# Algorithms and Data Structures

## Modelling with Linear Programs

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In practice: There are many fantastic LP solvers (e.g., CPLEX, Gurobi, whatever-your-favourite-library-of-your-favourite-programming-language-uses, etc).

It suffices to formulate/model/express a problem as an LP and then ask one of those solvers for the solution.

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What is the price of one unit of product $j$? $\sigma_j$

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What is the revenue from all the units of of $j$? $c_j \cdot x_j$

What is the revenue in total? $\sum_{i=1}^{n} c_j x_j$

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2. We cannot produce more than the raw material allows:

$$\sum_{j=1}^{m} \alpha_{ij} x_j \leq b_i \text{ for } i = 1,\ldots,m$$

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**Maximise** $\sum_{i=1}^{n} c_j x_j$

**subject to** $\sum_{j=1}^{n} \alpha_{ij} \cdot x_j \leq b_i$ **for all** $i = 1,\ldots,m$

$x_j \geq 0$ **for all** $j = 1,\ldots,n$

# Integer Linear programming

$$\text{maximise} \qquad \sum_{j=1}^{n} c_j x_j$$

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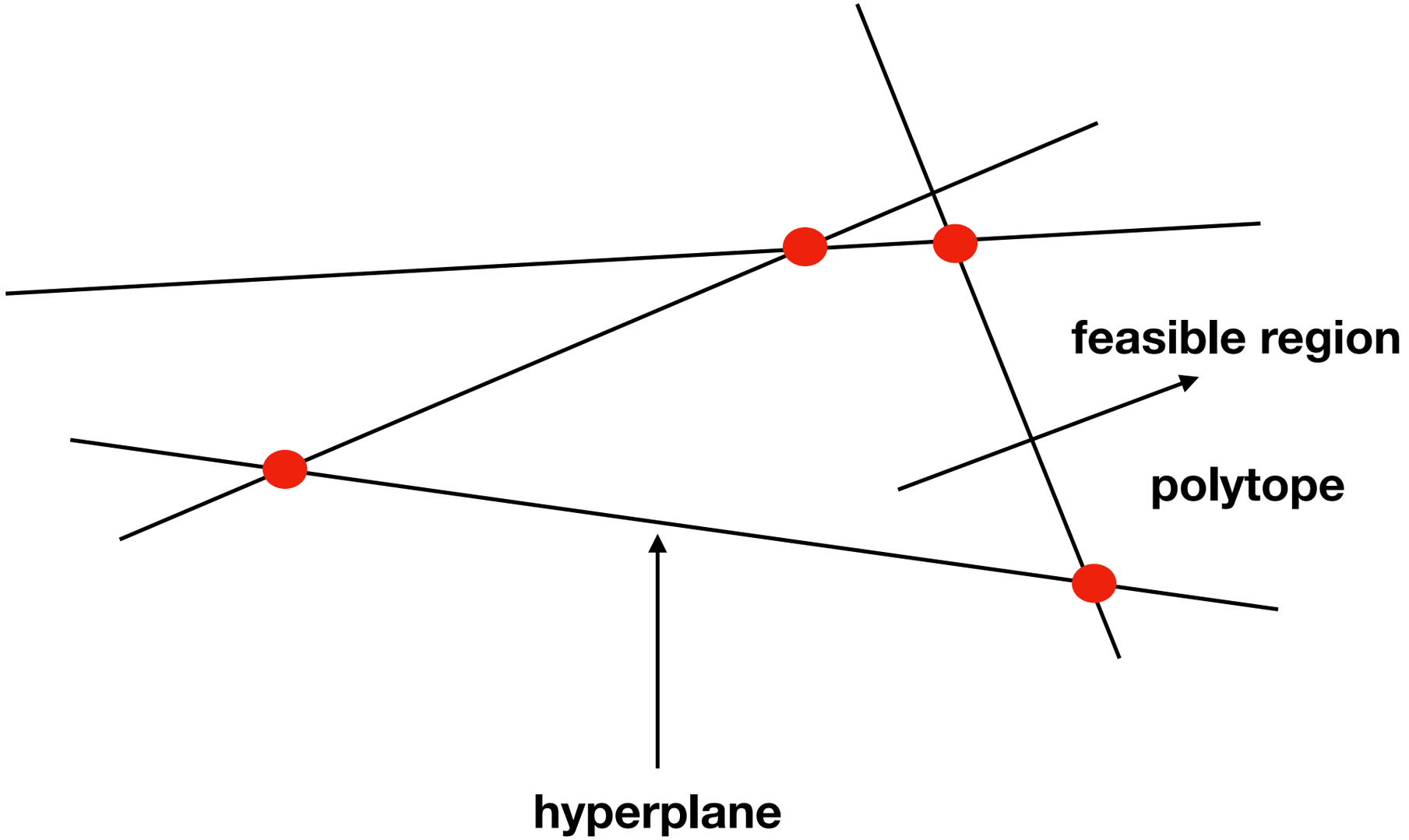
# Feasible region




candidate optimal solution

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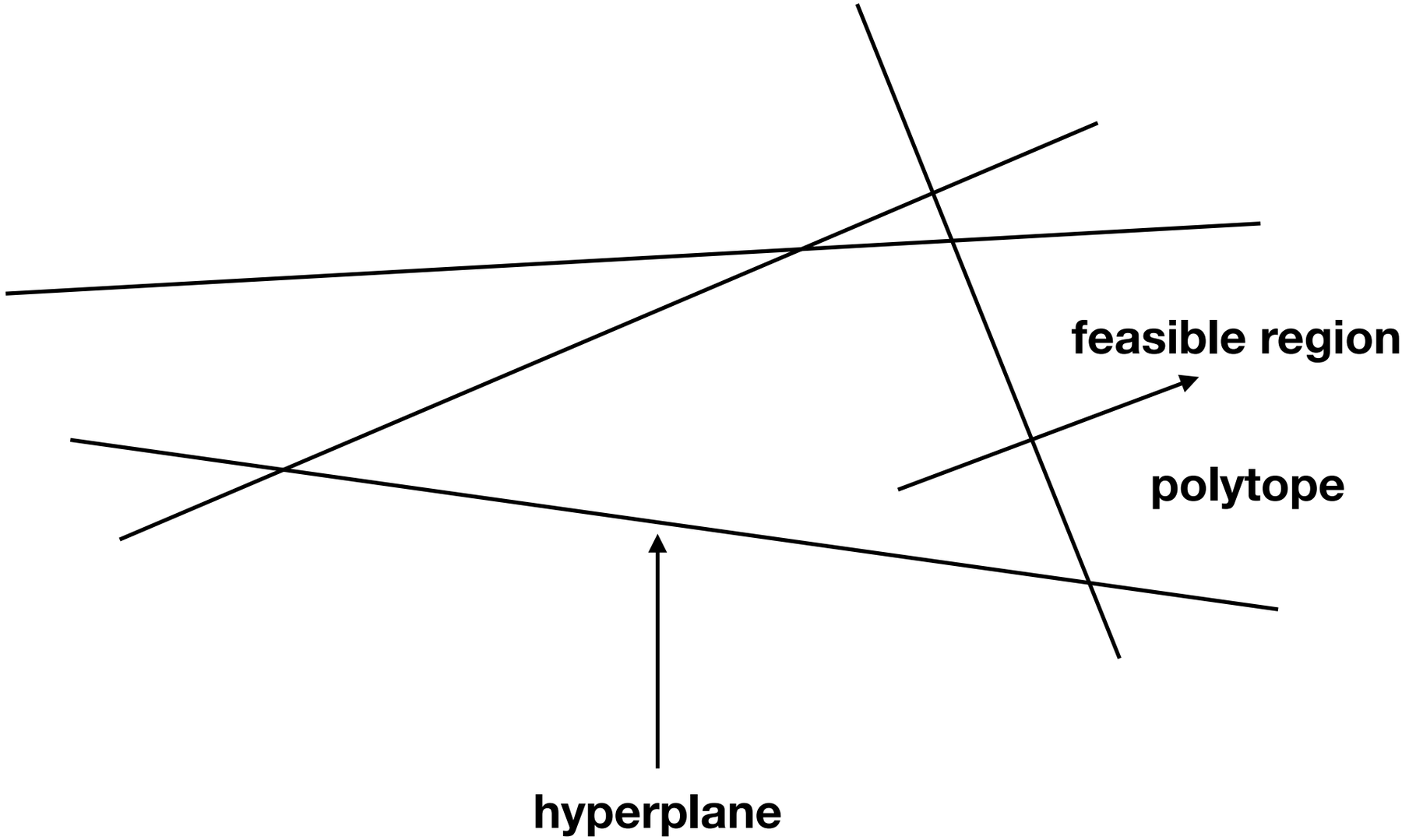

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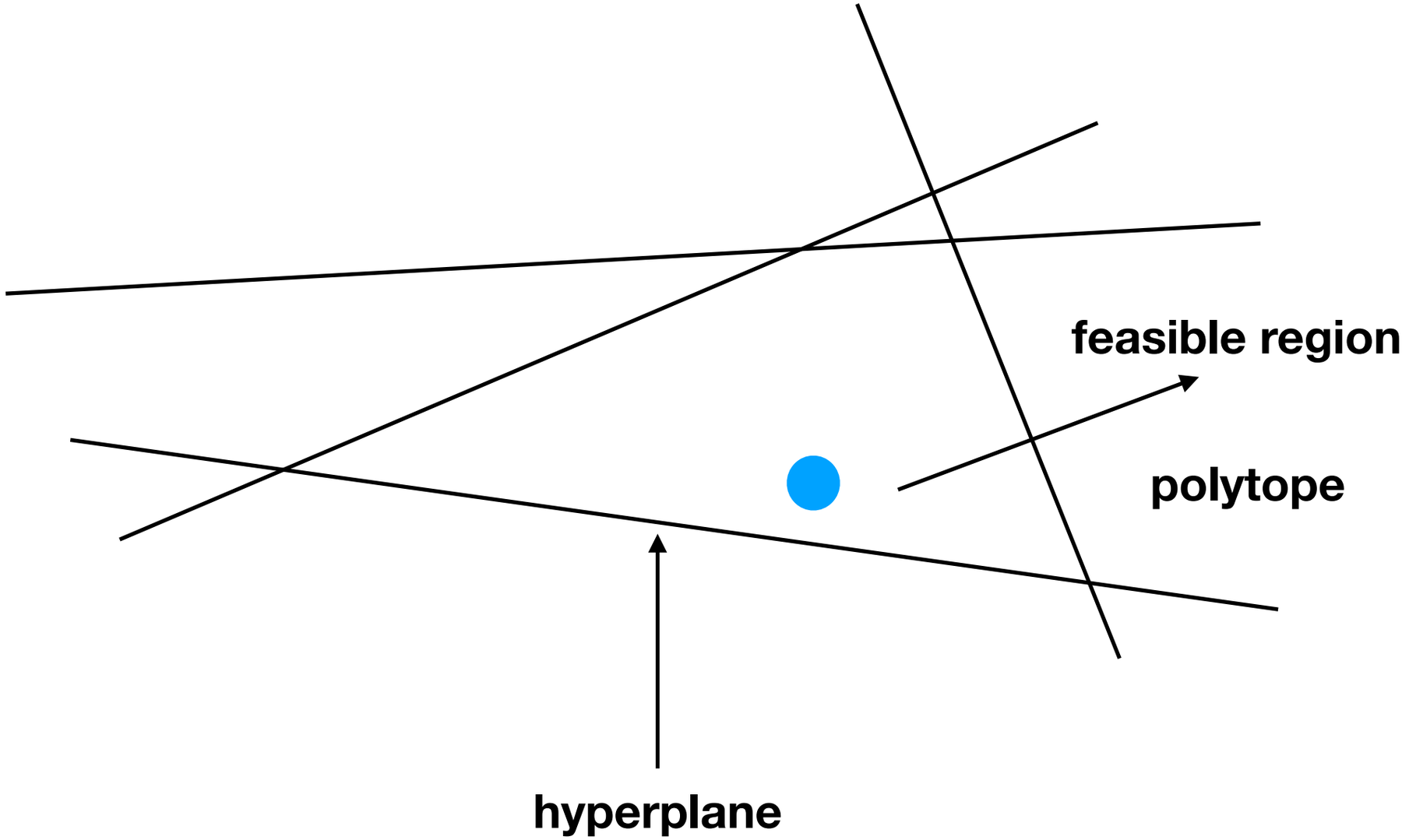

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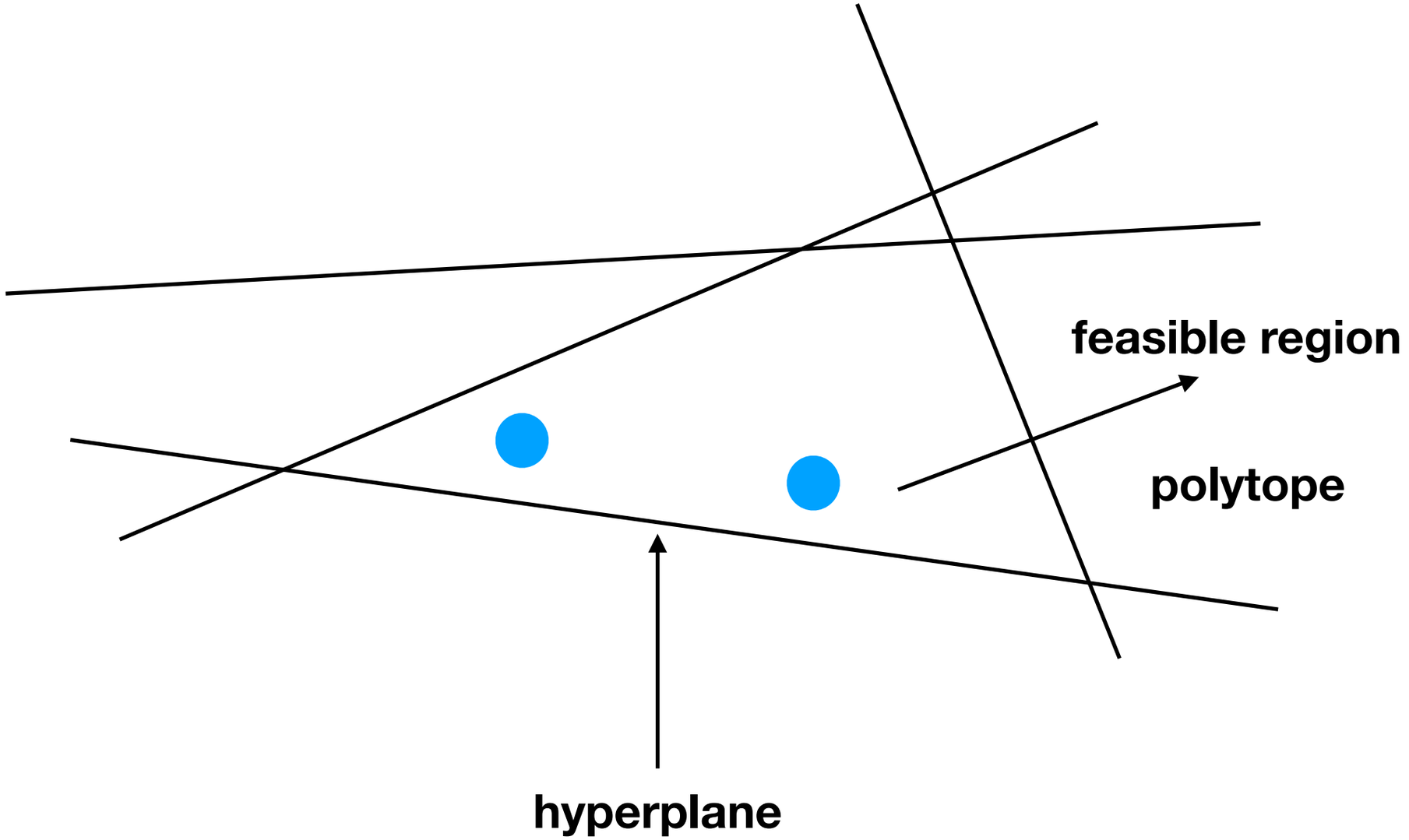

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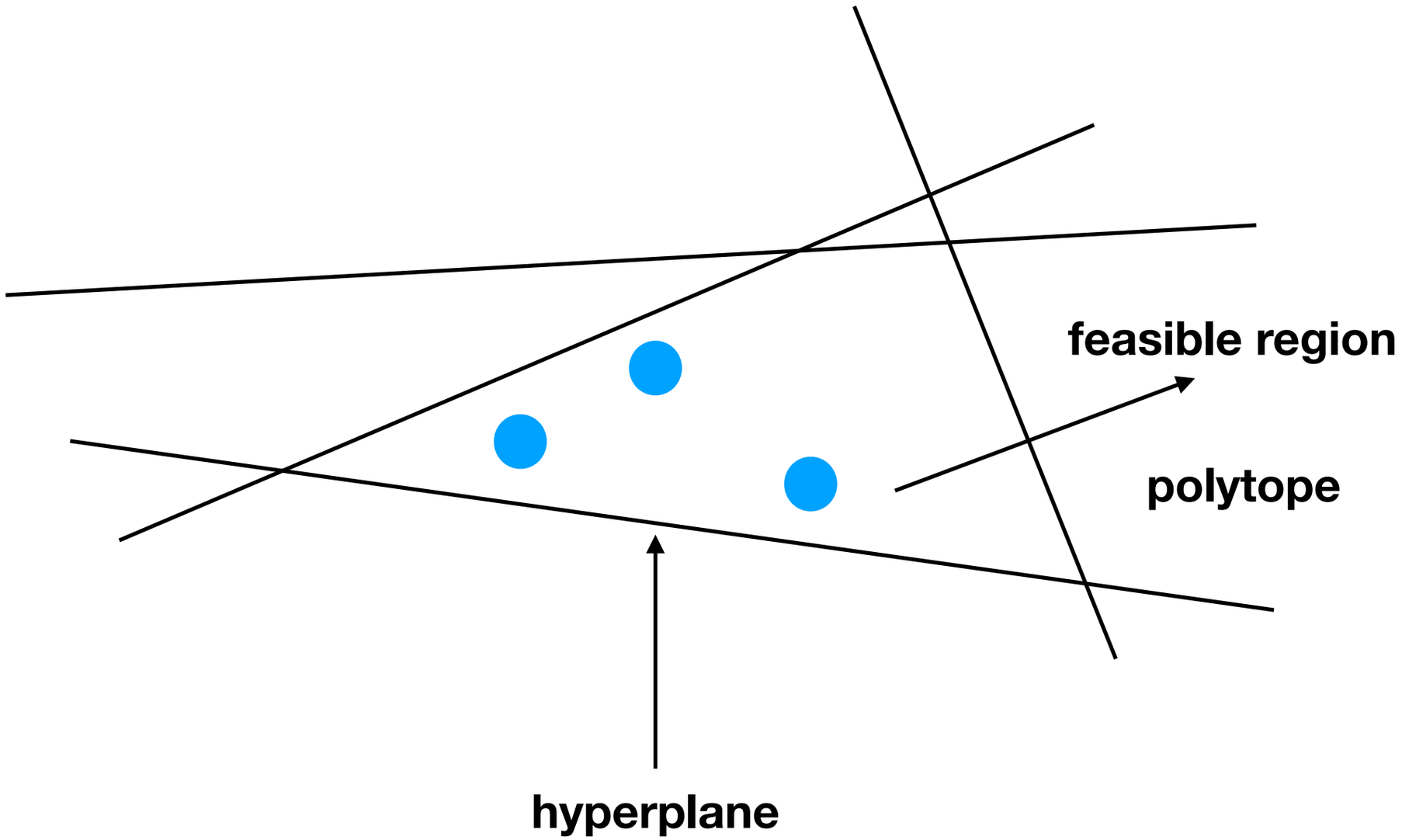

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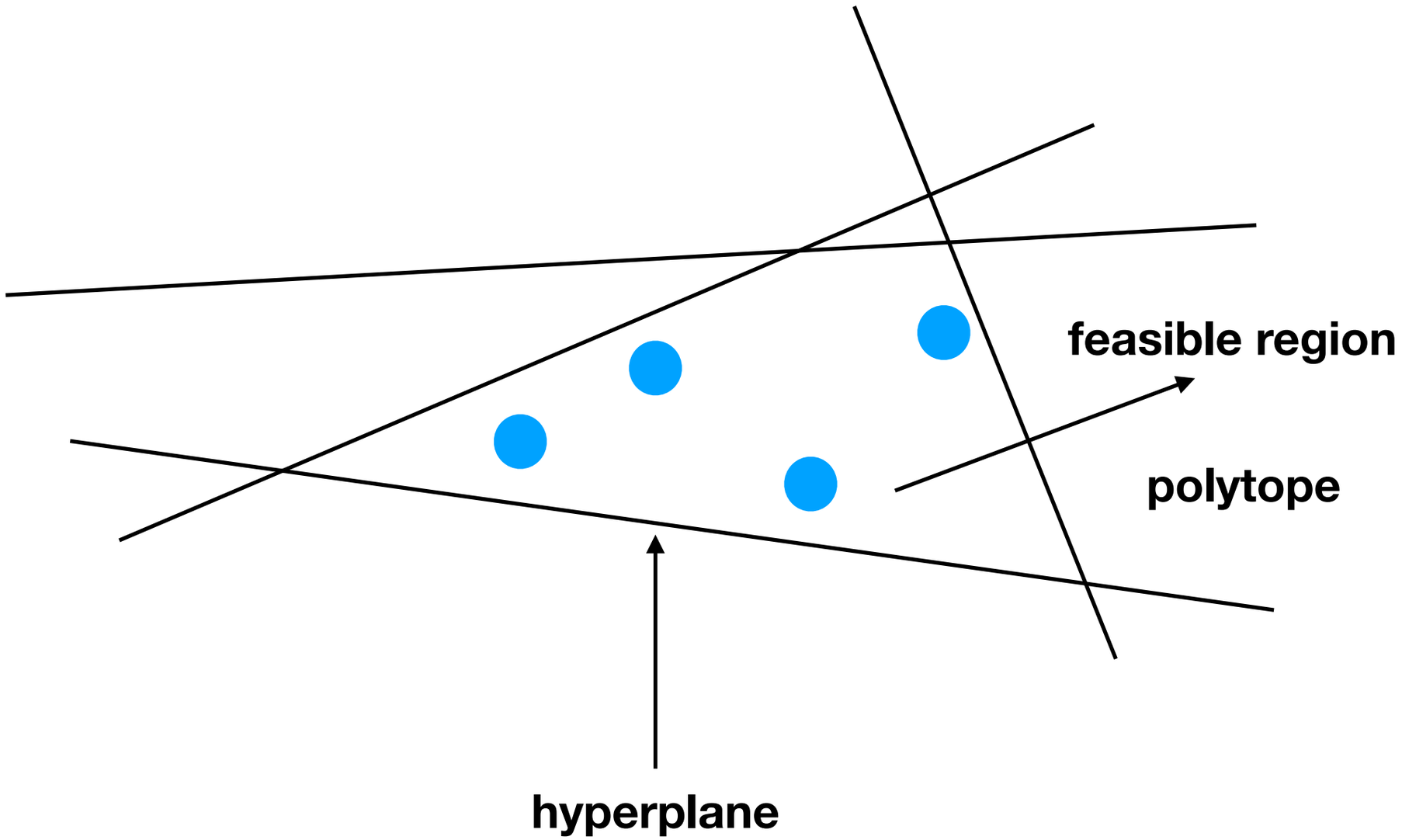

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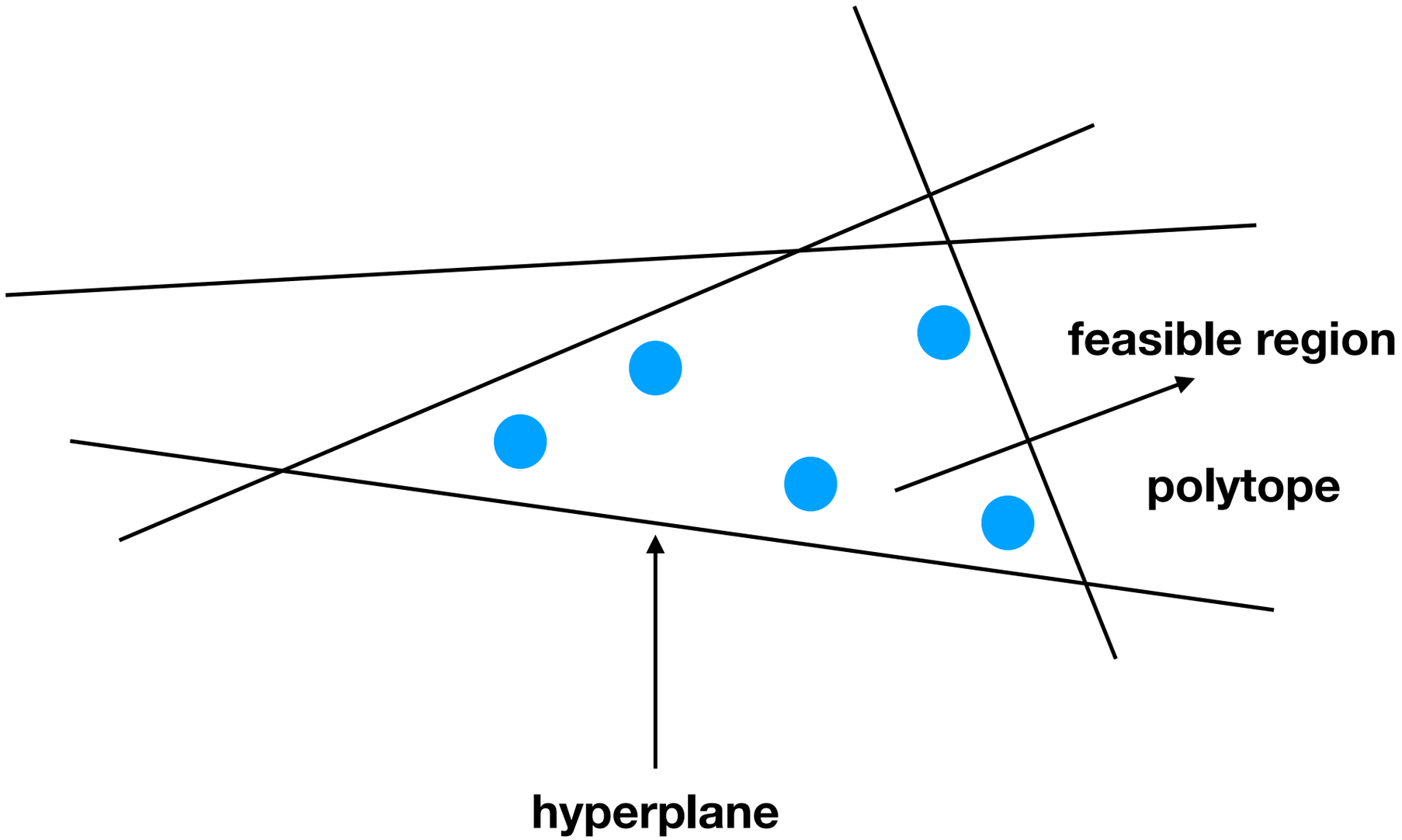

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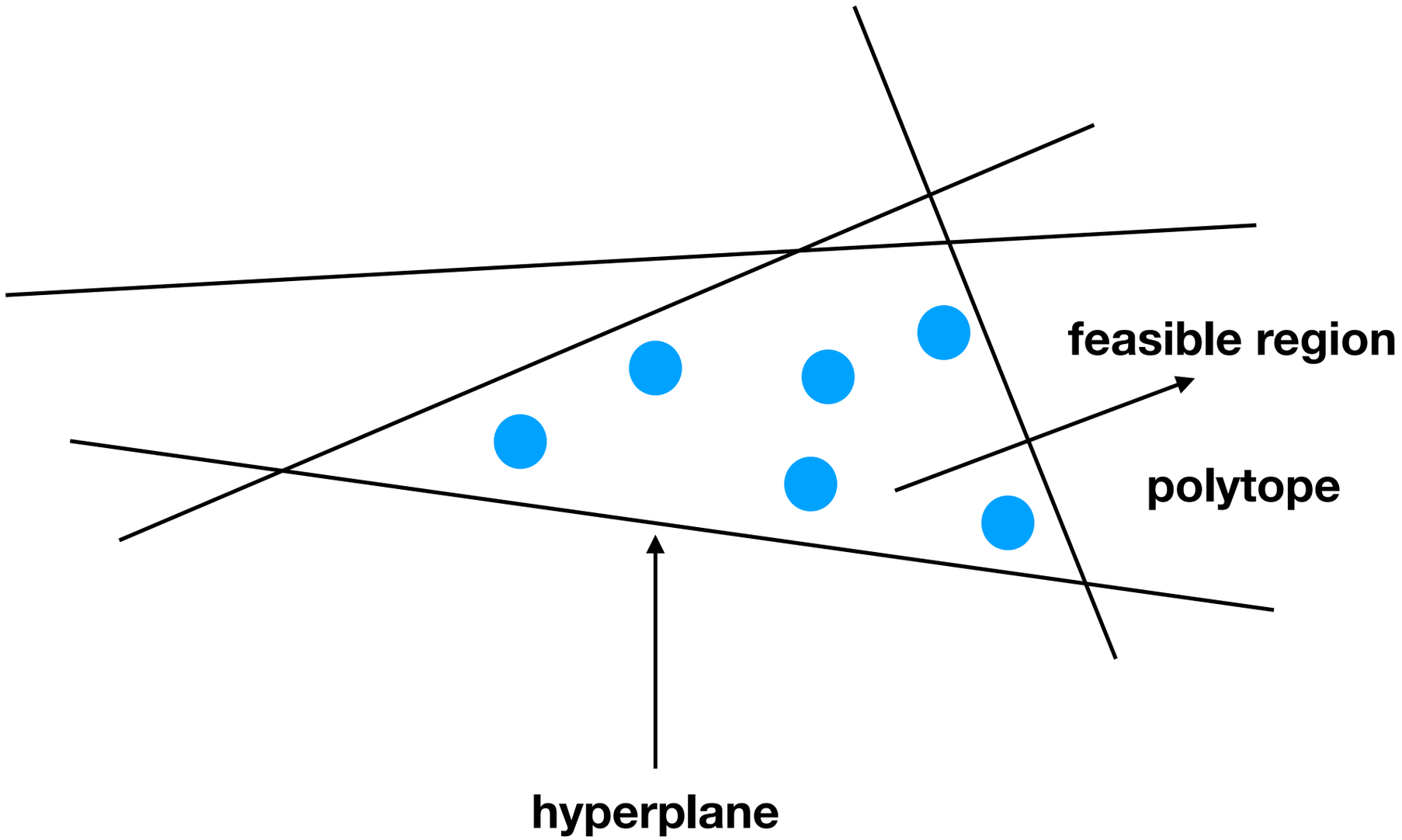

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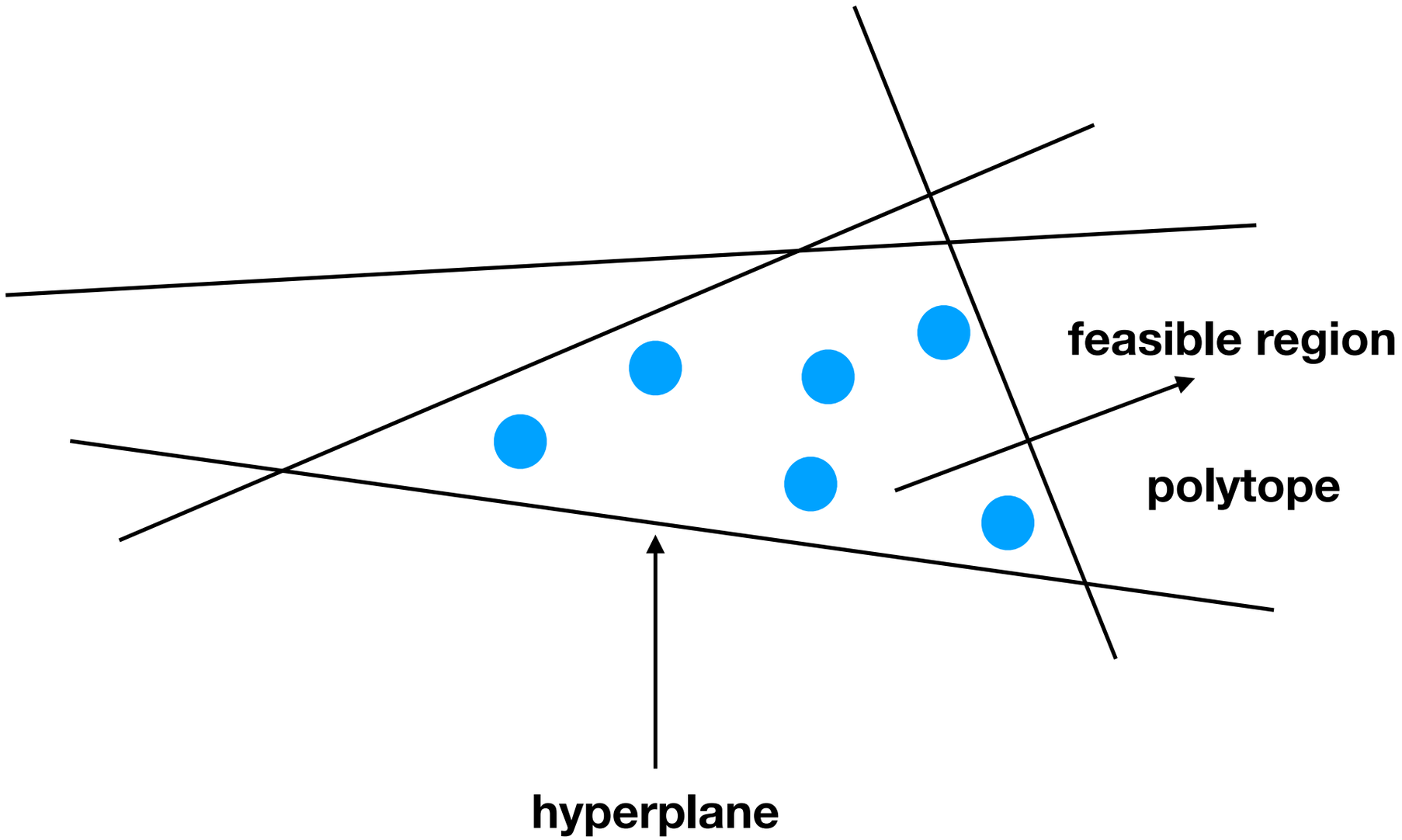


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Modelling as ILPs is a very useful skill.

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We would like to find a subset of the routes such that each leg is included in exactly one route.

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Here, we will let $x_j = 1$ if we select route $j$ and $x_j = 0$ otherwise.

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# Our ILP formulation

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**subject to** ?

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How many are these?

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**Minimise** $\sum_{j=1}^{n} c_j x_j$

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$x_j \in \{0,1\} \quad \textbf{for } j = 1,\ldots,n$

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Each job $i$ has a processing time $t_{ij}$ on machine $j$.

Each machine processes one job after another, but different machines run in parallel.

We would like to find a way to assign the jobs to the machines such that we minimise the *makespan*, i.e., the completion time of the last machine to finish.

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Here, we will let $x_{ij} = 1$ if we assign job $i$ to machine $j$ and $x_{ij} = 0$ otherwise.

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not linear!

# Our ILP formulation

**Minimise** $\max_{j=1,\ldots,m} \sum_{i=1,\ldots,n} t_{ij} x_{ij}$

**subject to** ?

(Let’s put that aside for a second…)

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call this $T$

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What is the relationship between $\sum_{i=1,\ldots,n} t_{ij}x_{ij}$ and $T$ for any $j$?

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The salesman would like to minimise the total distance travelled.

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The total number of possible tours is equal to the permutation of $n-1$ elements, i.e., $(n-1)!$

Enumeration is obviously too slow. We will use an ILP formulation approach instead and rely on our clever solvers to be faster than enumeration.

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Alternative interpretation: Think of the map as a fully connected graph with a node for every city and an edge between every two cities. Then $x_{ij} = 1$ if and only if the edge $(i, j)$ is being used by the tour.

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e.g. $t_3 = 4$ means that city 3 was visited 4th during the tour.

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Putting them together: $t_j \geq t_i + 1 - n(1 - x_{ij})$

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# No subtours

Assume that we instead have two disjoint subtours.

Consider one of these subtours that does not include city $0$, and let $r$ be the number of cities visited by this subtour.

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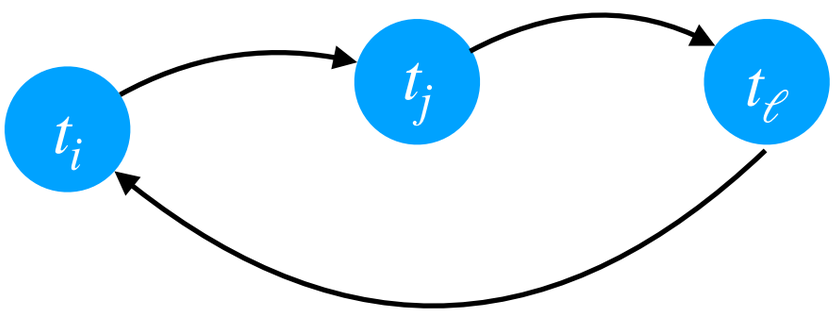

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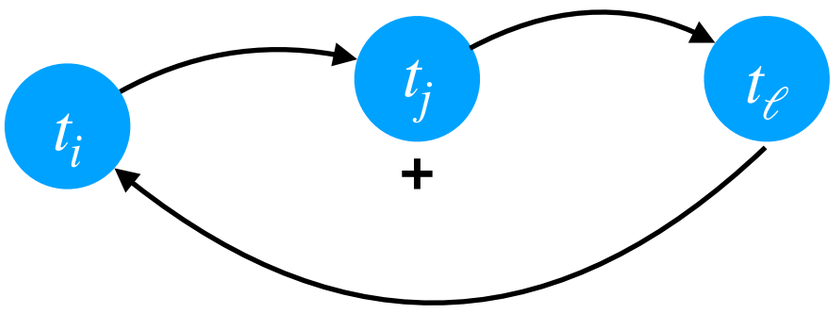

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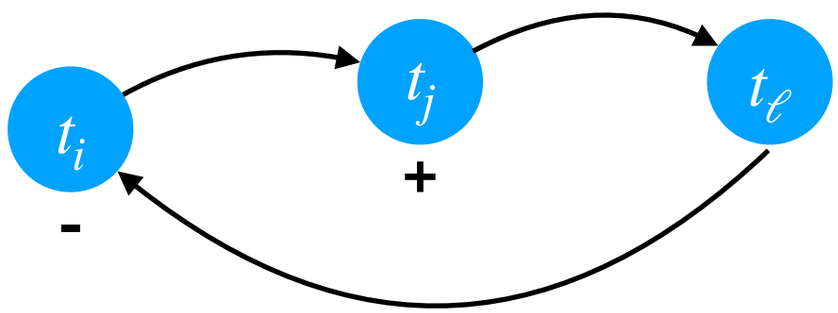

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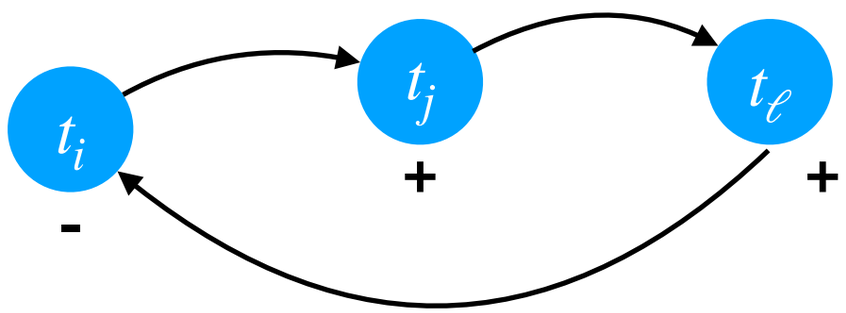

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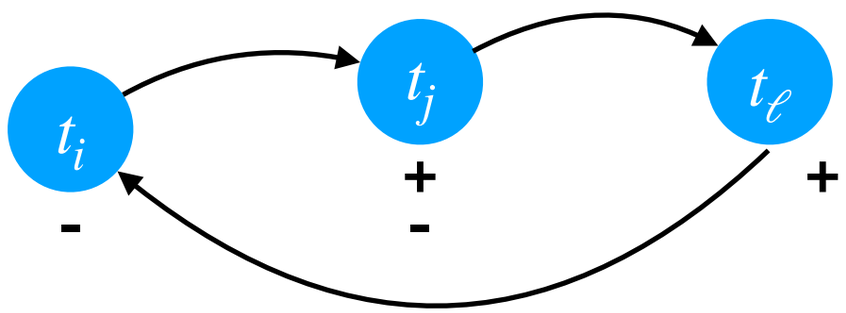

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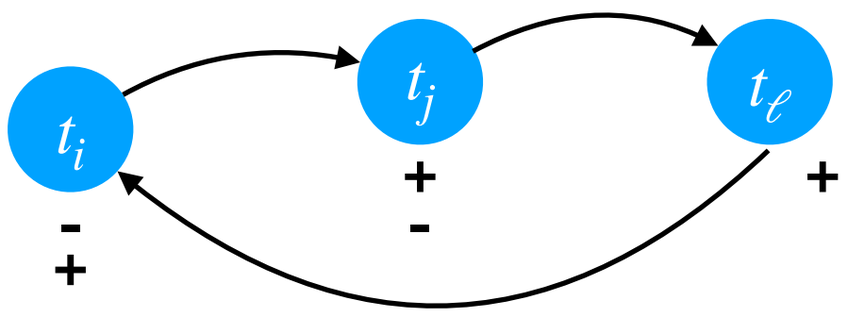

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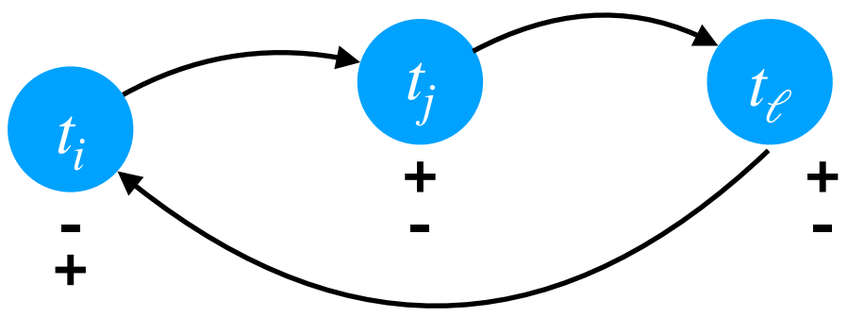

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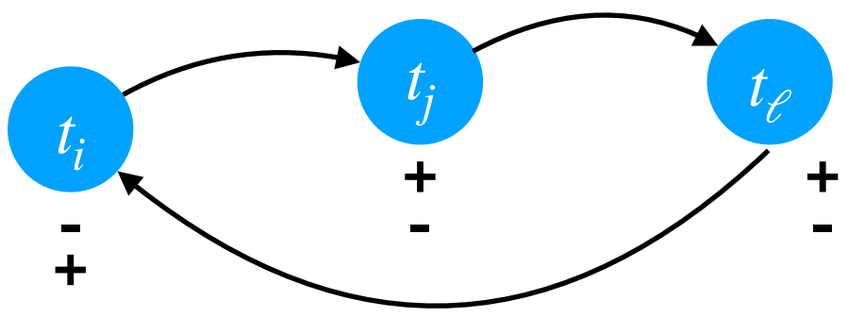


LHS = $X$

RHS = $X + r$

contradiction