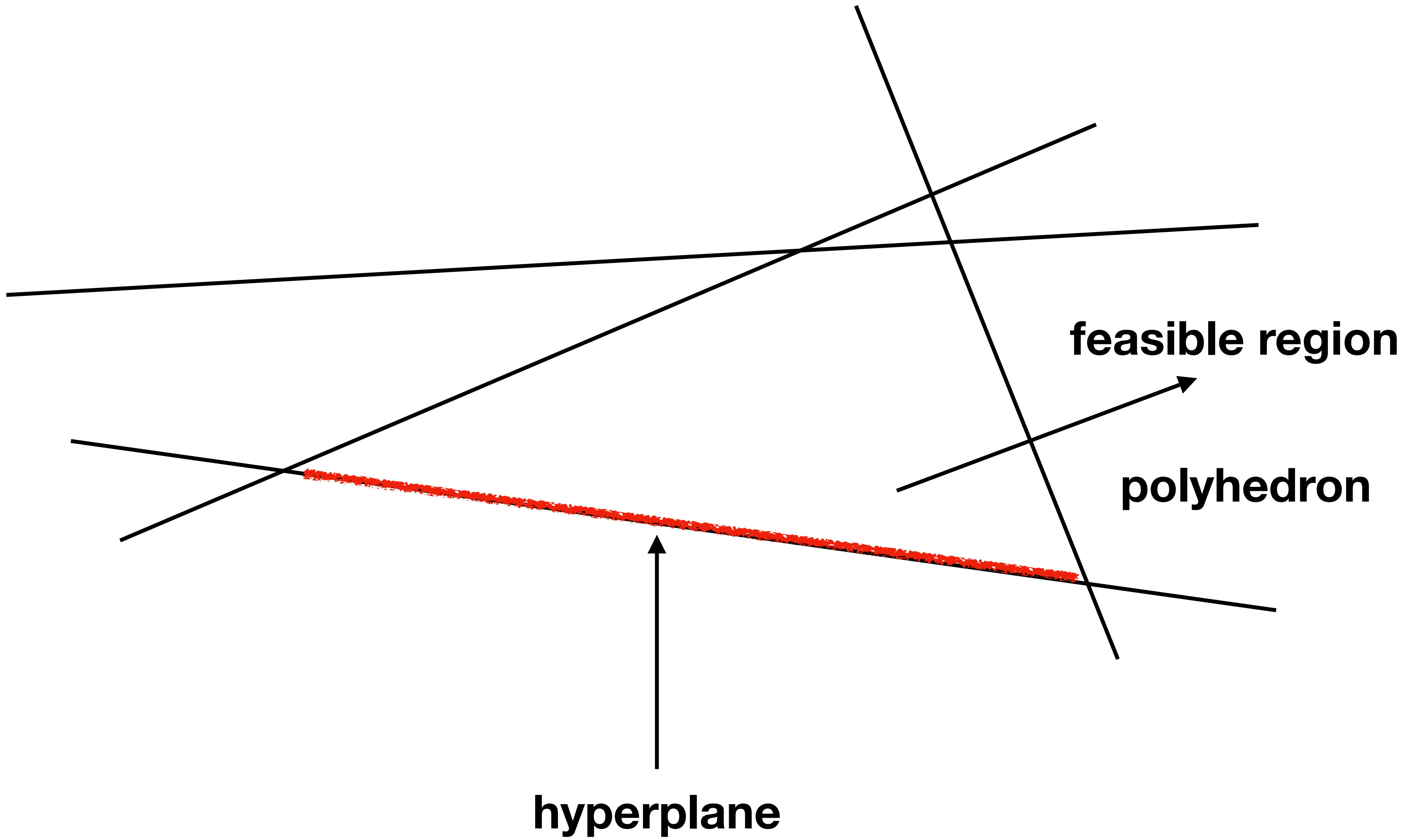

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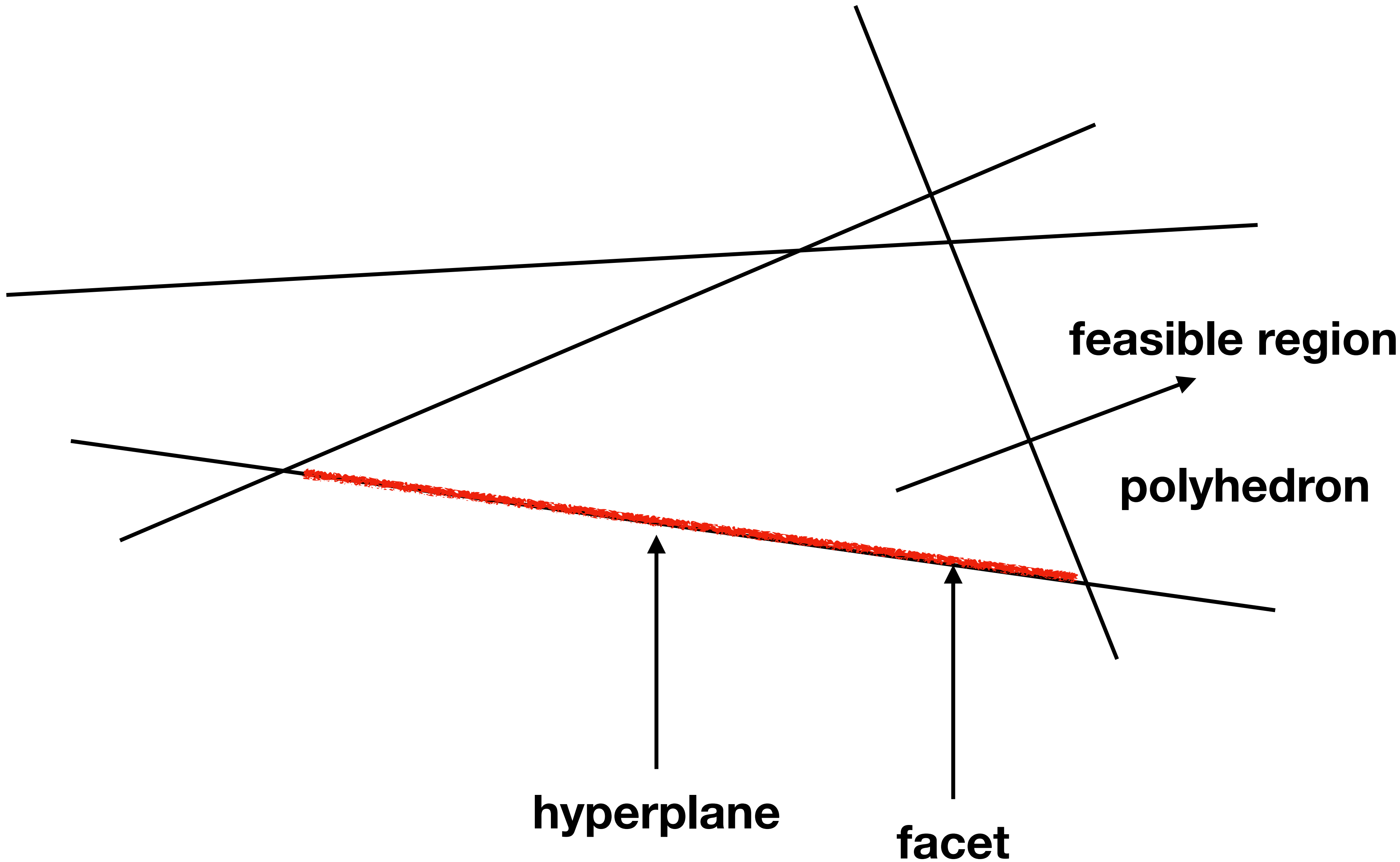

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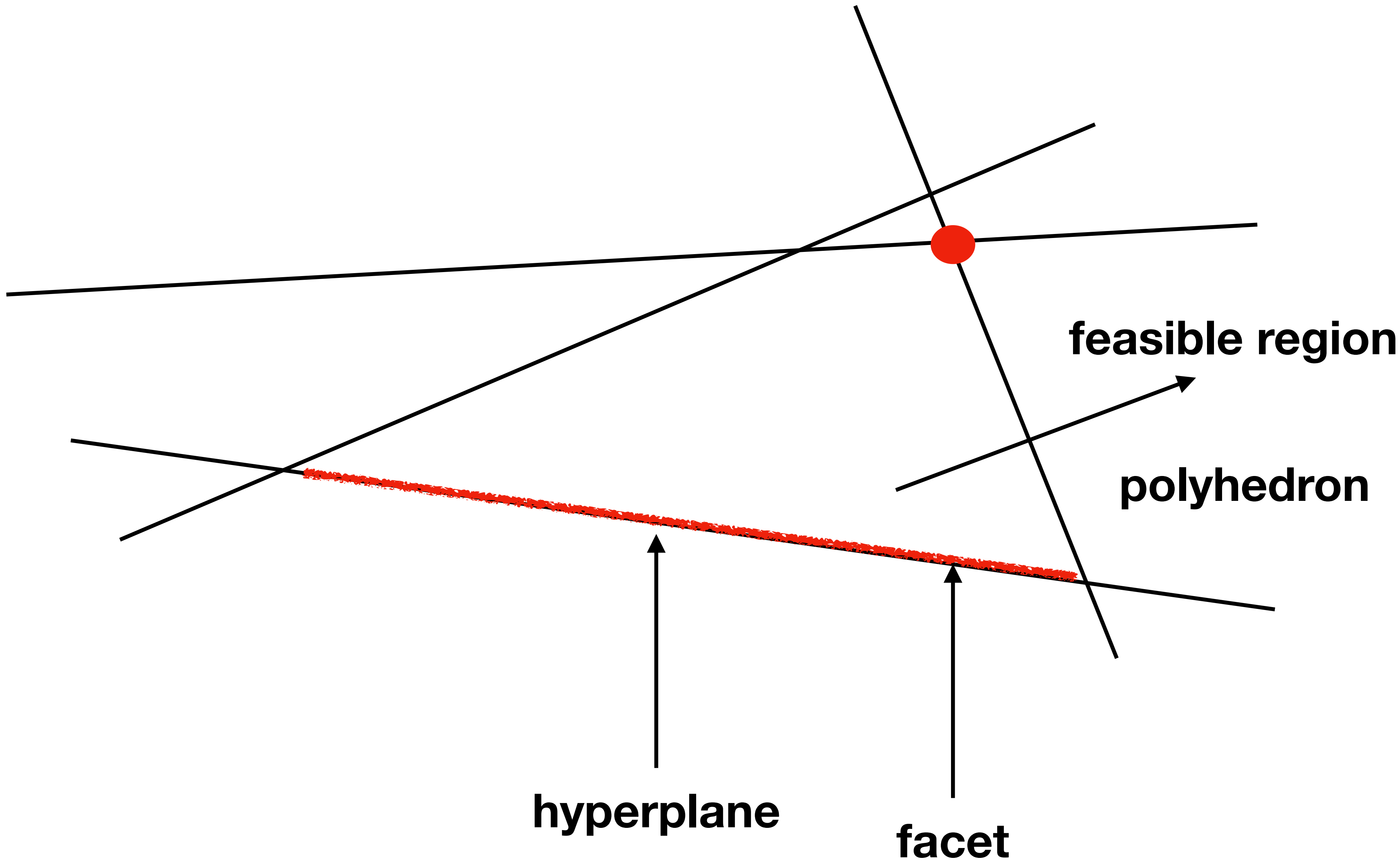

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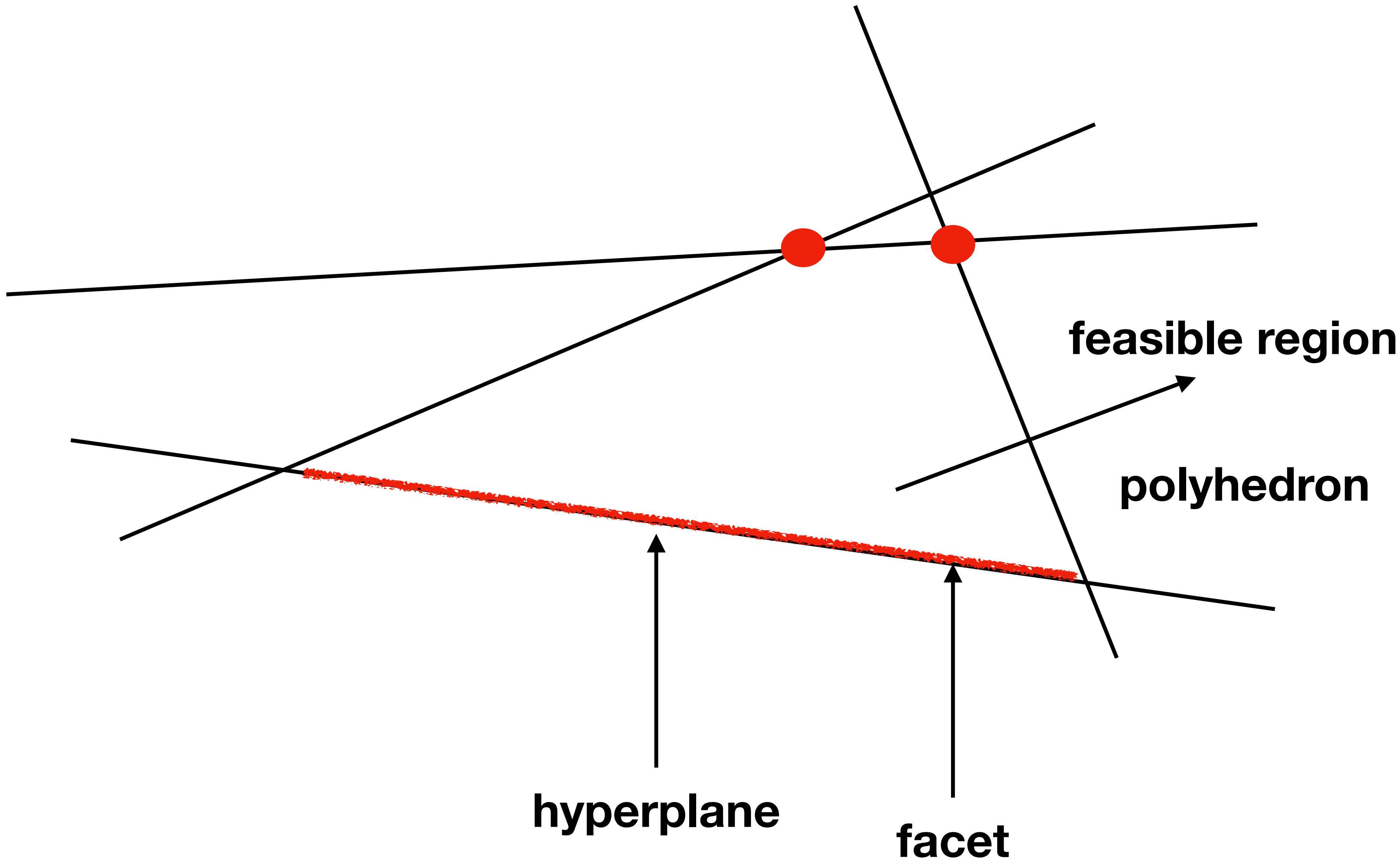

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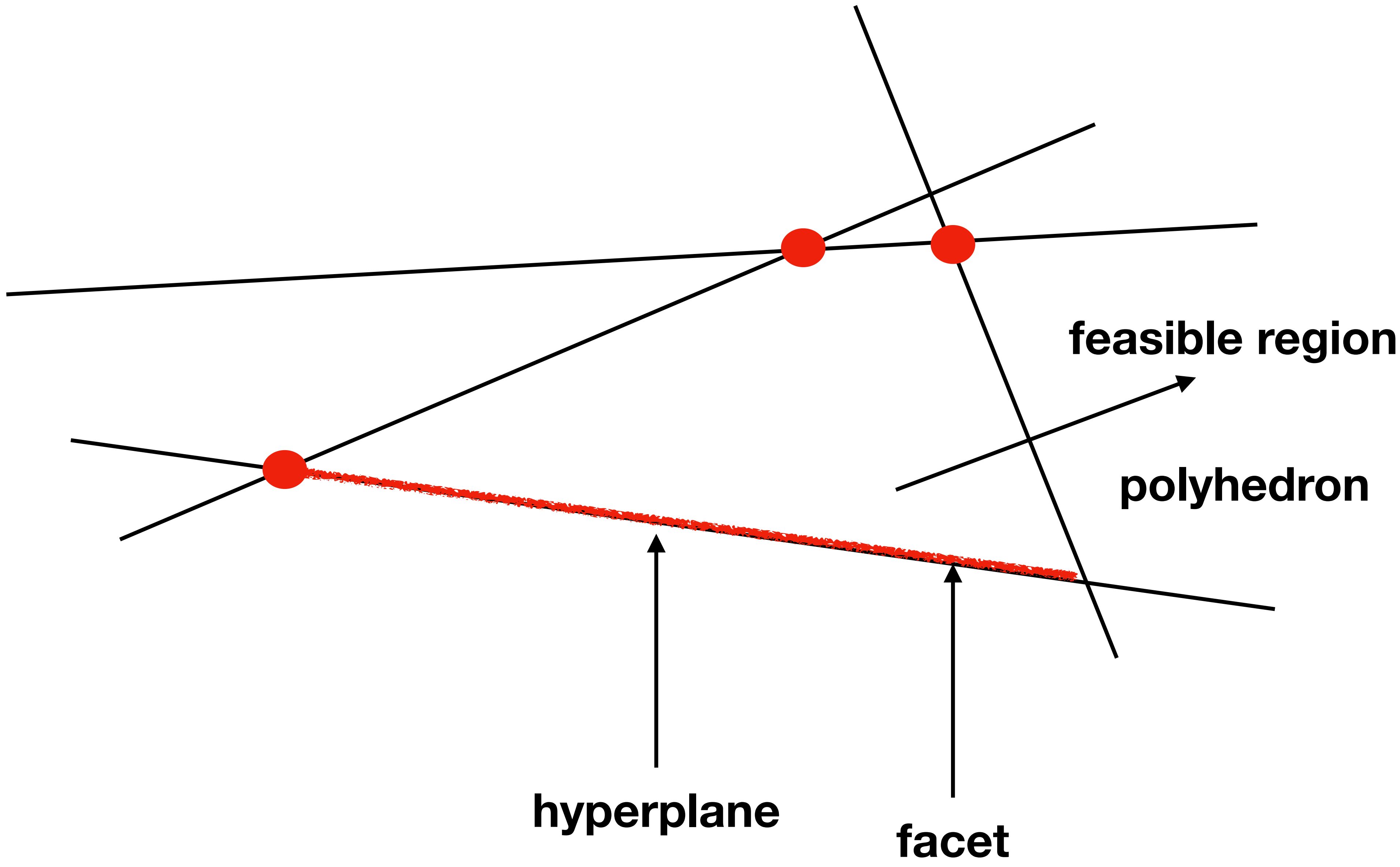

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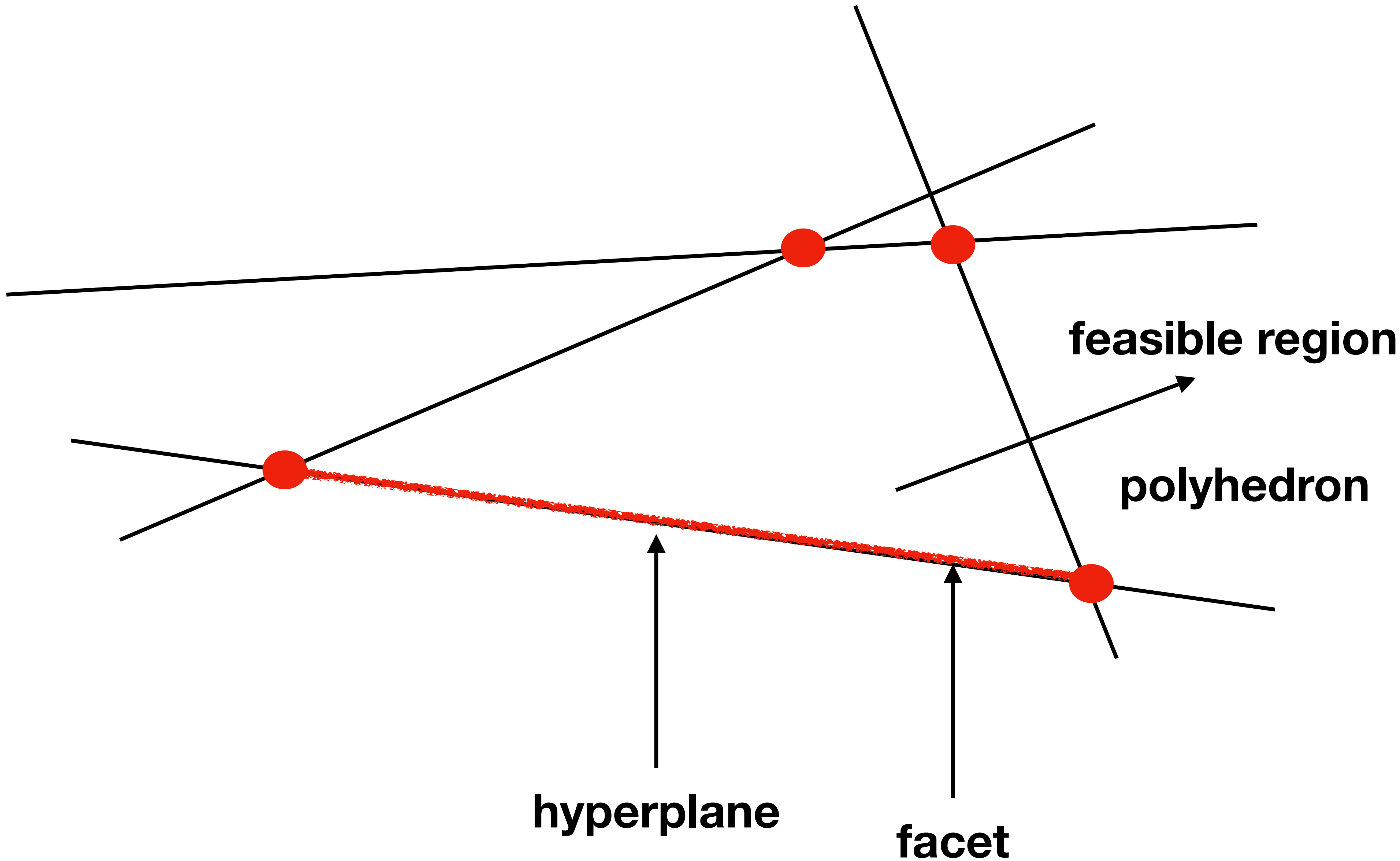

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A facet corresponds to a constraint satisfied with equality, e.g., $x_1 + 2x_3 = 3$

In terms of the dictionary, a facet corresponds to the corresponding variable (original or slack) being 0.

Geometry

Consider an LP with three variables x_1, x_2, x_3.

In terms of the dictionary, a facet corresponds to the corresponding variable (original or slack) being 0.

An edge corresponds to two variables being 0.

A vertex corresponds to three variables being 0.

Geometry

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$$\left.\begin{array}{r} 2x_1 + 3x_2 + x_3 \leq 5 \\ 4x_1 + x_2 + 2x + 3 \leq 11 \\ 3x_1 + 4x_2 + 2x_3 \leq 8 \end{array}\right\}$$

$$x_1, x_2, x_3 \geq 0$$

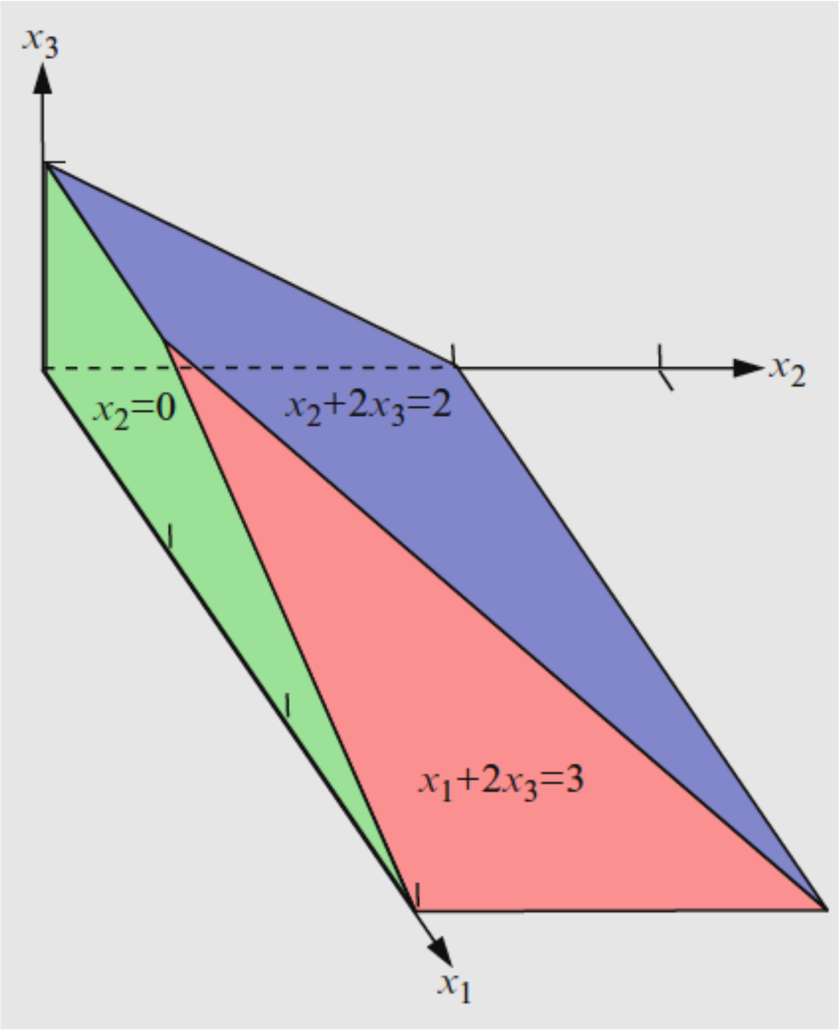
$w_1 = w_2 = w_3 = 0$
corresponds to the intersection of
these three hyperplanes.

Example

Maximise $x_1 + 2x_2 + 3x_3$

subject to

$$x_1 + 2x_3 \leq 3$$
$$x_2 + 2x_3 \leq 2$$
$$x_1, x_2, x_3 \geq 0$$



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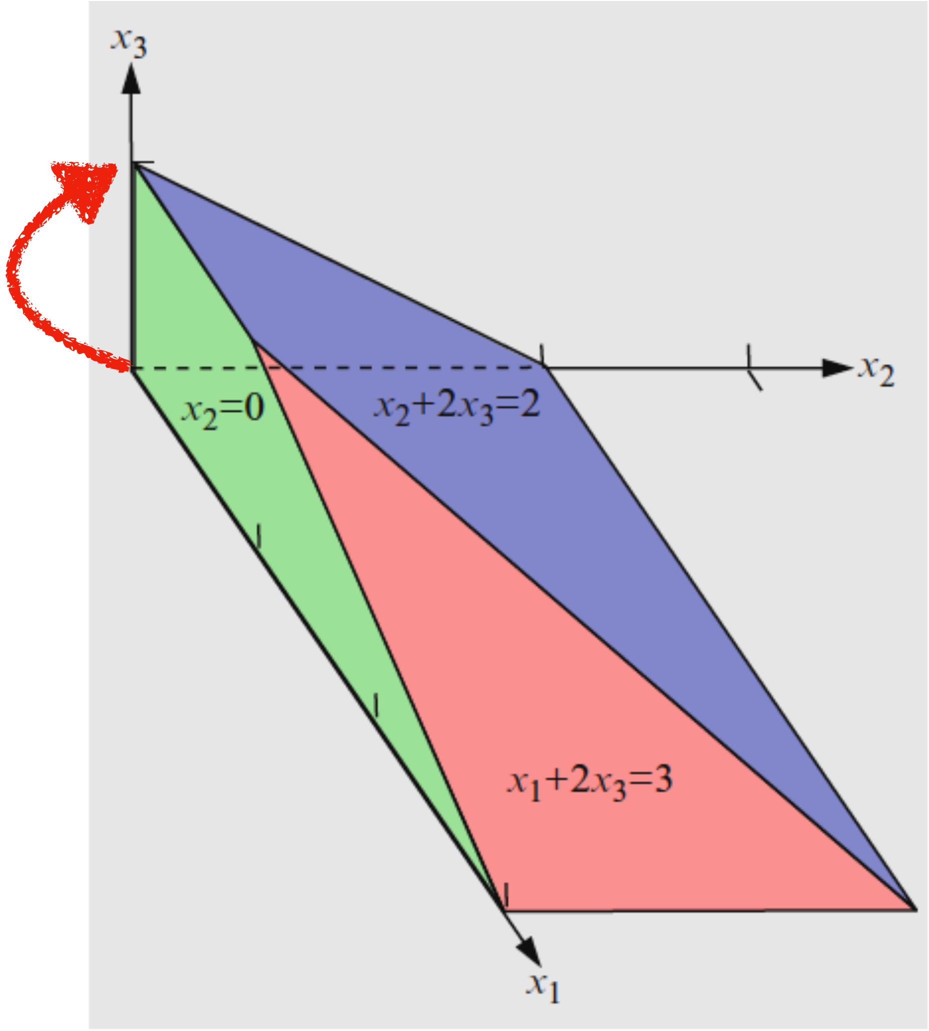
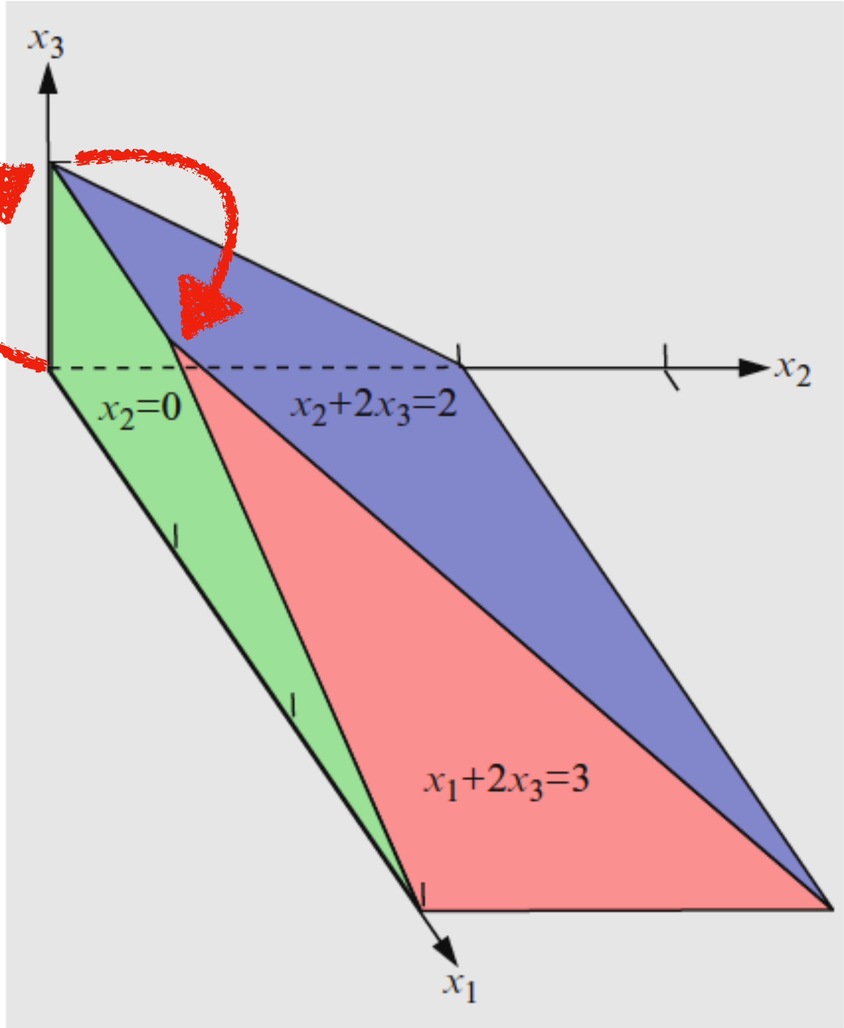


Example

Maximise $x_1 + 2x_2 + 3x_3$

subject to

$$x_1 + 2x_3 \leq 3$$
$$x_2 + 2x_3 \leq 2$$
$$x_1, x_2, x_3 \geq 0$$



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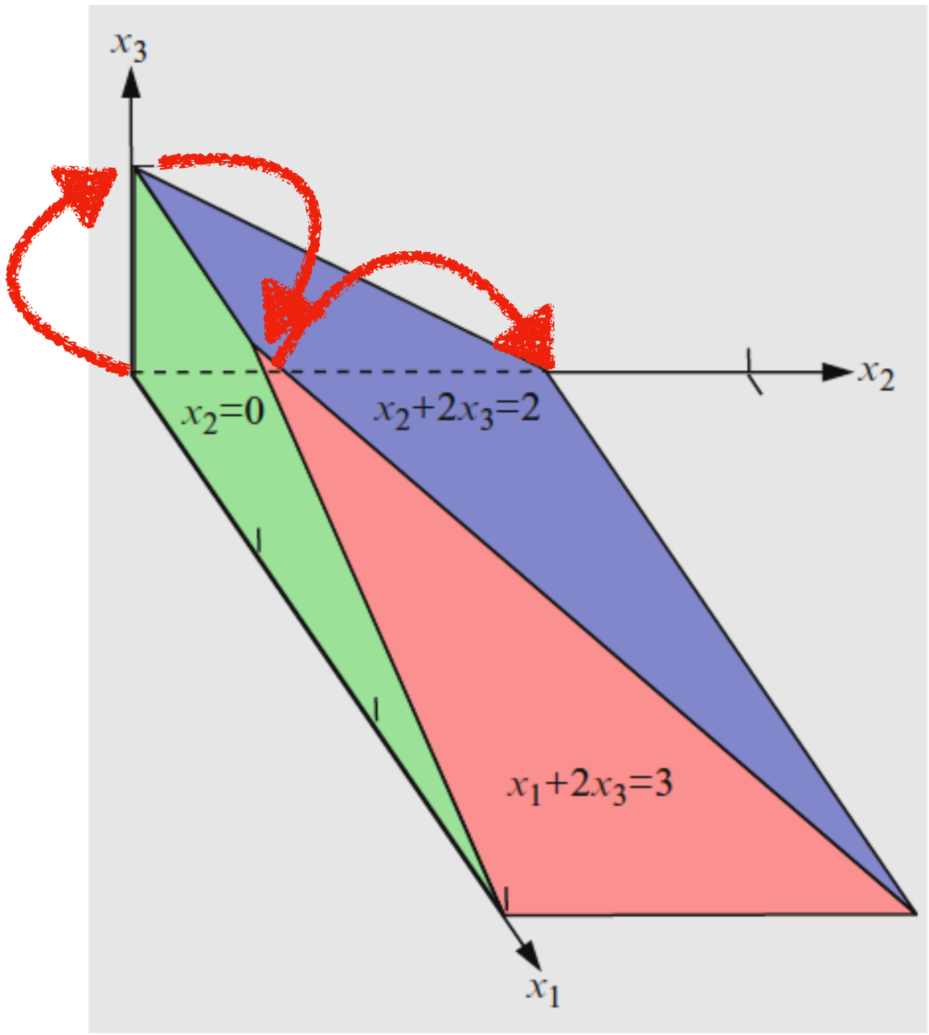

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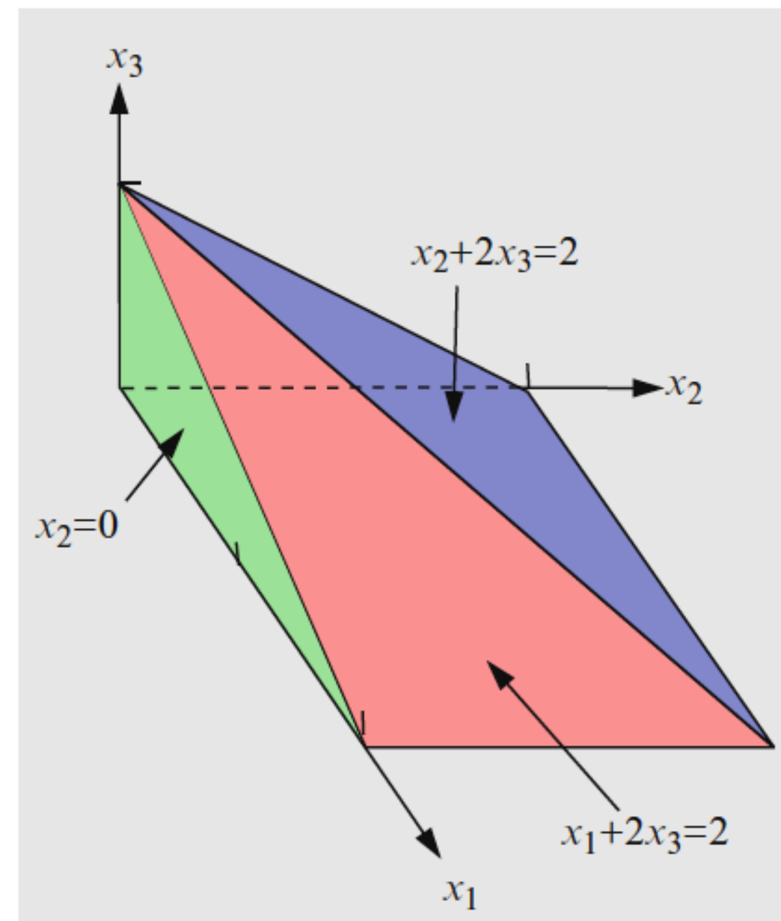

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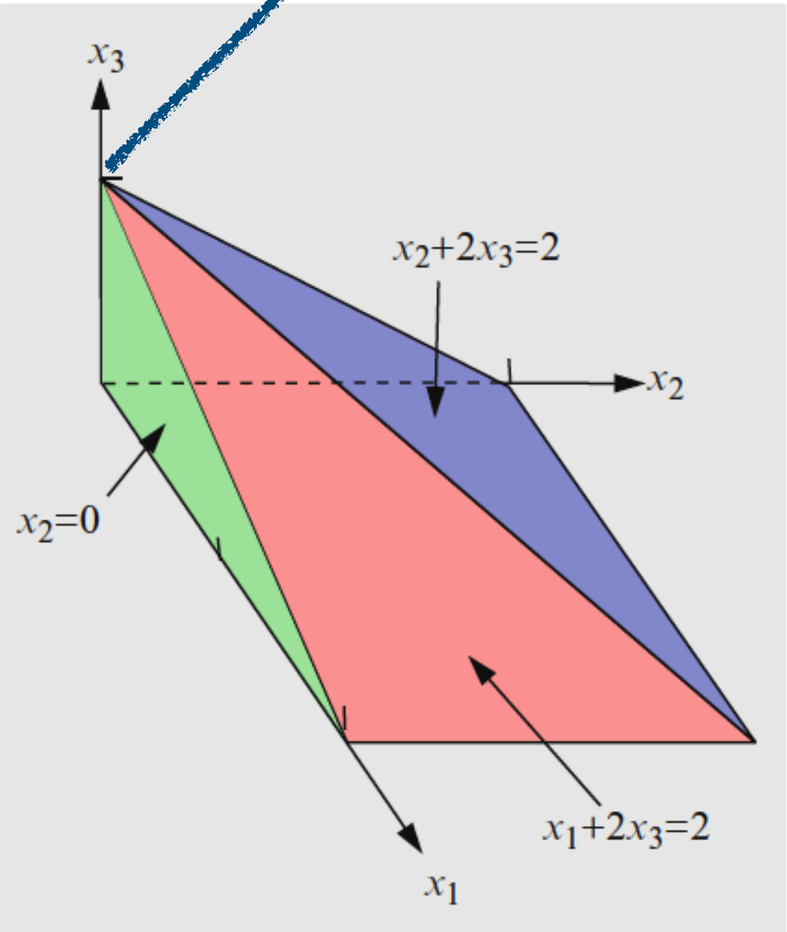
intersection of four facets



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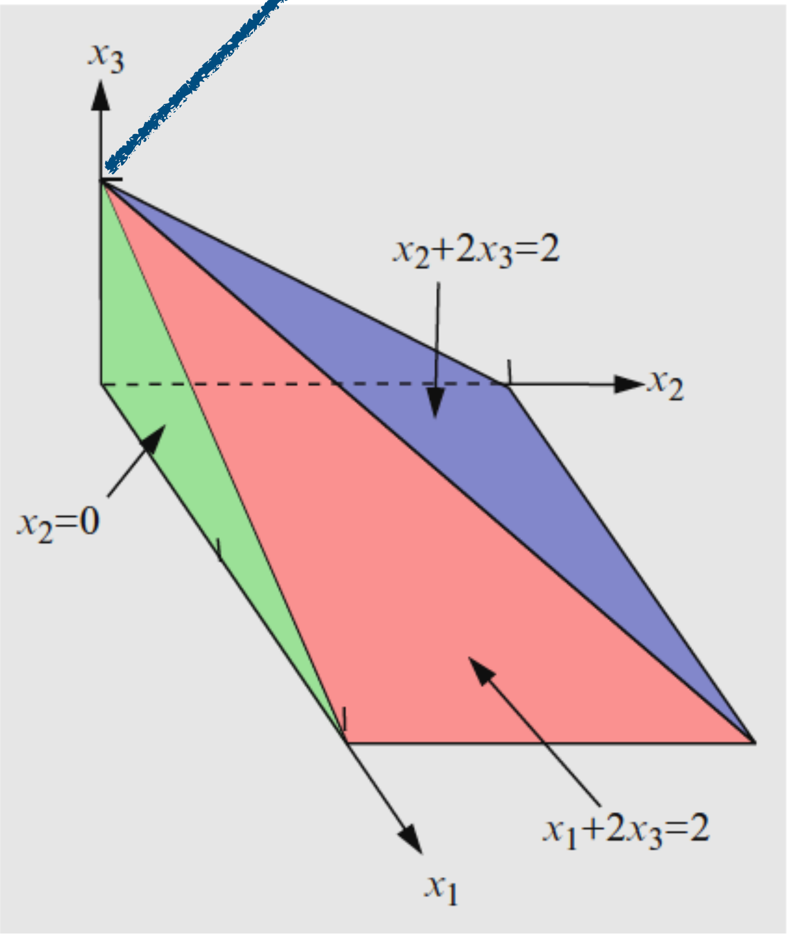

The intersection point corresponds to the same solution of the LP.

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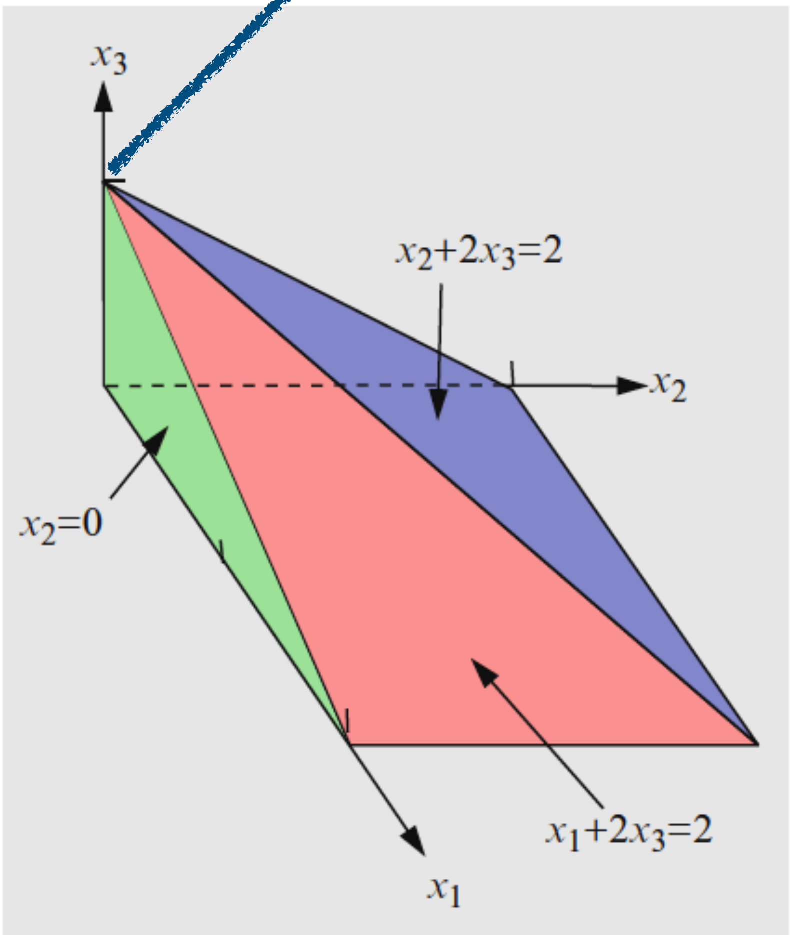

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But it corresponds to four different basic feasible solutions/dictionaries.

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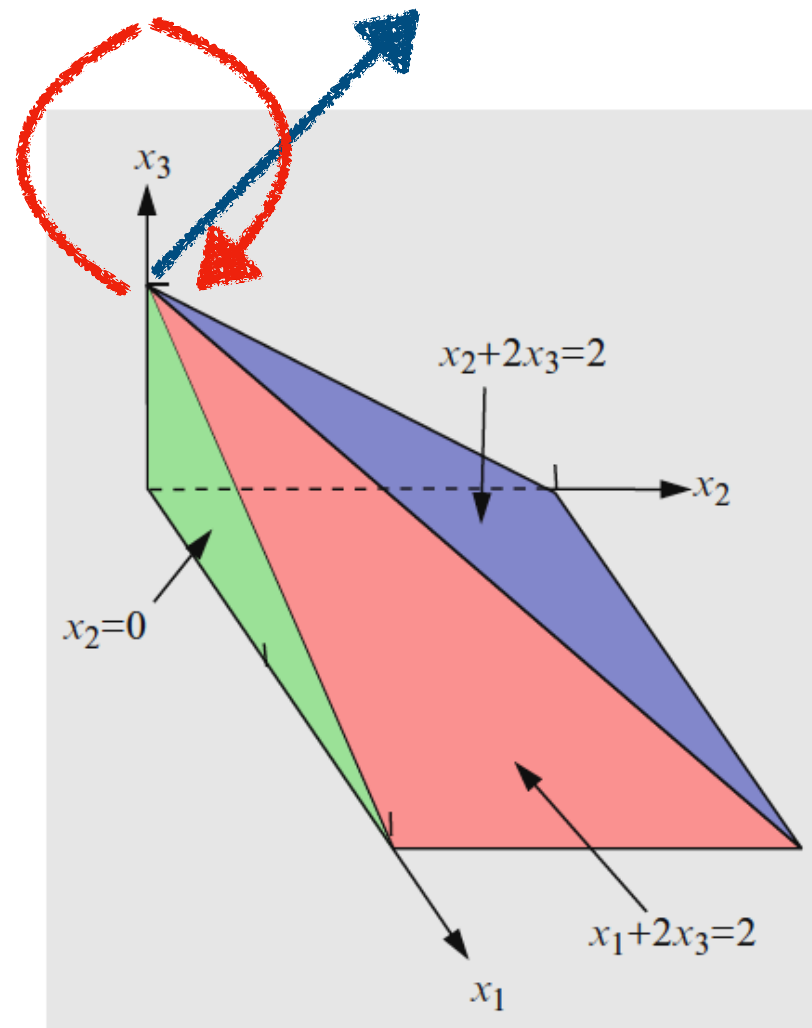
intersection of four facets



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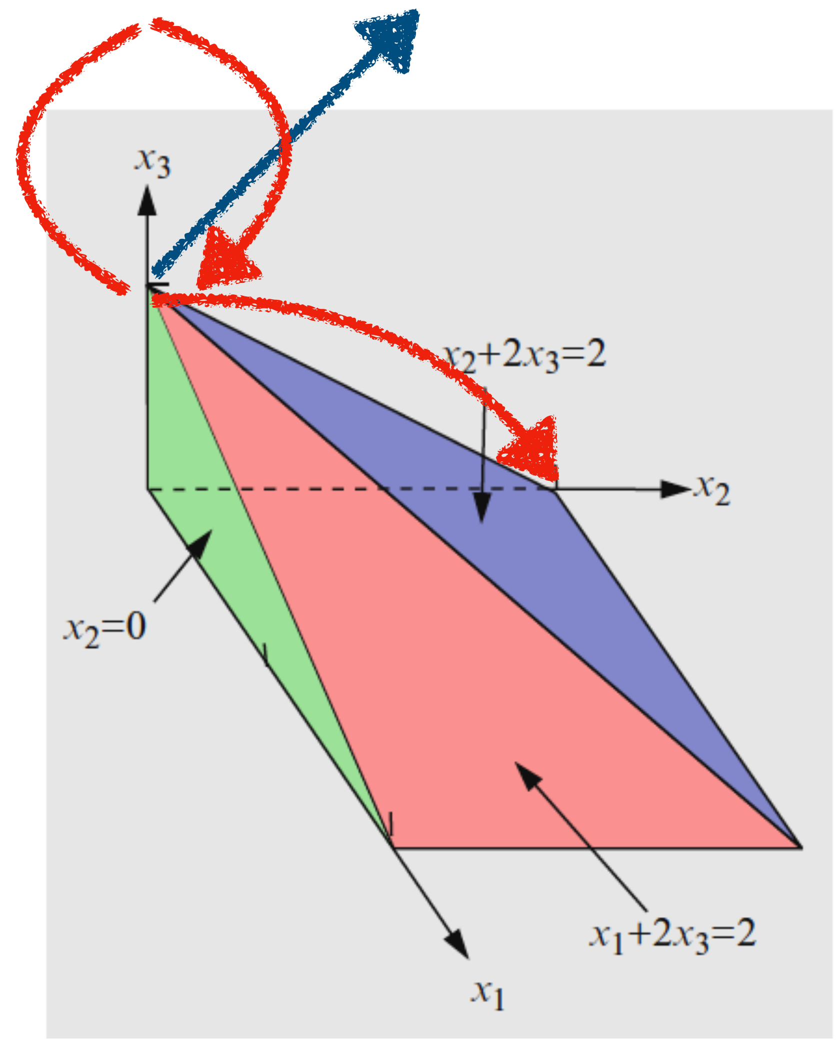

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But it corresponds to four different basic feasible solutions/dictionaries.

Termination

Theorem: If the simplex method does not cycle, it terminates.

Proof: A dictionary is determined by which variables are basic and which are non-basic.

There only $\binom{n+m}{m} = \frac{(n+m)!}{n!m!}$ possibilities.

Simplex Running Time

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More good news: "*Beyond the worst-case analysis*" shows that the algorithm is also efficient in theory.

Even more good news: We have other algorithms that run in worst-case polynomial running time (Ellipsoid Method, Interior Point Methods).

Duality

Suppose that we have a linear program, which we will refer to as *the primal*.

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Maximisation becomes minimisation.

The two linear programs will have a very important connection.

The Primal

Maximise

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \leq b_m$$
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$x_1, \ldots, x_n \geq 0$

Maximise $c^\top x$

Subject to

$Ax \leq b$

$x_1, \ldots, x_n \geq 0$

The Dual

Minimise

$$b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

Subject to

$$a_{1,1} y_1 + a_{1,2} y_2 + \ldots + a_{1,m} y_m \geq c_1$$
$$a_{2,1} y_1 + a_{2,2} y_2 + \ldots + a_{2,m} y_m \geq c_2$$
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$\vdots$

$a_{n,1} y_1 + a_{n,2} y_2 + \ldots + a_{n,m} y_m \geq c_n$

$y_1, \ldots, y_m \geq 0$

Minimise $b^\top y$

Subject to

$A^\top y \geq c$

$y_1, \ldots, y_m \geq 0$

Side by side

Maximise

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \leq b_1$$
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Minimise

$$b_1y_1 + b_2y_2 + \ldots + b_my_m$$

Subject to

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Side by side

n variables
m constraints

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Side by side

n variables
m constraints

Maximise

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

Subject to

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$\leq$ becomes $\geq$

m variables
n constraints

Minimise

$$b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

Subject to

$$a_{1,1}y_1 + a_{1,2}y_2 + \ldots + a_{1,m}y_m \geq c_1$$
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Side by side

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Maximise

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Subject to

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m variables
n constraints

Minimise

$$b_1y_1 + b_2y_2 + \ldots + b_my_m$$

Subject to

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$$y_1, \ldots, y_m \geq 0$$

Matrix A gets transposed

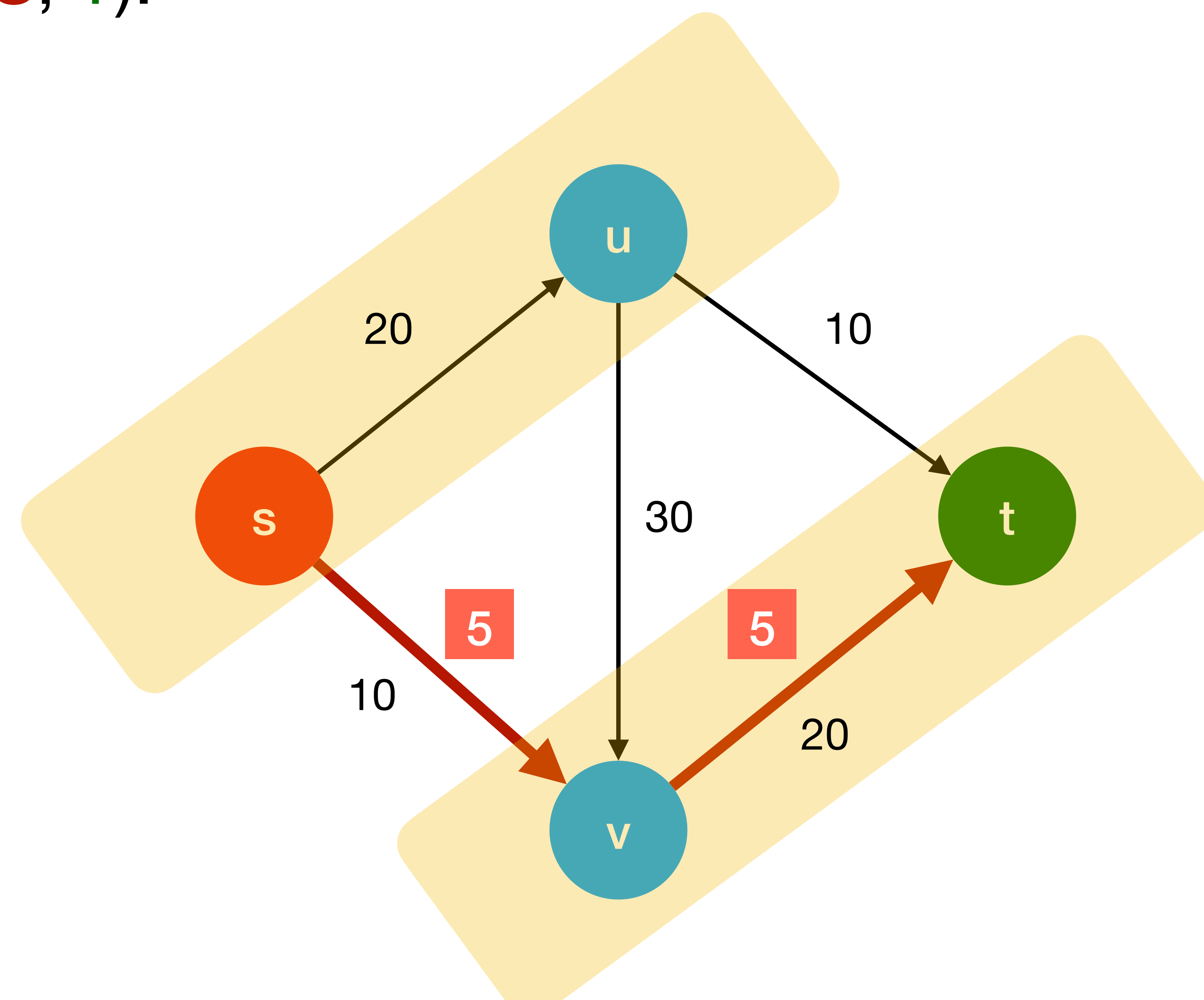
Weak Duality

Let x be any feasible solution to the Primal and let y be any feasible solution to the Dual. Then we have that

$$\text{value}(x) \leq \text{value}(y)$$

Weak Duality

Fact 3: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then v(f) ≤ c(S, T).



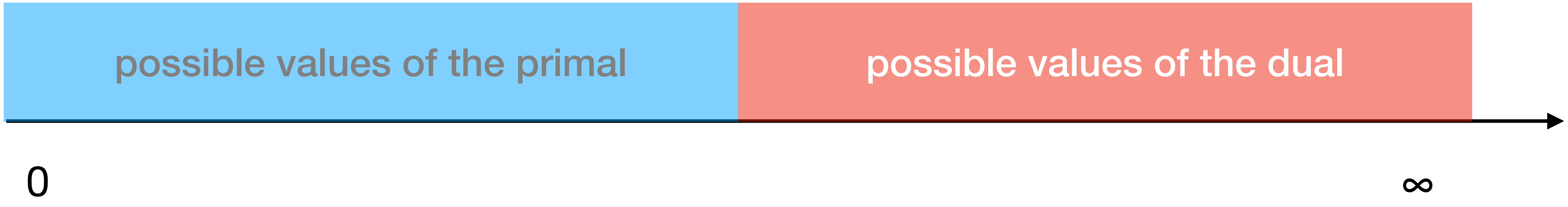
Strong Duality

Let x be any feasible solution to the Primal and let y be any feasible solution to the Dual. If

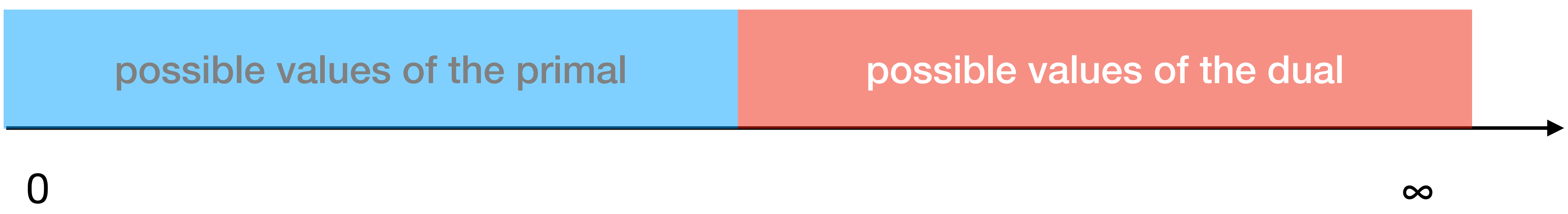
$$\text{value}(x) = \text{value}(y)$$

then x and y are both optimal solutions.

Pictorially

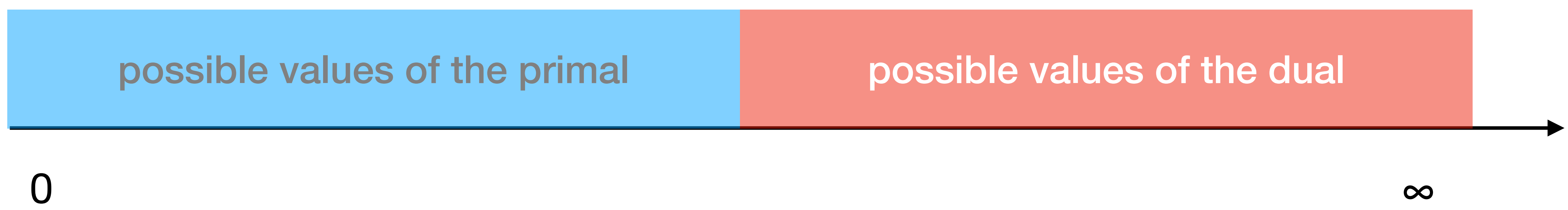


Pictorially



How can we prove that a solution x to the primal is maximum?

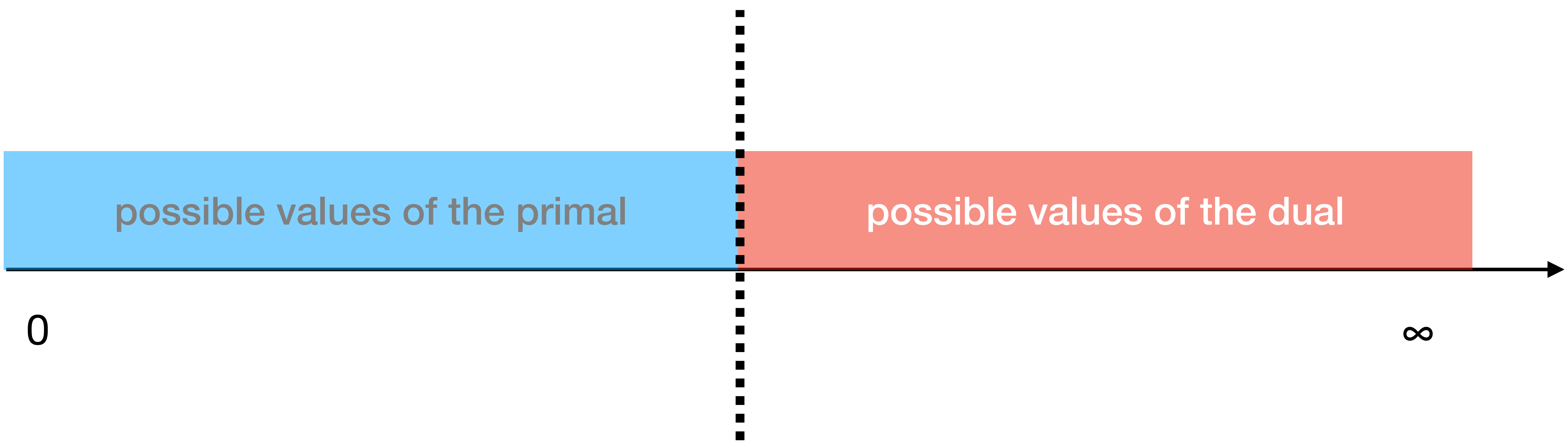
Pictorially



How can we prove that a solution x to the primal is maximum?

Find a solution y to the dual with value(y) = value(x)

Pictorially



How can we prove that a solution x to the primal is maximum?

Find a solution y to the dual with value(y) = value(x)

Strong Duality (complete statement)

Theorem (Strong Duality, von Neumann 1947): One of the following is true:

1. Both the primal and the dual are feasible, and let $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^m$ be any optimal solutions to the primal and the dual, respectively. Then $c^\top x^* = b^\top y^*$.

2. The primal is infeasible and the dual is unbounded.

3. The primal is unbounded and the dual is infeasible.

4. Both the primal and the dual are infeasible.

LP-Duality and the Simplex Method

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Let $y_j^* = -\hat{c}_{n+j}$, where $\hat{c}_{n+j}$ is the coefficient of the slack variable x_{n+j}.

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<u>Interesting observation:</u> Consider the final dictionary of the simplex method at which we compute an optimal solution x^* to the primal.

Let $y_j^* = -\hat{c}_{n+j}$, where $\hat{c}_{n+j}$ is the coefficient of the slack variable x_{n+j}.

The values y^* obtained that way are an optimal solution to the dual!

The Final Dictionary

Maximise $\zeta = 13 - w_1 - 2x_2 - w_3$

subject to

$$
\begin{aligned}
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

$$w_1 = 0,\ x_2 = 0\ w_3 = 0 \qquad x_1 = 2,\ w_2 = 1,\ x_3 = 1$$

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\begin{array}{rcrlrlrl}
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w_2 &=& 1 & +2w_1 & +5 & x_2 & & \\
x_3 &=& 1 & +3w_1 & + & x_2 & -2 & w_3
\end{array}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3 \geq 0$$

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Sanity check

Maximise $5x_1 + 4x_2 + 3x_3$

subject to

$2x_1 + 3x_2 + x_3 \leq 5$

$4x_1 + x_2 + 2x + 3 \leq 11$

$3x_1 + 4x_2 + 2x_3 \leq 8$

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Minimise $5y_1 + 11y_2 + 8y_3$

subject to

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Maximise $5x_1 + 4x_2 + 3x_3$

$y^* = (1,0,1)$

subject to

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$5 * 1 + 8 * 1 = 13$

subject to

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Complementary Slackness

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Proposition (Complementary Slackness): Let x^* and y^*be feasible solutions to the primal and the dual respectively. Then x^* and y^* are both optimal if and only if both of the following hold:

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On an example

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$2 * 1 + 4 * 0 + 3 * 1 = 5$ $2y_1 + 4y_2 + 3y_3 \geq 5$
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So for each $j = 1, \ldots, n$, we have $\left((A^\top y^*)_j - c_j\right) \cdot x_j^* = 0$.

Similarly for the case of $\left((Ax^*)_i - b_i\right) \cdot y_i^* = 0$

The Max-Flow Min-Cut Theorem

Theorem: In every flow network, the value of the maximum flow is *equal* to the capacity of the minimum cut.

This is a consequence of the *strong duality theorem* for linear programs!

Maximum Flow as an LP

We can write the maximum flow problem as a linear problem.

"Maximise the flow, subject to capacity and flow conservation constraints".

$$
\begin{array}{lrll}
\text{maximise} & \displaystyle\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} & & \\
\text{subject to} & f_{uv} \leq c_{uv}, & \text{for each} & u, v \in V \\
& \displaystyle\sum_{v\in V} f_{vu} = \sum_{v\in V} f_{uv}, & \text{for each } u \in V - \{s, t\} & \\
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\end{aligned}
$$

Maximum Flow as an LP

We can write the maximum flow problem as a linear problem.

"Maximise the flow, subject to capacity and flow conservation constraints".

total flow out of s

total flow into s

$$\text{maximise} \quad \boxed{\sum_{v \in V} f_{sv}} - \boxed{\sum_{v \in V} f_{vs}}$$

capacity constraint

$$\text{subject to} \quad \boxed{f_{uv} \le c_{uv}, \quad \text{for each} \quad u, v \in V}$$

flow conservation

$$\boxed{\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}, \quad \text{for each } u \in V - \{s, t\}}$$

non-negative flow

$$\boxed{f_{uv} \ge 0, \quad \text{for each} \quad u, v \in V}$$

Constructing the dual

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\begin{array}{lrll}
\text{minimise} & \displaystyle\sum_{(u,v)\in E} c_{uv} d_{uv} & & \\
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This is *1* if u is in S and v is in T
and *0* otherwise.

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\end{array}
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If u is in S and v is
in T, then d_{uv} must be *1*.

$$d_{uv} \in \{0,1\}, \quad \text{for each } (u,v)\in E$$
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Constructing the dual

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If v is in T then d_{sv} must be *1*.

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If u is in S then d_{ut} must be *1*.

Minimum Cut as an ILP

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\begin{array}{lrll}
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\end{array}
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LP-relaxation

An *LP-relaxation* of an Integer Linear Program is a linear program which is identical to the ILP, except all the integrality constraints have been removed ("*relaxed*"), or replaced with non-integral constraints.

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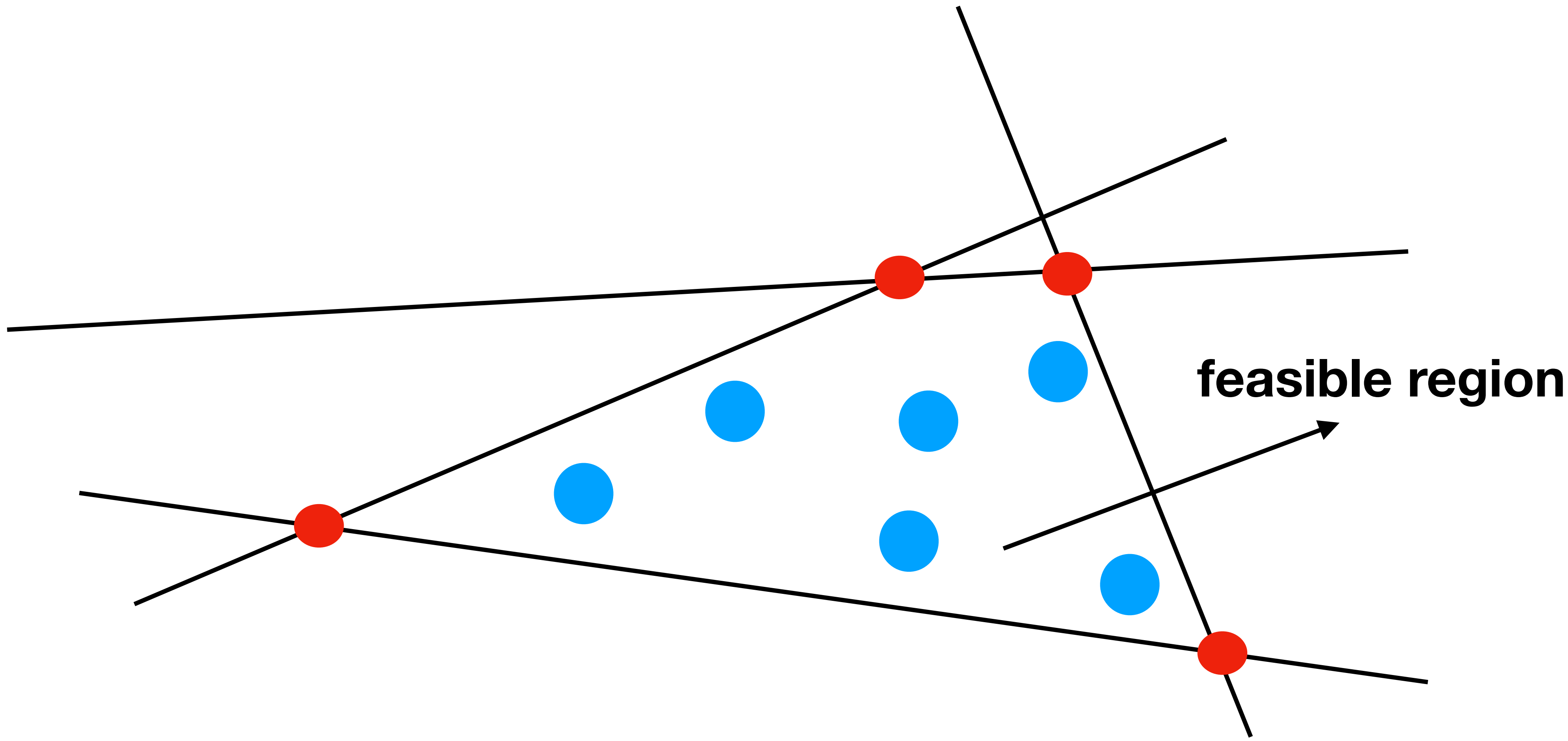
ILP vs LP-relaxation



candidate optimal solution for ILP

candidate optimal solution for LP-relaxation

ILP vs LP-relaxation

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For a maximisation problem:

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The optimal value of the ILP is not larger than the optimal value of the LP-relaxation.

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The optimal value of the ILP is not larger than the optimal value of the LP-relaxation.

The ratio

max_value(LP-relaxation) / max_value(LP)

is called the integrality gap of the LP-formulation.

Let’s put some facts together.

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In other words, the Min-Cut LP has *an integer optimal solution*.

Back to Maximum Flow

What if we wanted an integer flow instead of any flow?

$$
\begin{array}{llll}
\text{maximise} & \displaystyle\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs} & & \\
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& && f_{uv} \geq 0, && \text{for each } u, v \in V
\end{aligned}$$

$$\color{red}{f_{uv} \in \mathbb{R}, \quad \text{for each } u, v \in V}$$

Back to Maximum Flow

Does the LP-relaxation of this ILP always have an integer solution?

$$
\begin{array}{lrll}
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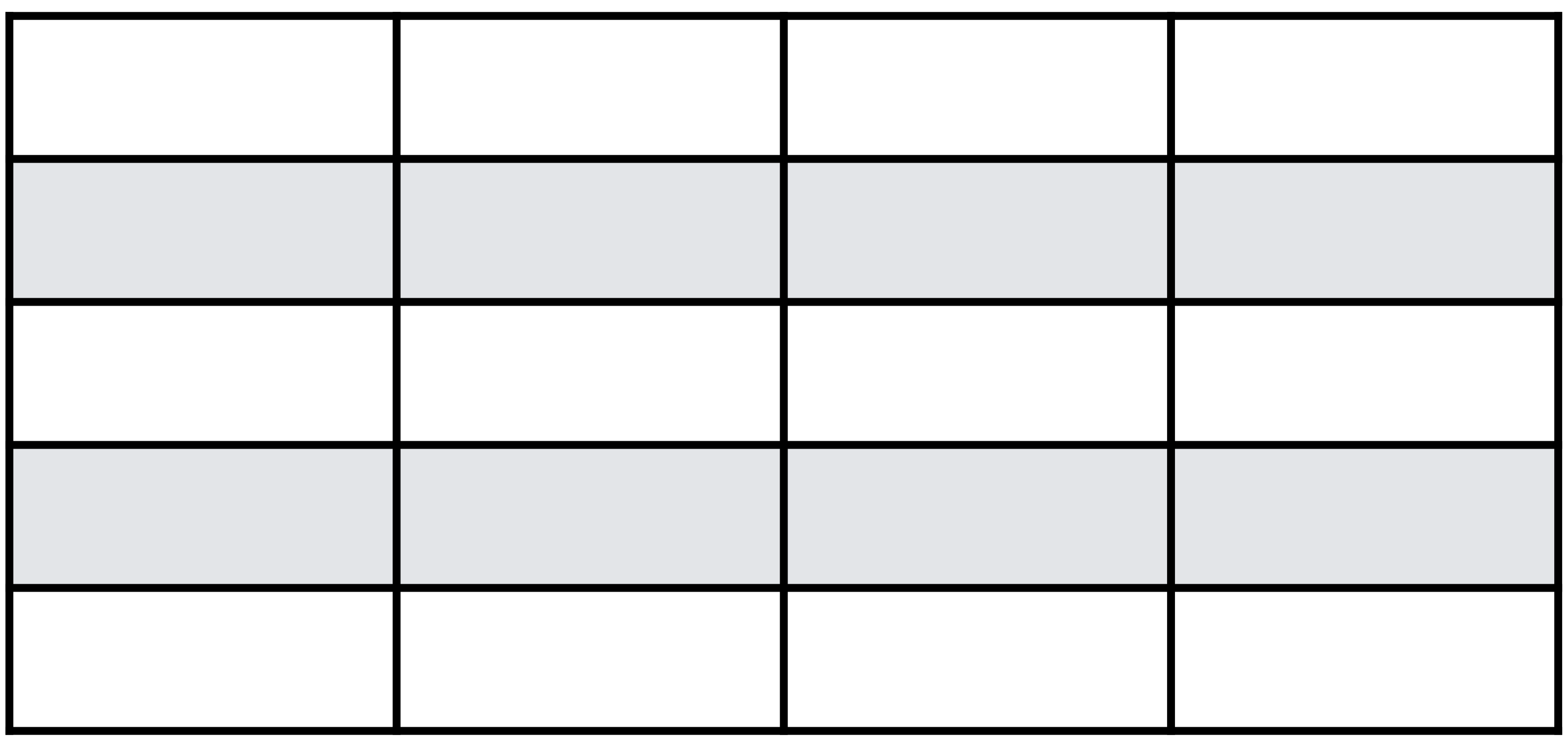
For Max-Flow, it finds the maximum fractional flow.

For Min-Cut, it finds the minimum "fractional" cut.

If we solve those LP-relaxations, we will get *integer solutions*.

Totally Unimodular Matrices

Let A be a real *m* x *n* matrix. Suppose that every square submatrix of A has determinant in {*0*, *+1*, *-1*}. Then A is *totally unimodular*.



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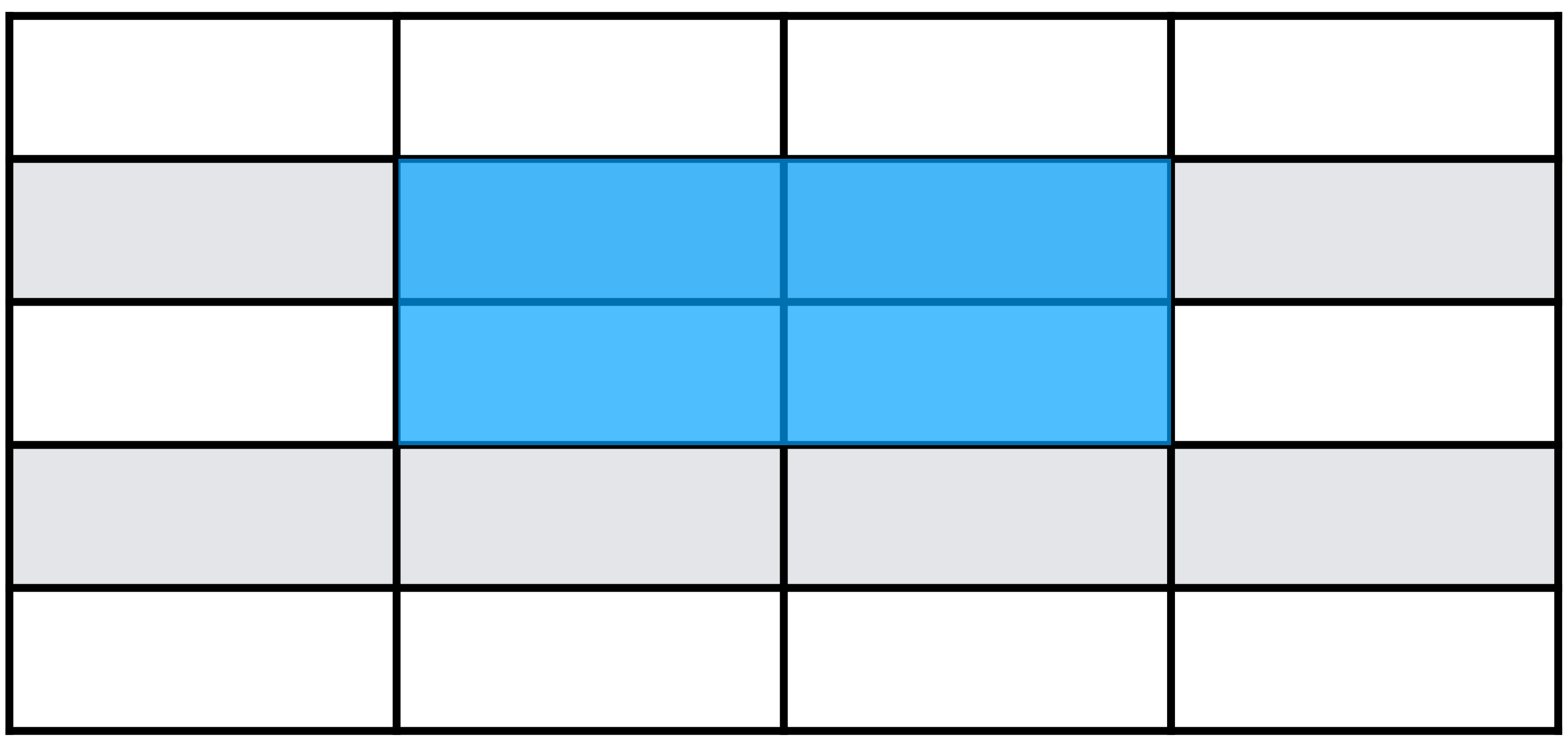

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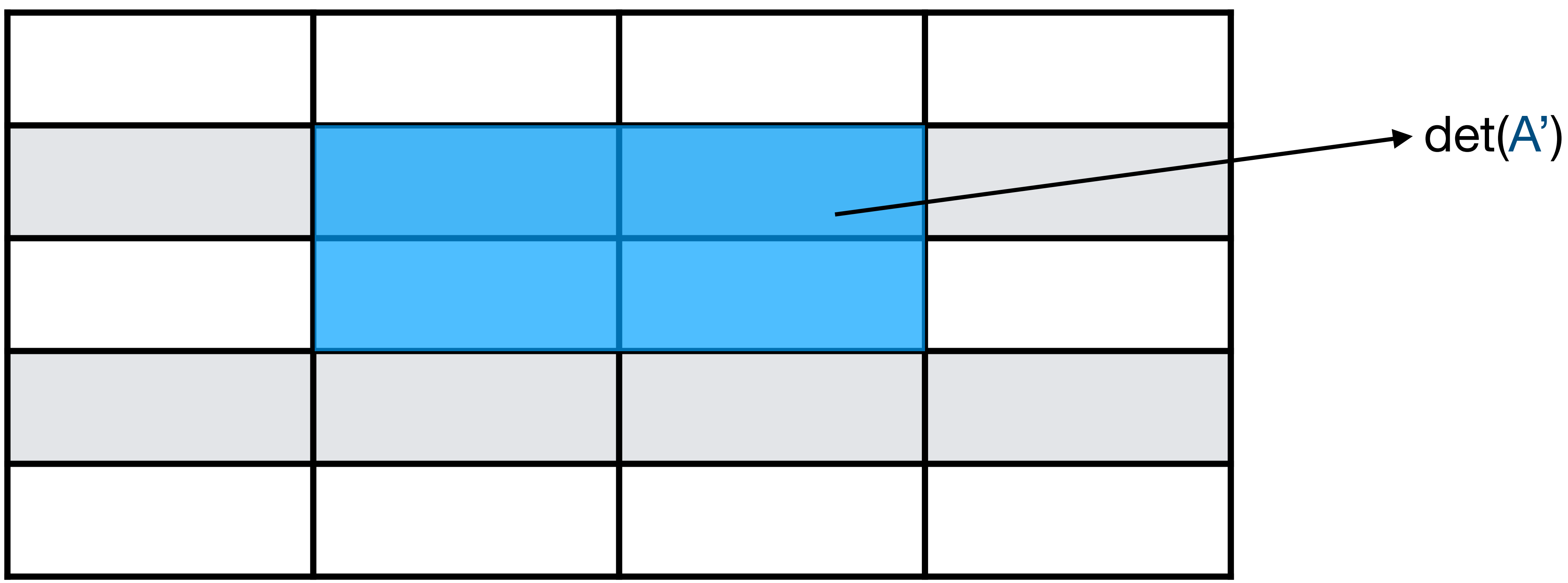

Total Unimodularity

If the constraint matrix A is totally unimodular and b is an *integer vector*, then the LP has an *integer solution*.

The Max-Flow and Min-Cut LP-relaxations admit integer solutions because their constraint matrices are totally unimodular.

$$\begin{aligned} \text{maximise} \quad & c^{\mathrm{T}} x \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \end{aligned}$$

To be more precise

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Lemma: Suppose A is a totally unimodular matrix and b is an integer vector. Then every extreme point of

$P = \{x: Ax < b\}$ is integral.

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Lemma: Suppose A is a totally unimodular matrix and b is an integer vector. Then every extreme point of

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Claim: Suppose A is totally unimodular. Then the matrix $A' = (A\ {-A}\ I\ {-I})^T$ is also totally unimodular.

Corollary: Suppose A is a totally unimodular matrix and b is an integer vector. Then every extreme point of

$P = \{x: Ax = b, 0 \leq x \leq c\}$ is integral.

Back to maximum flow

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\begin{array}{lrll}
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$0 \le x \le c$

Back to maximum flow

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0 ≤ x ≤ c

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$Ax = 0$

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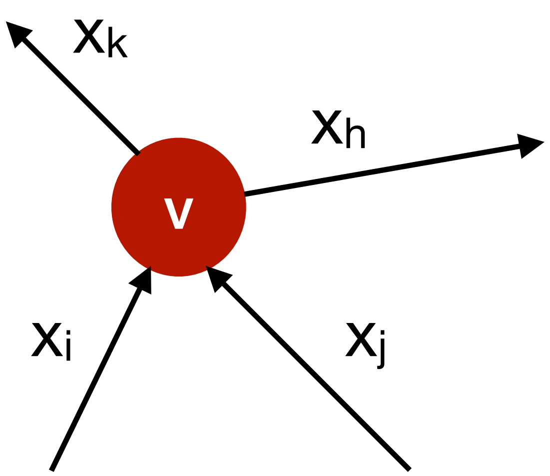
& && f_{uv} \ge 0, \quad \text{for each} \quad u, v \in V
\end{aligned}$$



$0 \le x \le c$

$Ax = 0$

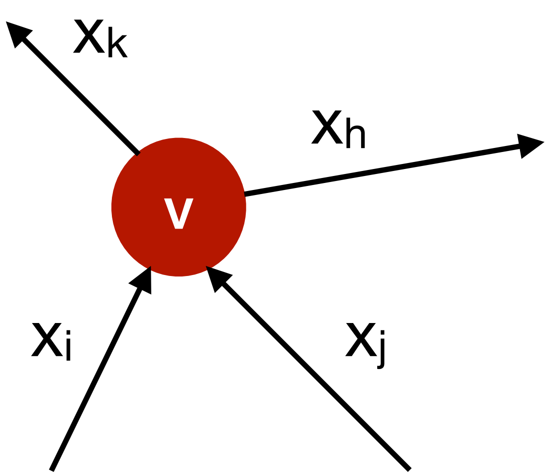
Back to maximum flow

$$\text{maximise} \quad \sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$$

$$\text{subject to} \quad f_{uv} \le c_{uv}, \quad \text{for each} \quad u, v \in V$$

$$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}, \quad \text{for each } u \in V - \{s, t\}$$

$$f_{uv} \ge 0, \quad \text{for each} \quad u, v \in V$$



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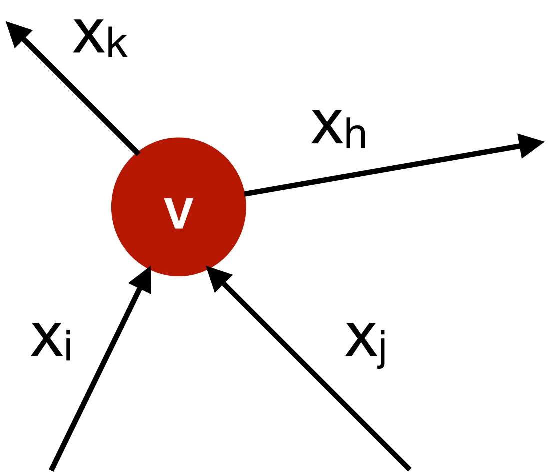
$x_k + x_h - x_i - x_j = 0$

Back to maximum flow

$$
\begin{array}{lll}
\text{maximise} & \displaystyle\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} & \\
\text{subject to} & \boxed{f_{uv} \le c_{uv}, \quad \text{for each} \quad u, v \in V} & \\
& \boxed{\displaystyle\sum_{v\in V} f_{vu} = \sum_{v\in V} f_{uv}, \quad \text{for each } u \in V - \{s, t\}} & \\
& \boxed{f_{uv} \ge 0, \quad \text{for each} \quad u, v \in V} &
\end{array}
$$



$x_k + x_h - x_i - x_j = 0$

$0 \le x \le c$

$Ax = 0$

$A_{vi} = A_{vj} = -1$

$A_{vk} = A_{vh} = 1$

Back to maximum flow

- Consider the *incidence matrix* of the flow network (without s and t):
  - $A_{ij} = 1$ if edge j **starts** at node i in G_f.
  - $A_{ij} = -1$ if edge j **ends** at node i in G_f.
  - $A_{ij} = 0$ otherwise.

Nodes/Edges		*j*	*k*
i		1	-1

Back to maximum flow

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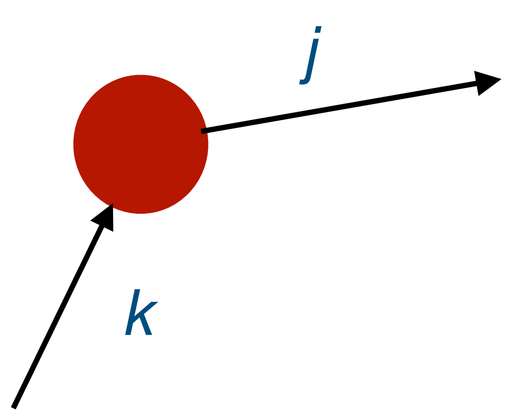

Nodes/Edges		*j*	*k*
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Back to maximum flow

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Consider the *incidence matrix* of the flow network (without s and t):

This is precisely the matrix A of the max flow LP.

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Back to maximum flow

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It suffices to prove that A is totally unimodular, by Corollary.

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Back to maximum flow

Consider the *incidence matrix* of the flow network (without s and t):

This is precisely the matrix A of the max flow LP.

It suffices to prove that A is totally unimodular, by Corollary.

Lemma: The incidence matrix of any directed graph is totally unimodular.

Nodes/Edges		*j*	*k*
i		1	-1