

Algorithms and Data Structures

Introduction to Dynamic Programming and Chain Matrix
Multiplication

Dynamic Programming

An technique for solving **optimisation problems**.

Term attributed to Bellman (1950s).

“Programming” as in “Planning” or “Optimising”.

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Solve the subproblems from the smallest to the largest. When you solve a subproblem, **store the solution** (e.g., in an array) and use it to solve the larger subproblems.

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For that we will use the *standard* algorithm for matrix multiplication:

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RECTANGULAR-MATRIX-MULTIPLY( $A, B, C, p, q, r$ )
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1  for  $i = 1$  to  $p$ 
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2      for  $j = 1$  to  $r$ 
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3          for  $k = 1$  to  $q$ 
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The running time is the number of scalar multiplications, i.e., pqr .

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If instead we do $(A_1 \cdot (A_2 \cdot A_3))$ we have $100 \cdot 5 \cdot 50 = 25000$ scalar multiplications + $10 \cdot 100 \cdot 50 = 50000$ scalar multiplications for a total of 75000 .

Full Parenthesisation

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We can think of the input as a sequence of dimensions $\langle p_0, p_1, p_2, \dots, p_n \rangle$.

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If not, there is a cheaper parenthesisation of $A_i \cdot \dots \cdot A_k$. We can use that one in the optimal parenthesisation of $A_i \cdot \dots \cdot A_j$ instead to obtain an overall lower cost, *a contradiction*.

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Similarly for the the way to parenthesise the “suffix” $A_{k+1} \cdot \dots \cdot A_j$.

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We must consider all the possible splits, i.e., choices of k .

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Notice: $M[i, j]$ gives us the optimal costs, but not the optimal splits. To find the optimal splits, we define $S[i, j]$ to be the value k such that

$$M[i, j] = M[i, k] + M[k + 1, j] + p_{i-1}p_kp_j.$$

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We could now define a recursive algorithm straightforwardly using this.

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RECURSIVE-MATRIX-CHAIN( $p, i, j$ )
1  if  $i == j$ 
2      return 0
3   $m[i, j] = \infty$ 
4  for  $k = i$  to  $j - 1$ 
5       $q = \text{RECURSIVE-MATRIX-CHAIN}(p, i, k)$ 
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Running time?

The running time is given by the following recurrence relation:

$$T(n) \geq \begin{cases} 1, & \text{if } n = 1, \\ 1 + \sum_{k=1}^{n-1} (T(k) + T(n-k) + 1) & \text{if } n > 1. \end{cases}$$

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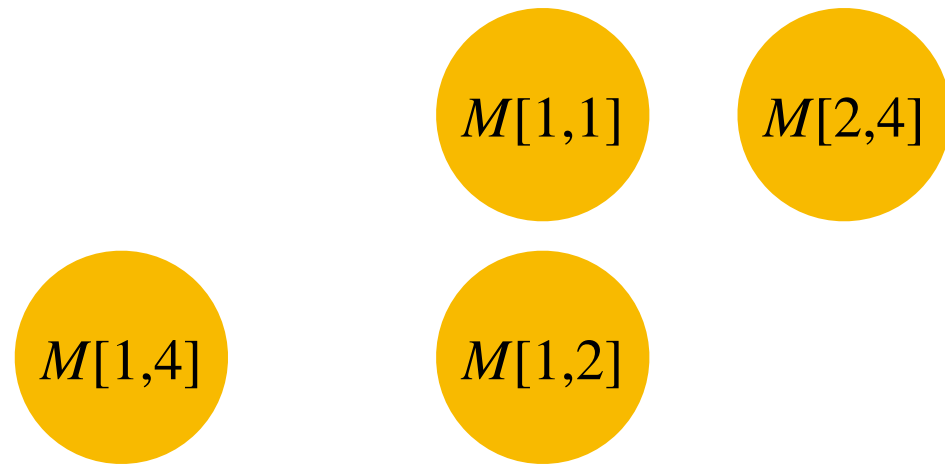
Computing the subproblems

$M[1,1]$

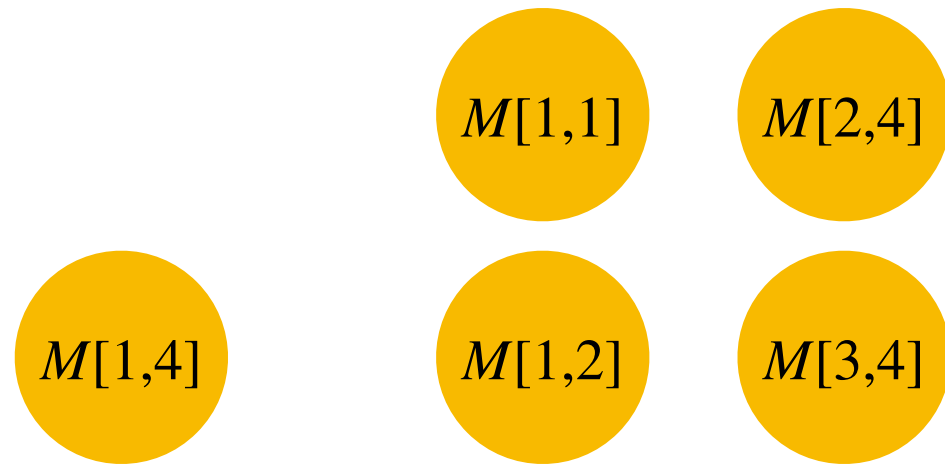
$M[2,4]$

$M[1,4]$

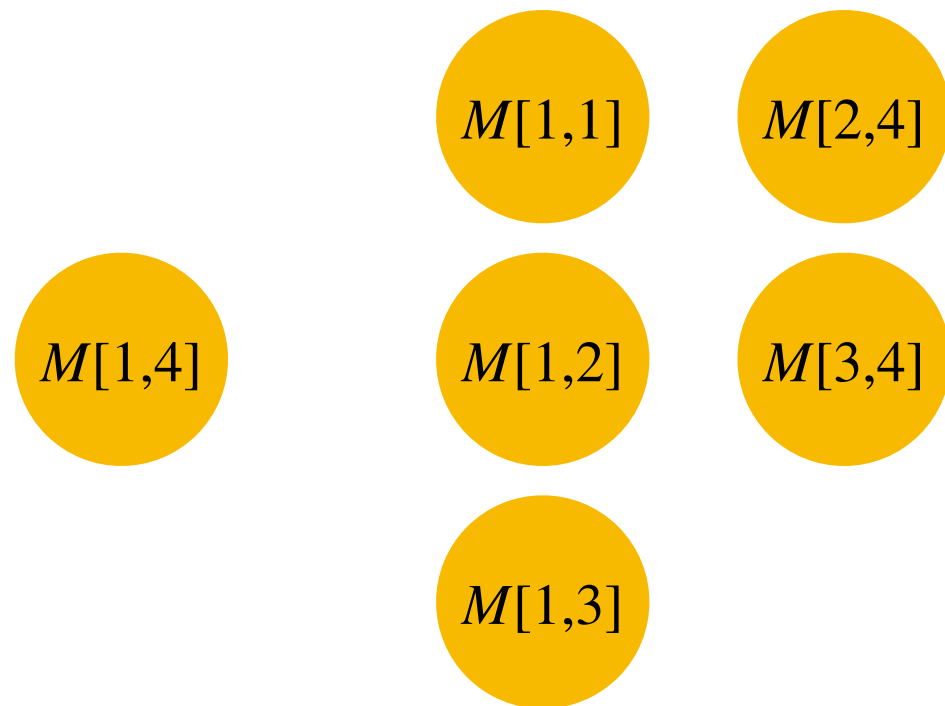
Computing the subproblems



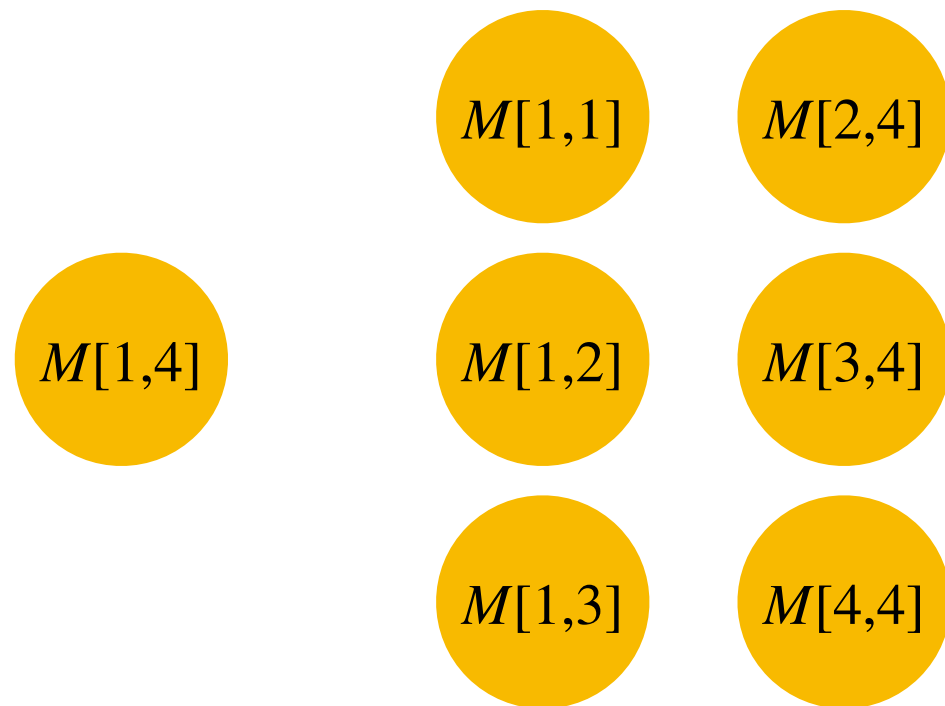
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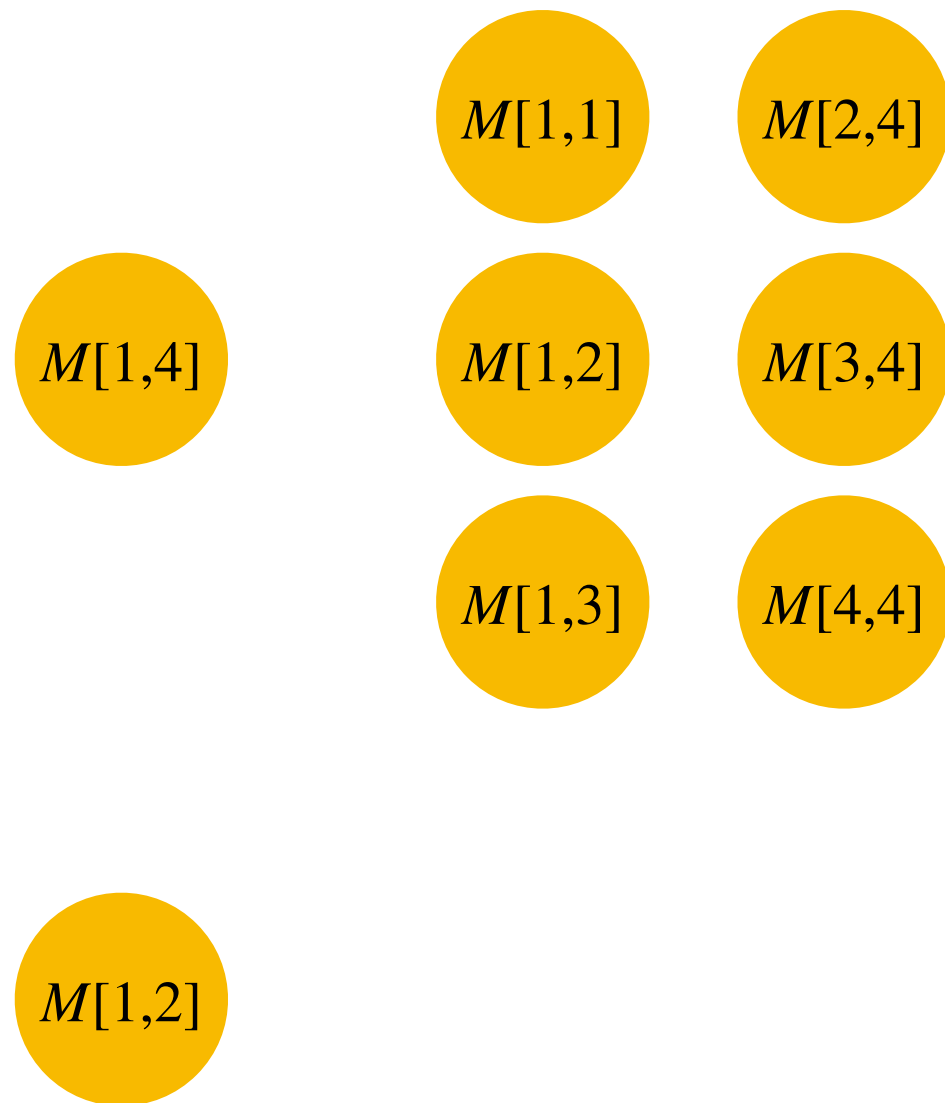
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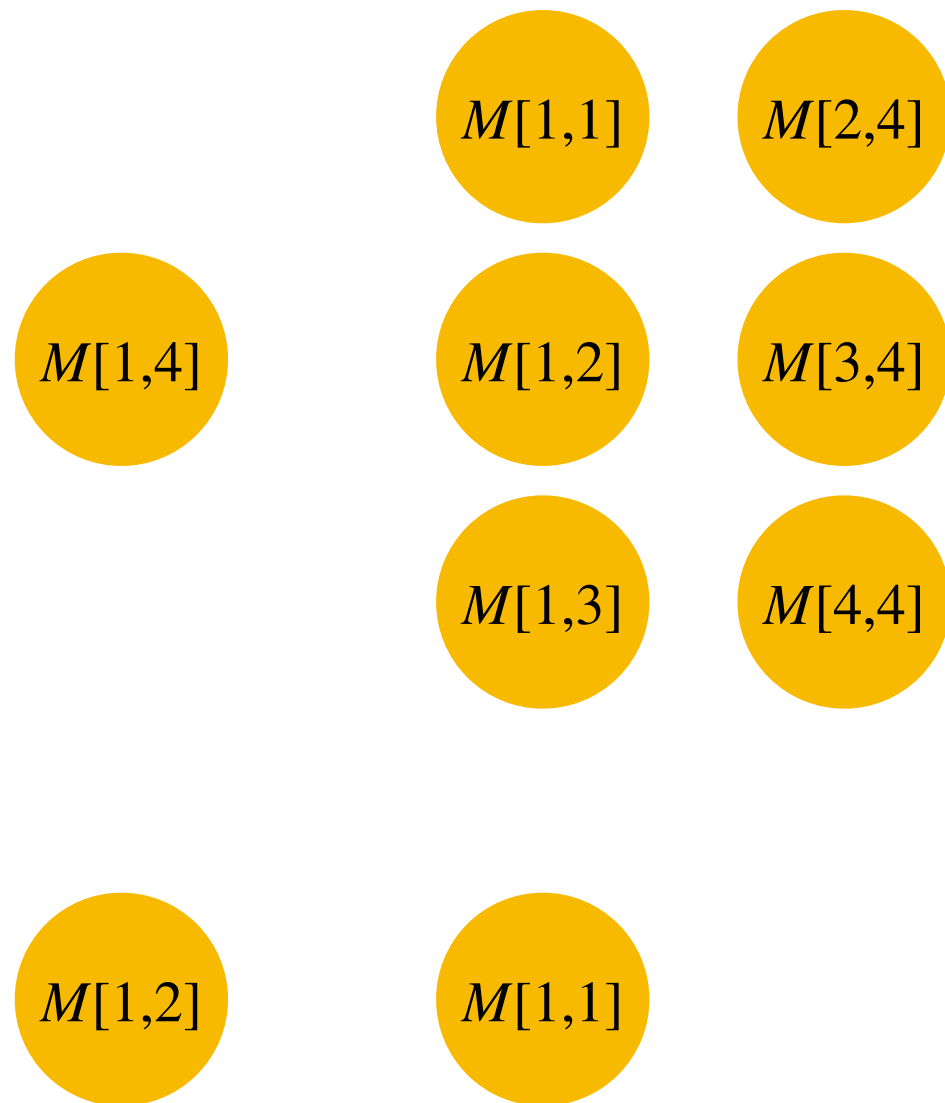
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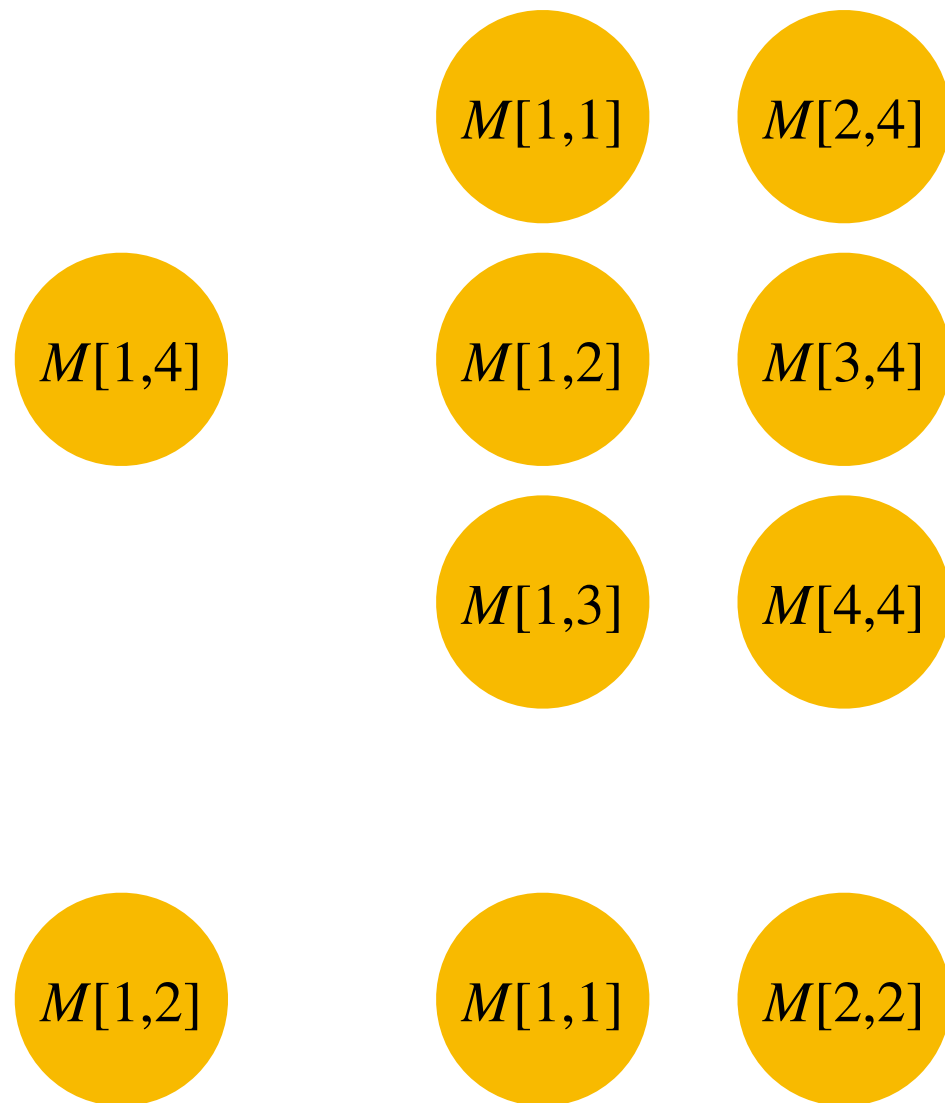
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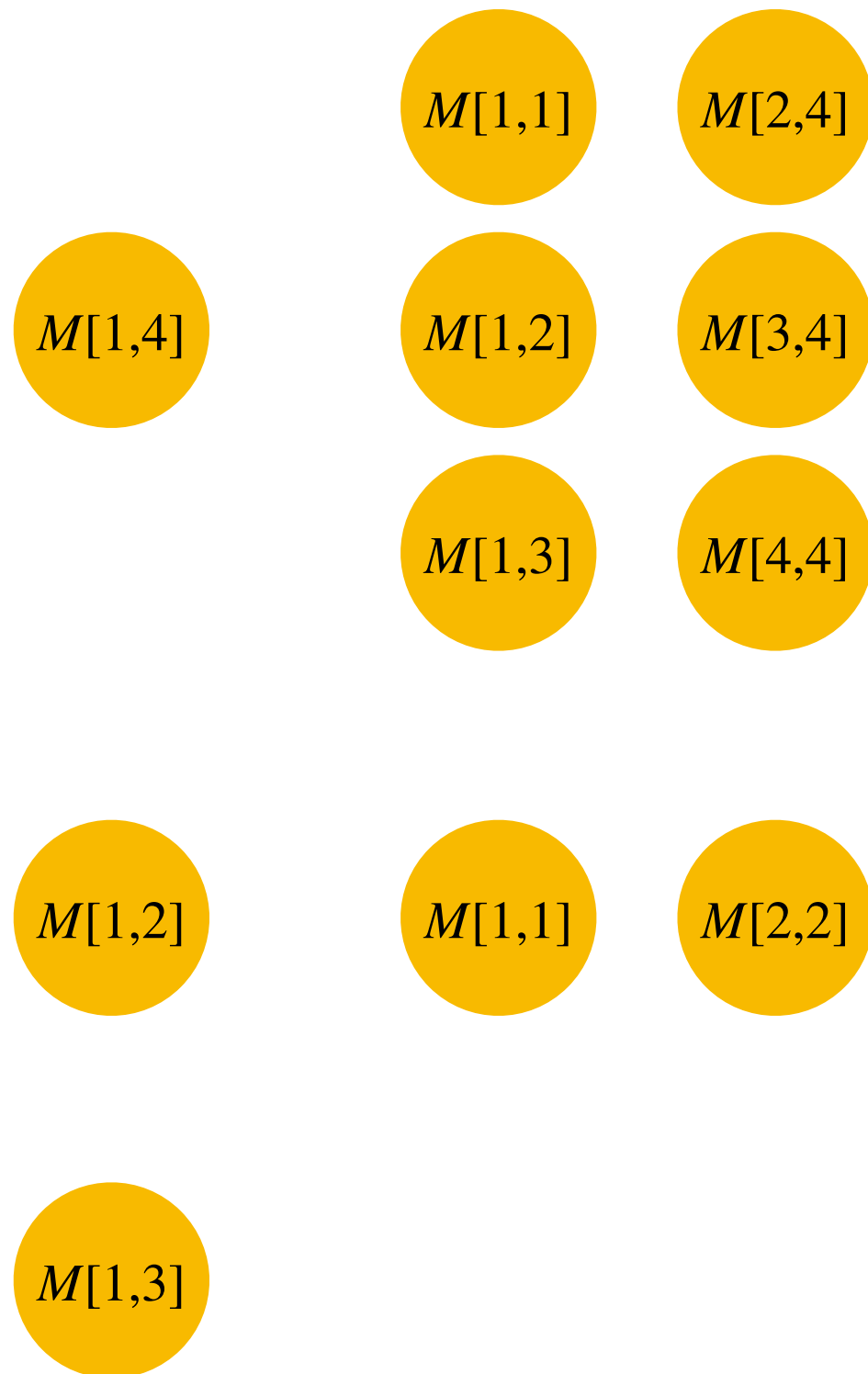
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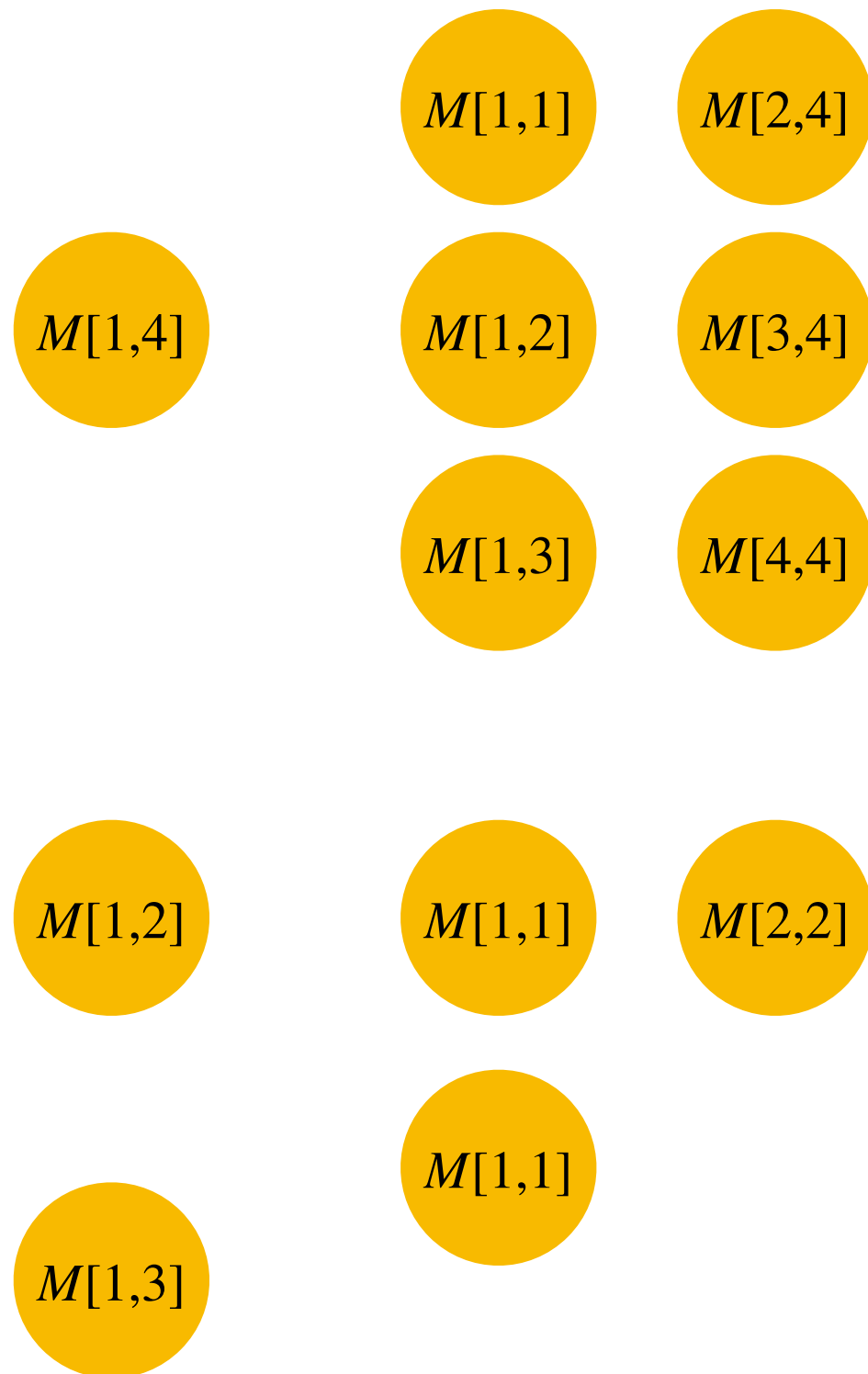
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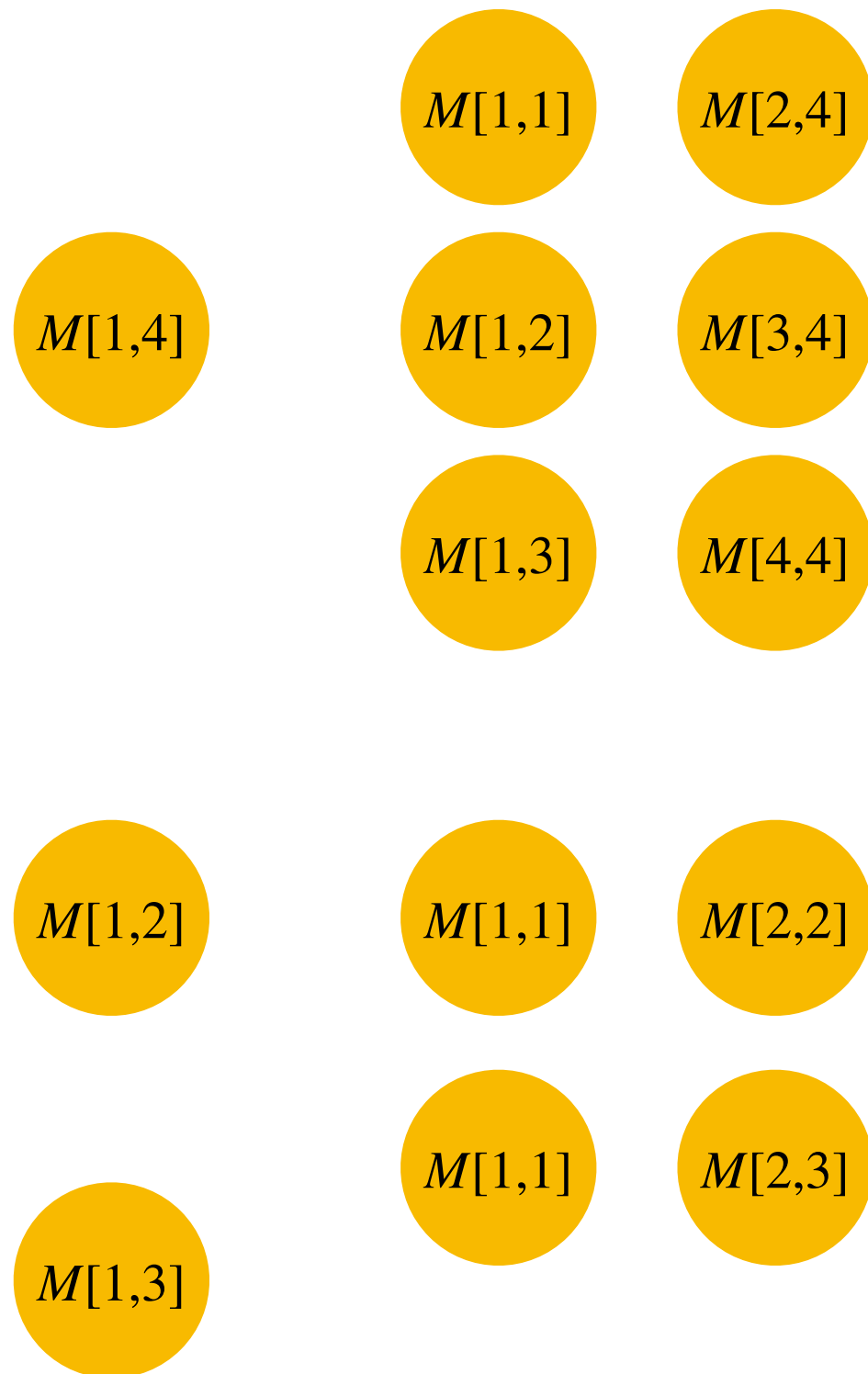
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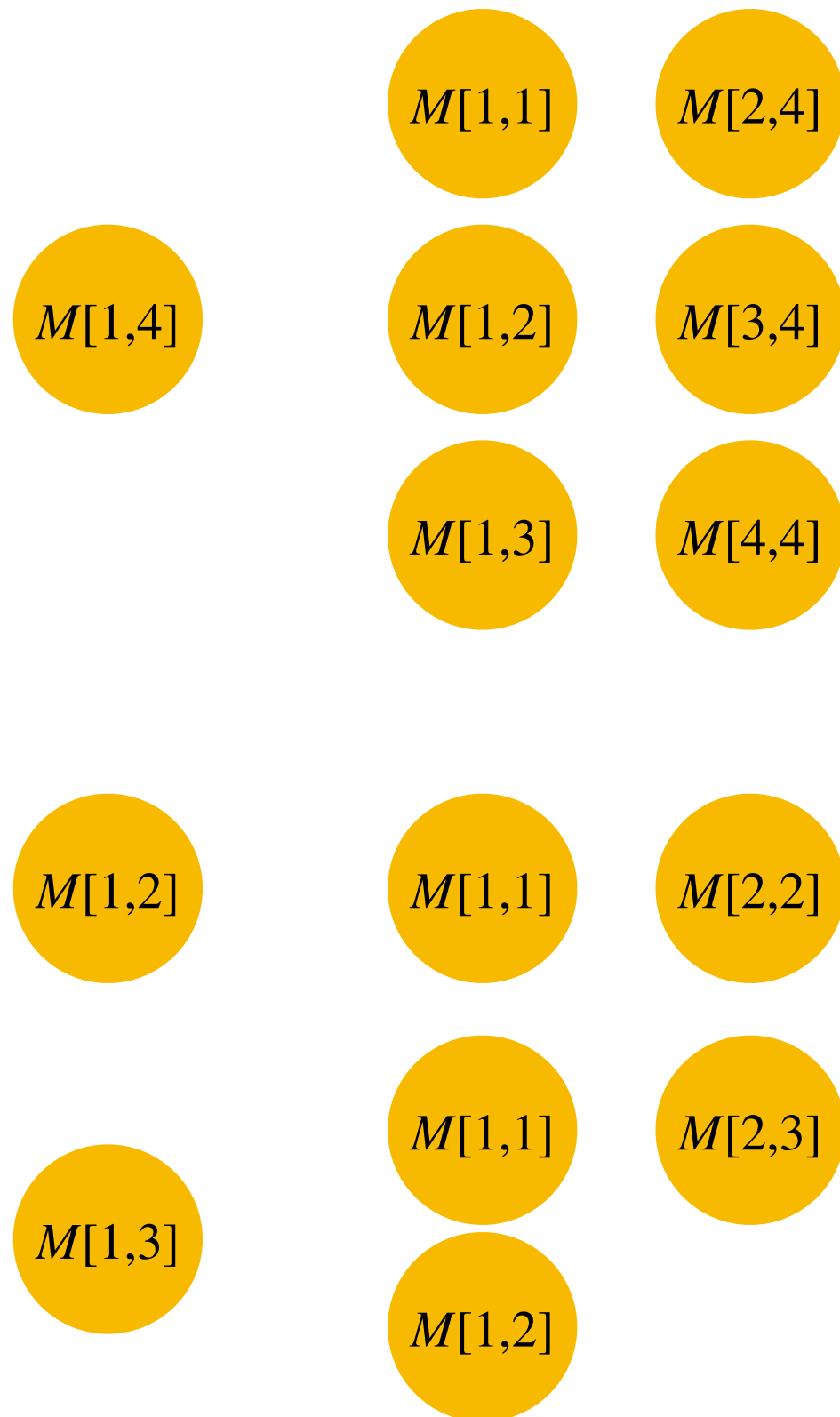
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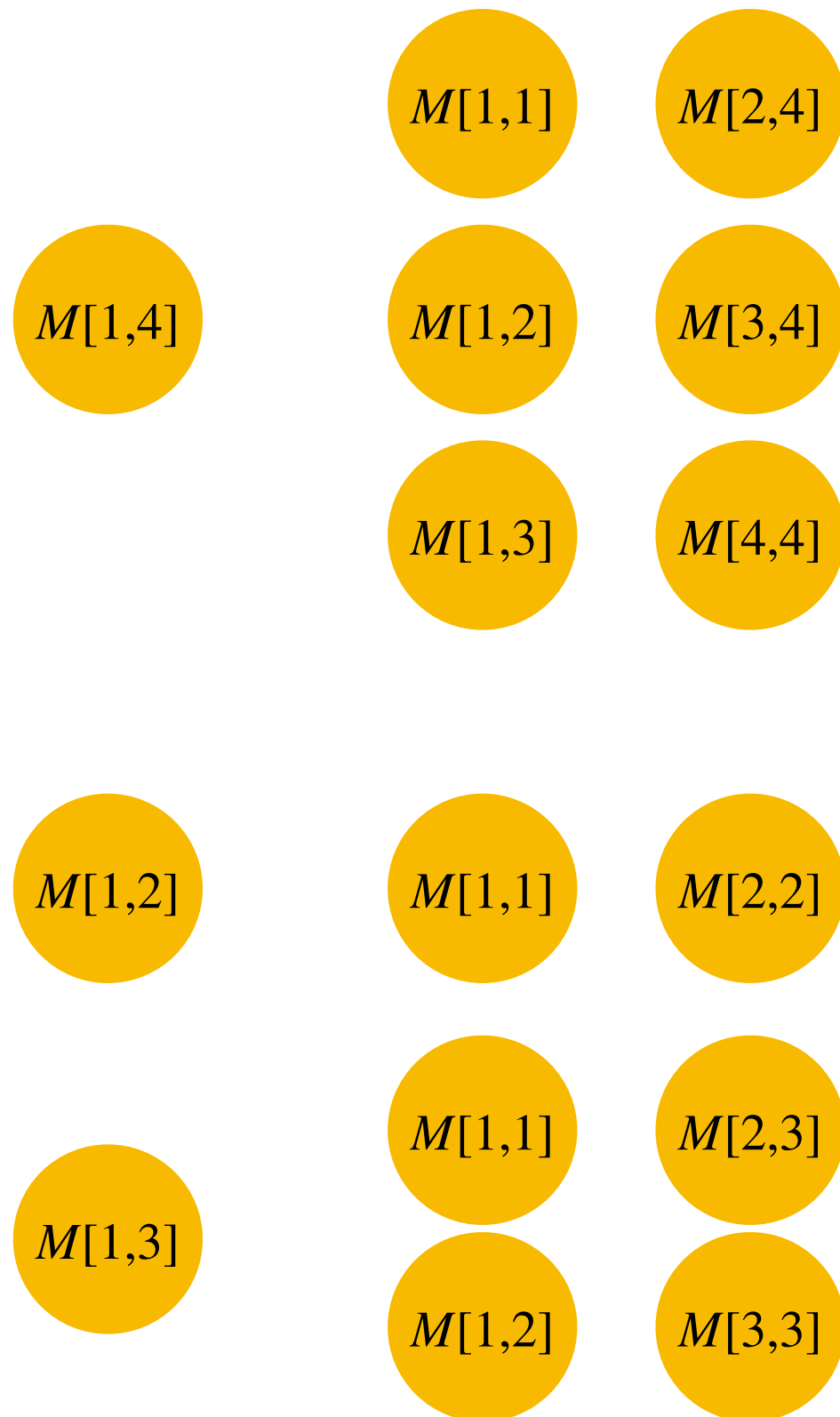
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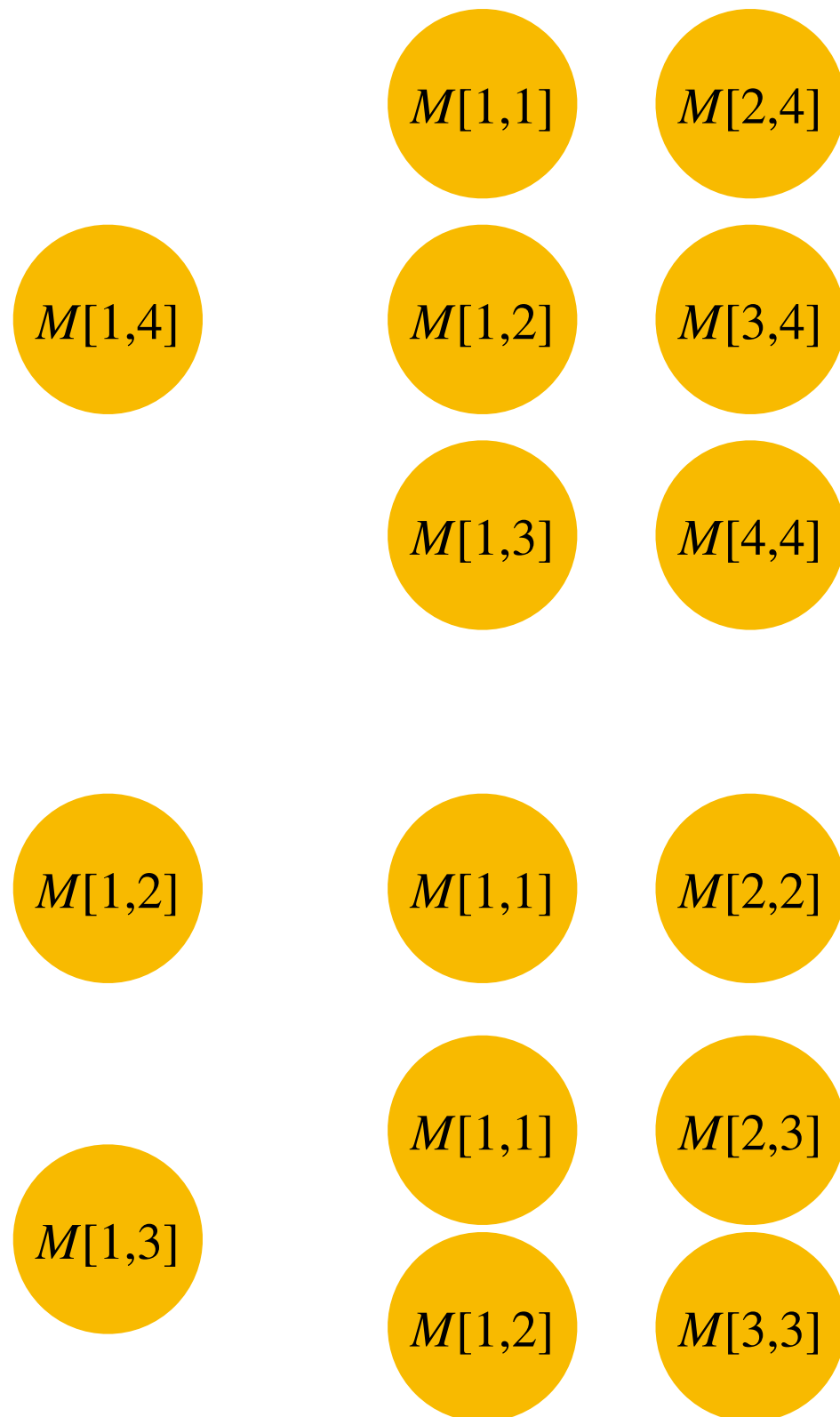
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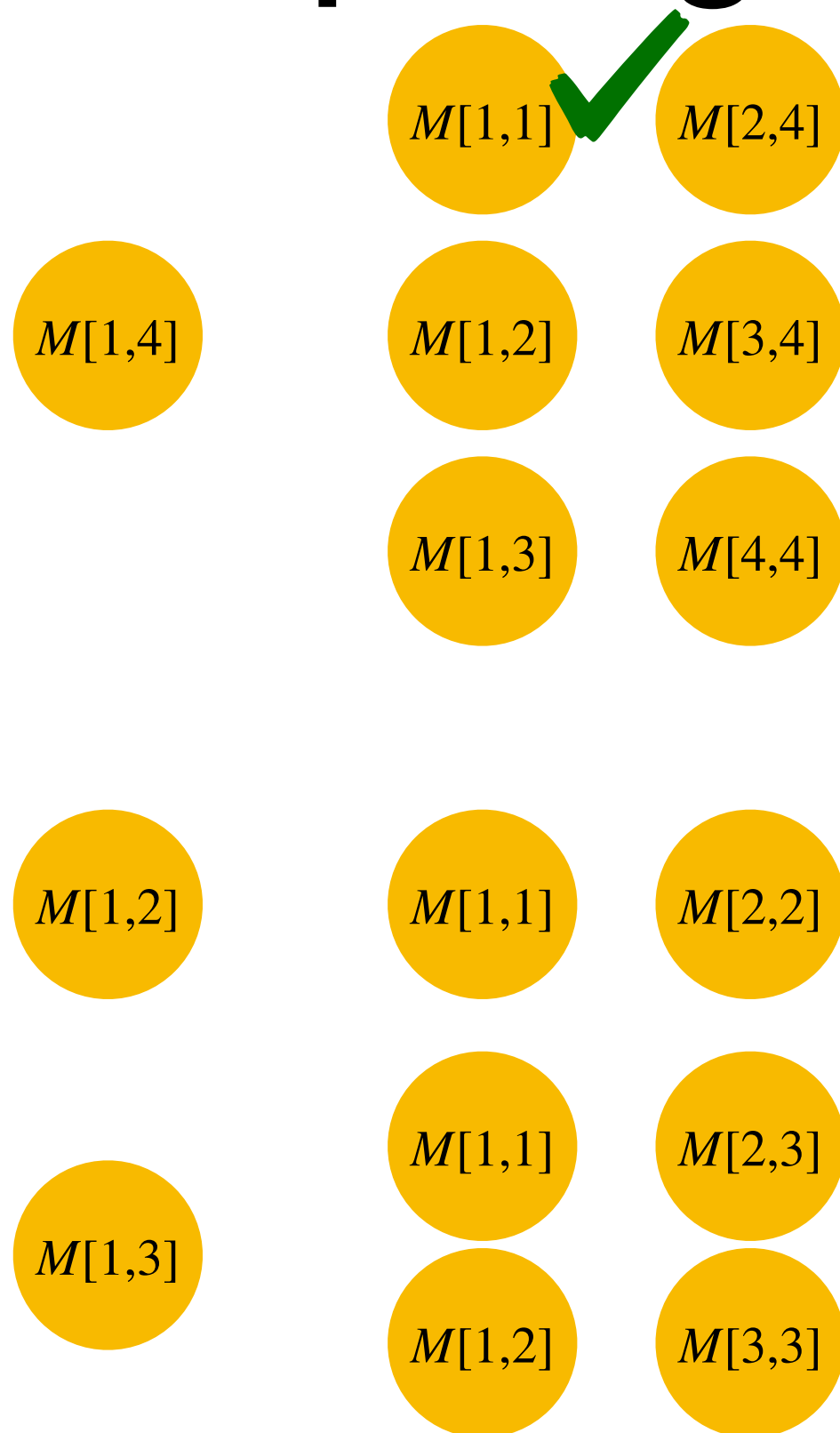


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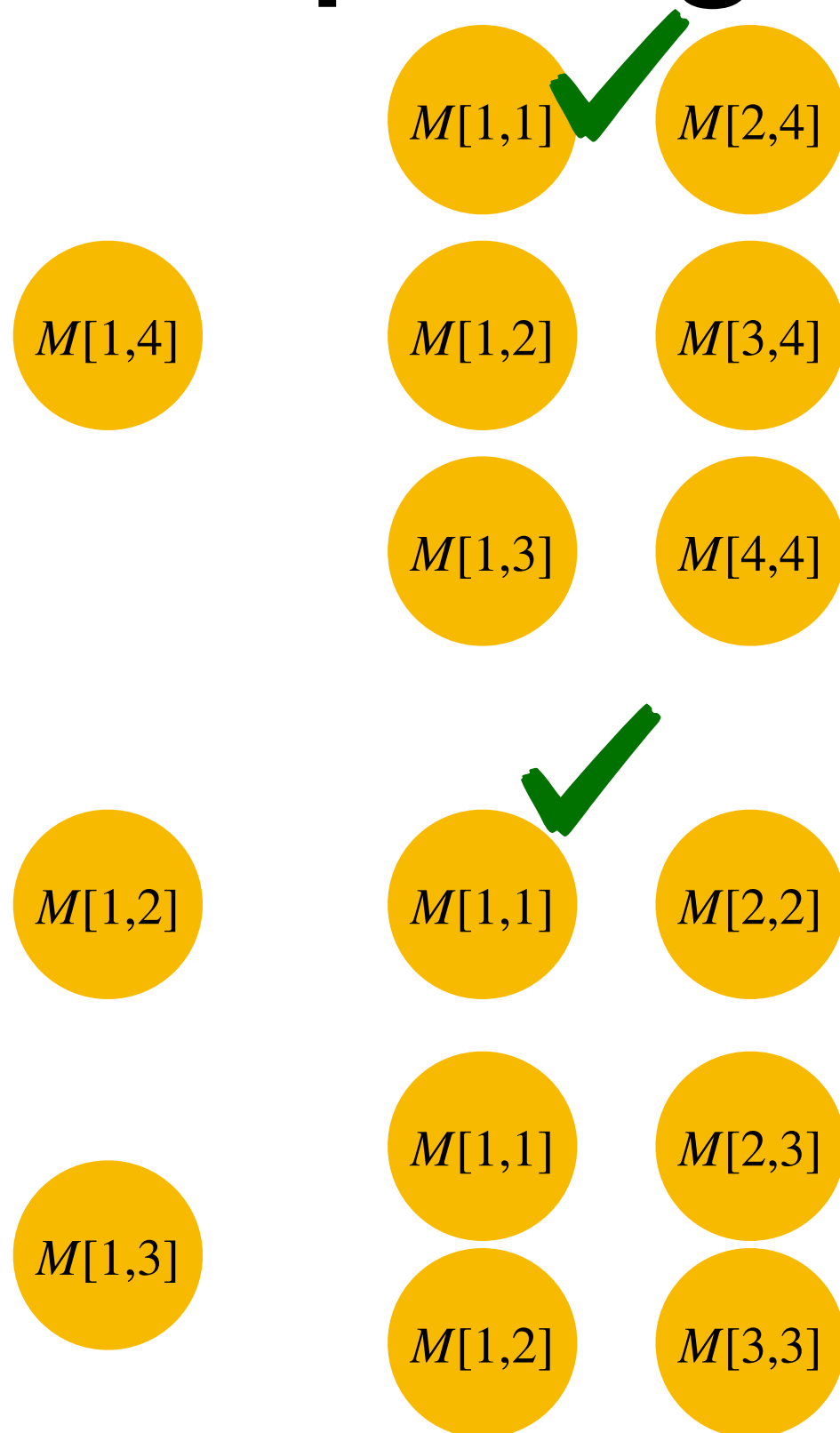
Do you observe something?

Computing the subproblems



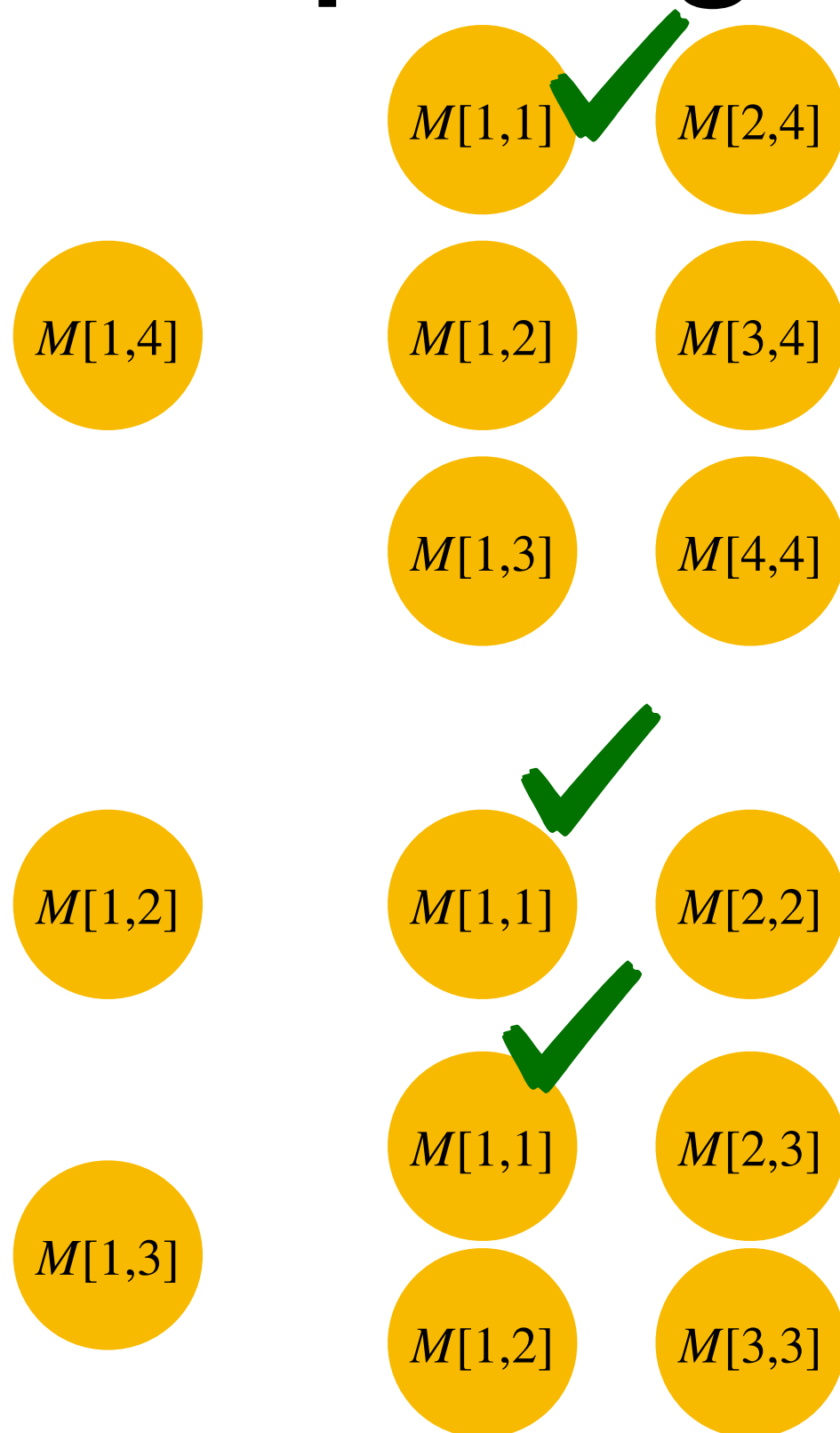
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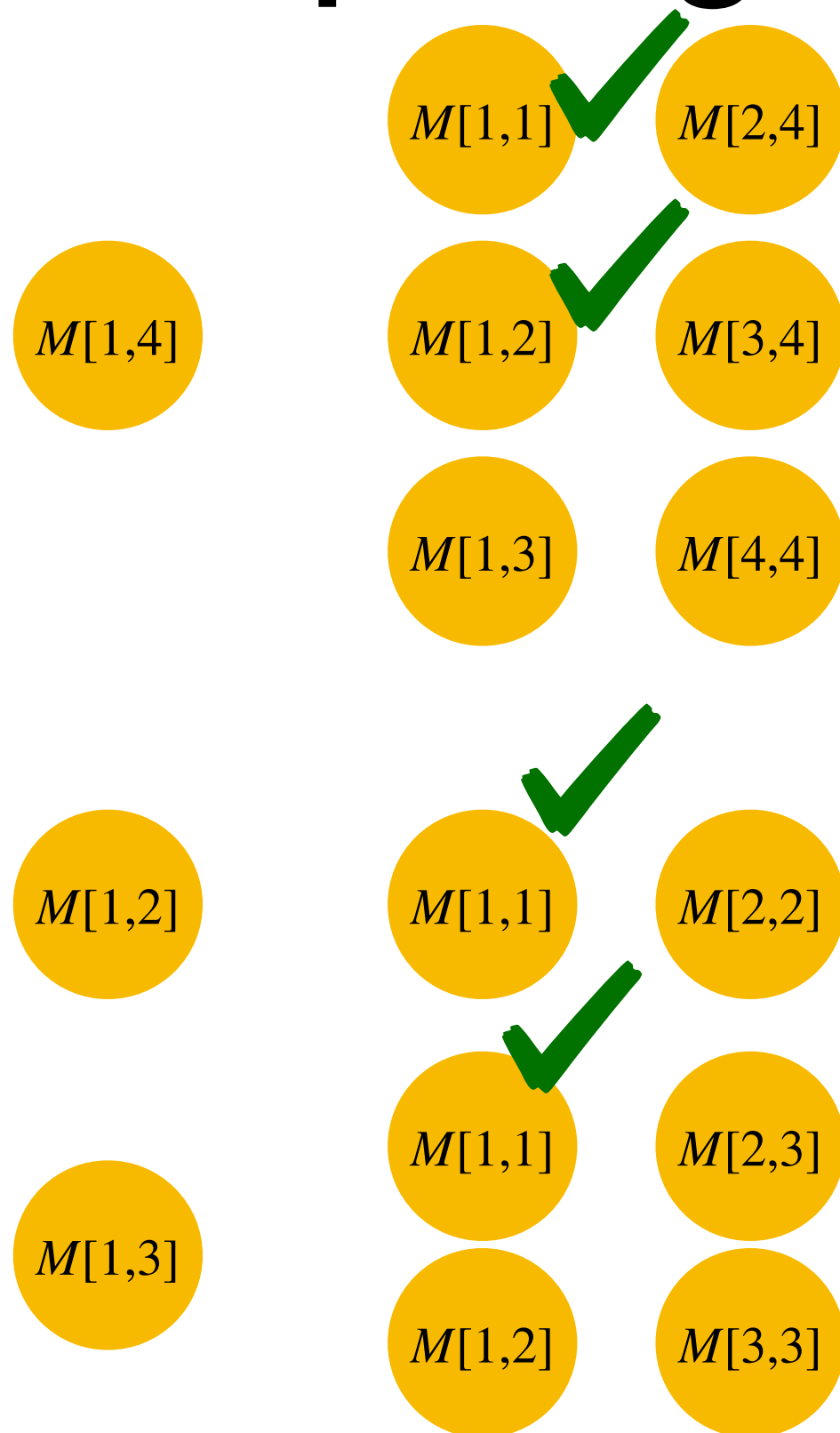
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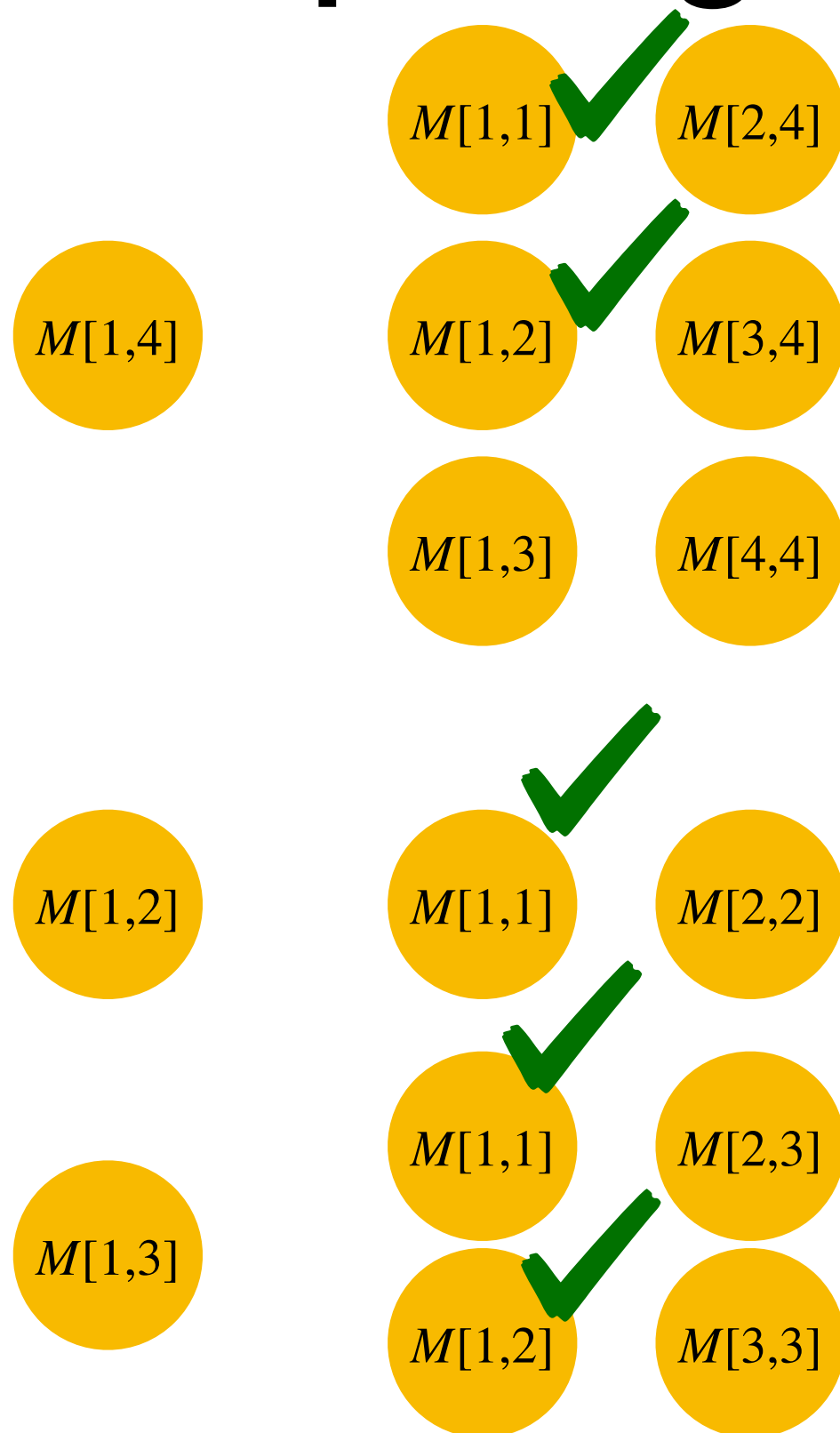
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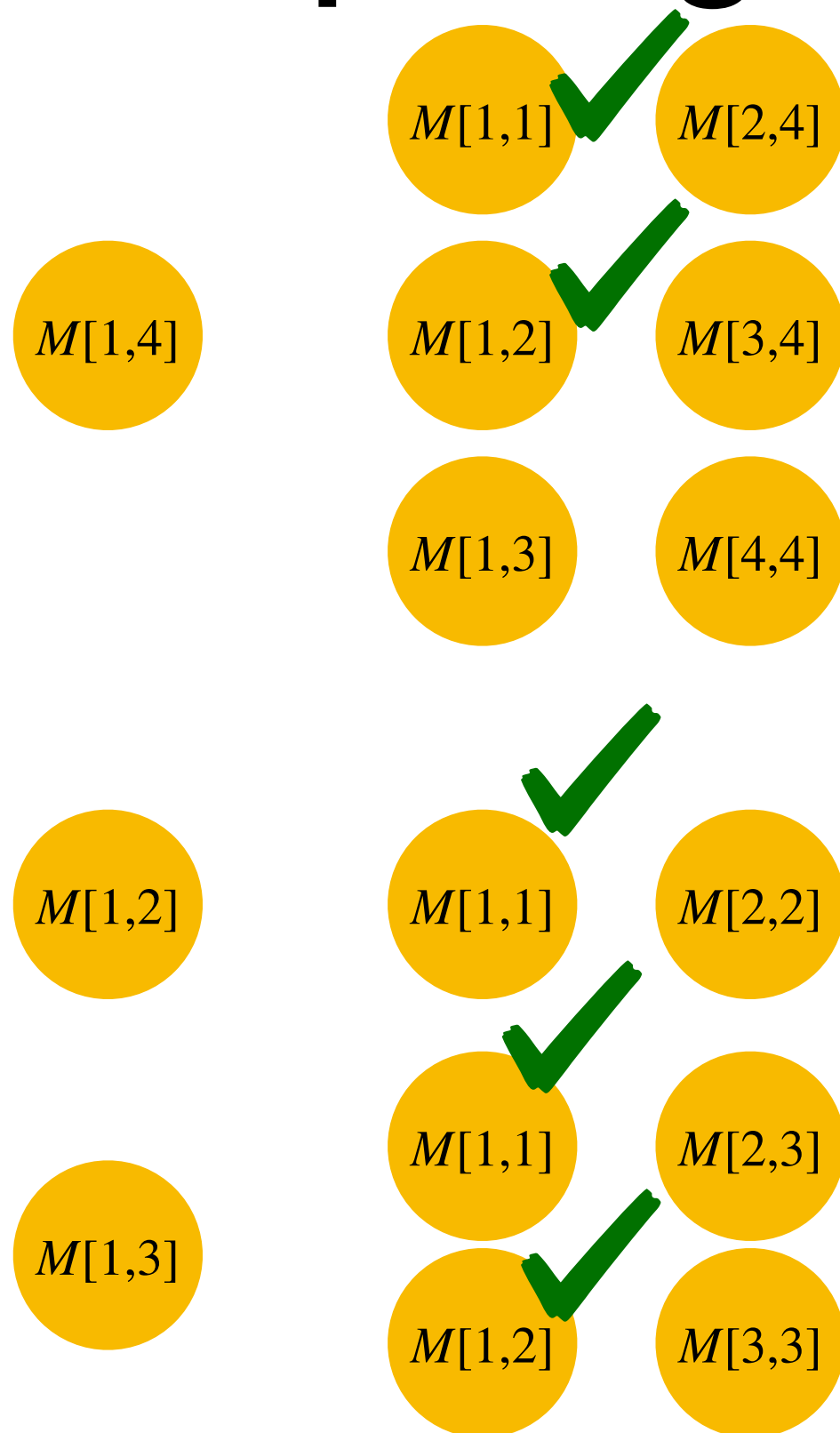
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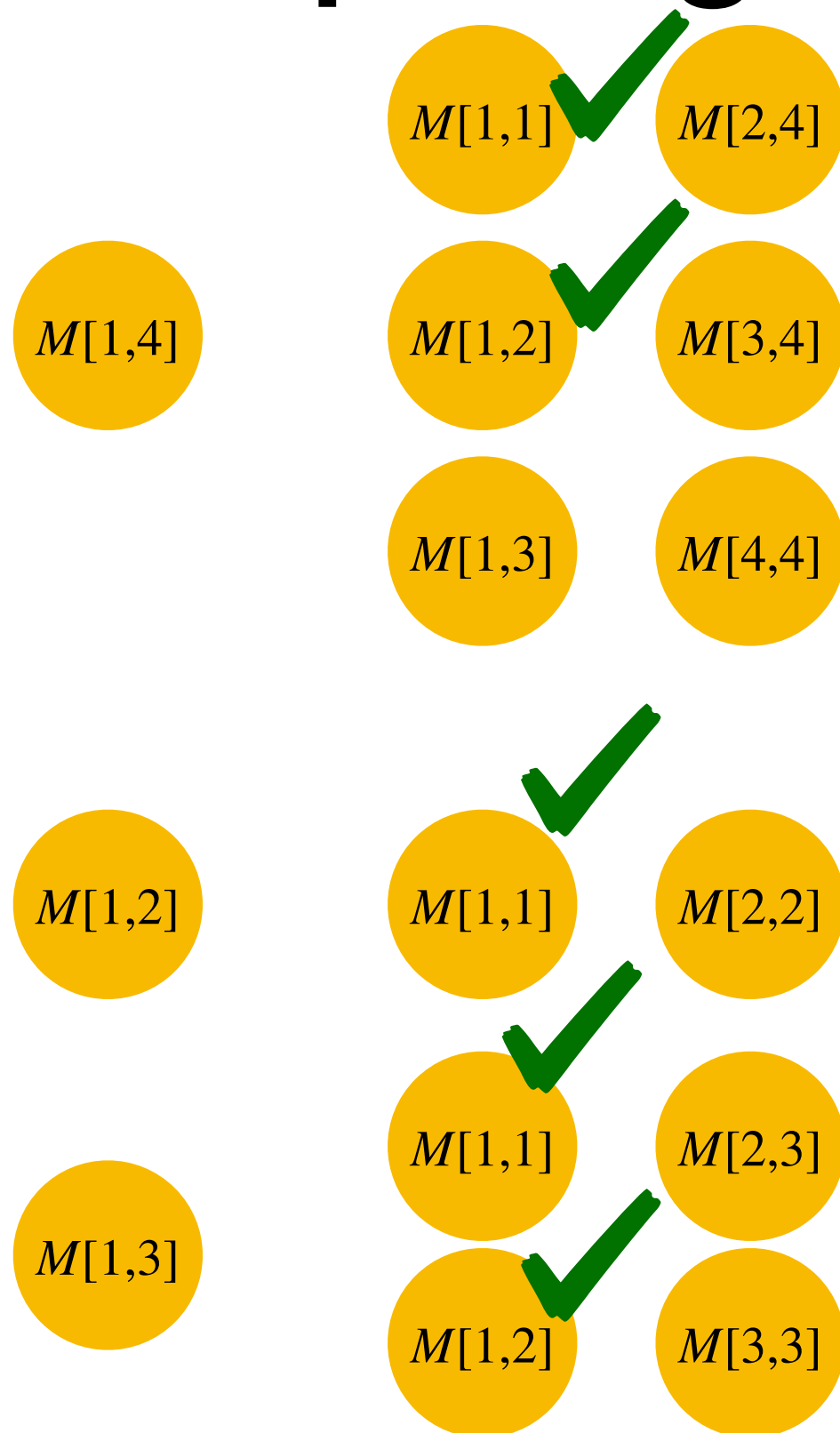
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How many distinct subproblems of the problem from i to j are there?

Computing the subproblems

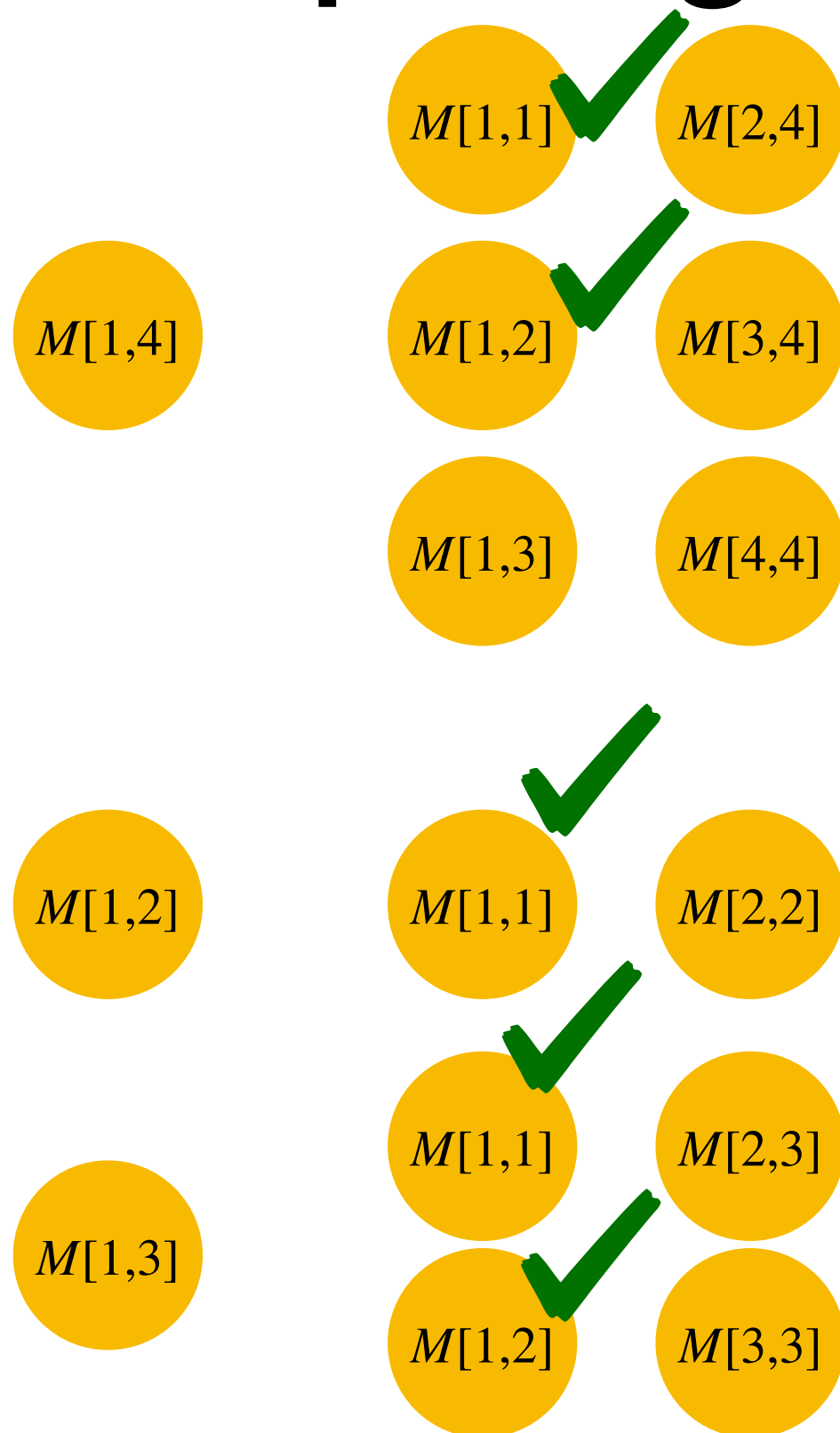


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Computing the subproblems



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The key is to *store* the calculation of the subproblems and reuse it.

A Dynamic Programming algorithm

A Dynamic Programming algorithm

To compute $M[i, j]$, we need the values of $M[i, k]$ and $M[k + 1, j]$ for all $i \leq k \leq j$.

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So we will have to compute those first.

We work in a *bottom-up manner*.

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$                                 // chain length 1
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Example

$$A_1 : 30 \times 35$$

$$A_2 : 35 \times 15$$

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The table M

0					
	0				
		0			
			0		
				0	
					0

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```

Example

$$A_1 : 30 \times 35$$

$$A_2 : 35 \times 15$$

$$A_3 : 15 \times 5$$

$$A_4 : 5 \times 10$$

$$A_5 : 10 \times 20$$

$$A_6 : 20 \times 25$$

Example

$$A_1 : 30 \times 35$$

$$p_0 p_1 p_2 = 30 \cdot 35 \cdot 15 = 15750$$

$$A_2 : 35 \times 15$$

$$A_3 : 15 \times 5$$

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The table M

0	15750				
	0				
		0			
			0		
				0	
					0

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```

Annotations:

- // chain length 1* (points to line 2)
- chains of length 1* (points to line 3)
- chains of length 2* (points to line 4)
- // l is the chain length* (points to line 4)
- // chain begins at A_i* (points to line 5)
- // chain ends at A_j* (points to line 6)
- // try $A_{i:k}A_{k+1:j}$* (points to line 8)
- // remember this cost* (points to line 11)
- // remember this index* (points to line 12)
- this goes up to 5* (points to line 5)

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 
```

Annotations:

- chains of length 1* (points to lines 2-3)
- chains of length 2* (points to line 4)
- l is the chain length* (points to line 4)
- chain begins at A_i* (points to line 5)
- chain ends at A_j* (points to line 6)
- try $A_{i:k}A_{k+1:j}$* (points to line 8)
- remember this cost* (points to line 11)
- remember this index* (points to line 12)
- this goes up to 5* (points to line 4)
- $i = 2$* (points to line 5)

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 
```

Annotations:

- chains of length 1* (points to line 3)
- chains of length 2* (points to line 4)
- // chain length 1* (points to line 2)
- // l is the chain length* (points to line 4)
- // chain begins at A_i* (points to line 5, $i = 2$)
- // chain ends at A_j* (points to line 6, $j = 3$)
- // try $A_{i:k}A_{k+1:j}$* (points to line 8)
- // remember this cost* (points to line 11)
- // remember this index* (points to line 12)
- this goes up to 5* (points to line 4)

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 
```

Annotations:

- // chain length 1* (points to line 2)
- chains of length 1* (points to line 3)
- chains of length 2* (points to line 4)
- // l is the chain length* (points to line 4)
- // chain begins at A_i* (points to line 5, $i = 2$)
- // chain ends at A_j* (points to line 6, $j = 3$)
- We are now computing $M[2,3]$* (points to line 7)
- // try $A_{i:k}A_{k+1:j}$* (points to line 8)
- // remember this cost* (points to line 11)
- // remember this index* (points to line 12)
- this goes up to 5* (points to line 4)

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 
```

Annotations:

- // chain length 1* (points to line 2)
- chains of length 1* (points to line 3)
- chains of length 2* (points to line 4)
- // l is the chain length* (points to line 4)
- // chain begins at A_i* (points to line 5, $i = 2$)
- // chain ends at A_j* (points to line 6, $j = 3$)
- We are now computing $M[2,3]$* (points to line 7)
- // try $A_{i:k}A_{k+1:j}$* (points to line 8, $k = 2$)
- // remember this cost* (points to line 11)
- // remember this index* (points to line 12)
- this goes up to 5* (points to line 4)

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 
```

Annotations:

- this goes up to 5* (points to line 4)
- // chain length 1* (points to line 2)
- chains of length 1* (points to line 3)
- chains of length 2* (points to line 4)
- // l is the chain length* (points to line 4)
- // chain begins at A_i* (points to line 5, $i = 2$)
- // chain ends at A_j* (points to line 6, $j = 3$)
- We are now computing $M[2,3]$* (points to line 7)
- // try $A_{i:k}A_{k+1:j}$* (points to line 8, $k = 2$)
- $M[2,2] + M[3,3] + p_1p_2p_3$* (points to line 9)
- // remember this cost* (points to line 11)
- // remember this index* (points to line 12)

Example

$$A_1 : 30 \times 35$$

$$A_2 : 35 \times 15$$

$$A_3 : 15 \times 5$$

$$A_4 : 5 \times 10$$

$$A_5 : 10 \times 20$$

$$A_6 : 20 \times 25$$

Example

$$A_1 : 30 \times 35$$

$$p_1 p_2 p_3 = 35 \cdot 15 \cdot 5 = 2625$$

$$A_2 : 35 \times 15$$

$$A_3 : 15 \times 5$$

$$A_4 : 5 \times 10$$

$$A_5 : 10 \times 20$$

$$A_6 : 20 \times 25$$

The table M

0	15750				
	0	2625			
		0			
			0		
				0	
					0

The table M

0	15750				
	0	2625			
		0	750		
			0	1000	
				0	5000
					0

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$                                 // chain length 1
3       $m[i, i] = 0$  } chains of length 1
4  for  $l = 2$  to  $n$                                 //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$                     // chain begins at  $A_i$ 
6           $j = i + l - 1$                         // chain ends at  $A_j$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$                     // try  $A_{i:k} A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$                 // remember this cost
12                  $s[i, j] = k$                 // remember this index
13  return  $m$  and  $s$ 
```

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$  } chains of length 1 // chain length 1
4  for  $l = 2$  to  $n$  chains of length 3 //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$  // chain begins at  $A_i$ 
6           $j = i + l - 1$  // chain ends at  $A_j$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$  // try  $A_{i:k} A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$  // remember this cost
12                  $s[i, j] = k$  // remember this index
13  return  $m$  and  $s$ 
```

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$  } chains of length 1 // chain length 1
4  for  $l = 2$  to  $n$  chains of length 3 //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$  // chain begins at  $A_i$   $i = 1$ 
6           $j = i + l - 1$  // chain ends at  $A_j$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$  // try  $A_{i:k} A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$  // remember this cost
12                  $s[i, j] = k$  // remember this index
13  return  $m$  and  $s$ 
```

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$                                 // chain length 1
3       $m[i, i] = 0$  } chains of length 1
4  for  $l = 2$  to  $n$  chains of length 3           //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$                    // chain begins at  $A_i$             $i = 1$ 
6           $j = i + l - 1$                        // chain ends at  $A_j$             $j = 3$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$                    // try  $A_{i:k} A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$                  // remember this cost
12                  $s[i, j] = k$                  // remember this index
13  return  $m$  and  $s$ 
```


A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$  } chains of length 1 // chain length 1
4  for  $l = 2$  to  $n$  chains of length 3 //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$  // chain begins at  $A_i$   $i = 1$ 
6           $j = i + l - 1$  // chain ends at  $A_j$   $j = 3$ 
7           $m[i, j] = \infty$  // We are now computing  $M[1,3]$ 
8          for  $k = i$  to  $j - 1$  // try  $A_{i:k}A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$  // remember this cost
12                  $s[i, j] = k$  // remember this index
13  return  $m$  and  $s$ 
```


A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$  } chains of length 1 // chain length 1
4  for  $l = 2$  to  $n$  chains of length 3 //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$  // chain begins at  $A_i$   $i = 1$ 
6           $j = i + l - 1$  // chain ends at  $A_j$   $j = 3$ 
7           $m[i, j] = \infty$  // We are now computing  $M[1,3]$ 
8          for  $k = i$  to  $j - 1$  // try  $A_{i:k}A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$  // remember this cost
12                  $s[i, j] = k$  // remember this index
13  return  $m$  and  $s$ 
```

$k = 1$

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- // chain length 1* (next to line 2)
- chains of length 1* (next to line 3)
- // l is the chain length* (next to line 4)
- // chain begins at A_i $i = 1$* (next to line 5)
- // chain ends at A_j $j = 3$* (next to line 6)
- We are now computing $M[1,3]$* (next to line 7)
- // try $A_{i:k}A_{k+1:j}$* (next to line 8)
- // remember this cost* (next to line 11)
- // remember this index* (next to line 12)

$$k = 1 \quad M[1,1] + M[2,3] + p_0p_1p_3 = 0 + 2625 + 5250 = 7875$$

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- // chain length 1* (next to line 2)
- chains of length 1* (next to line 3)
- chains of length 3* (next to line 4)
- // l is the chain length* (next to line 4)
- // chain begins at A_i $i = 1$* (next to line 5)
- // chain ends at A_j $j = 3$* (next to line 6)
- We are now computing $M[1,3]$* (next to line 7)
- // try $A_{i:k}A_{k+1:j}$* (next to line 8)
- // remember this cost* (next to line 11)
- // remember this index* (next to line 12)

$$k = 1 \quad M[1,1] + M[2,3] + p_0p_1p_3 = 0 + 2625 + 5250 = 7875$$

$$k = 2$$

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- // chain length 1* (next to line 2)
- chains of length 1* (next to line 3)
- chains of length 3* (next to line 4)
- // l is the chain length* (next to line 4)
- // chain begins at A_i $i = 1$* (next to line 5)
- // chain ends at A_j $j = 3$* (next to line 6)
- We are now computing $M[1,3]$* (next to line 7)
- // try $A_{i:k}A_{k+1:j}$* (next to line 8)
- // remember this cost* (next to line 11)
- // remember this index* (next to line 12)

$$k = 1 \quad M[1,1] + M[2,3] + p_0p_1p_3 = 0 + 2625 + 5250 = 7875$$

$$k = 2 \quad M[1,2] + M[3,3] + p_0p_2p_3 = 15750 + 0 + 2250 = 18000$$

The table M

0	15750	7875			
	0	2625			
		0	750		
			0	1000	
				0	5000
					0

The table M

0	15750	7875	9375		
	0	2625	4375		
		0	750	2500	
			0	1000	3500
				0	5000
					0

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$                                 // chain length 1
3       $m[i, i] = 0$  } chains of length 1
4  for  $l = 2$  to  $n$                                 //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$                     // chain begins at  $A_i$ 
6           $j = i + l - 1$                         // chain ends at  $A_j$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$                     // try  $A_{i:k}A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$                 // remember this cost
12                  $s[i, j] = k$                 // remember this index
13  return  $m$  and  $s$ 
```


A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$                                 // chain length 1
3       $m[i, i] = 0$  } chains of length 1
4  for  $l = 2$  to  $n$  chains of length  $l$            //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$                      // chain begins at  $A_i$ 
6           $j = i + l - 1$                          // chain ends at  $A_j$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$                      // try  $A_{i:k} A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$                      // remember this cost
12                  $s[i, j] = k$                      // remember this index
13  return  $m$  and  $s$ 
```


A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$  } chains of length 1 // chain length 1
4  for  $l = 2$  to  $n$  chains of length  $l$  //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$  // chain begins at  $A_i$   $i = 2$ 
6           $j = i + l - 1$  // chain ends at  $A_j$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$  // try  $A_{i:k} A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$  // remember this cost
12                  $s[i, j] = k$  // remember this index
13  return  $m$  and  $s$ 
```

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$  } chains of length 1 // chain length 1
4  for  $l = 2$  to  $n$  chains of length  $l$  //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$  // chain begins at  $A_i$   $i = 2$ 
6           $j = i + l - 1$  // chain ends at  $A_j$   $j = 5$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$  // try  $A_{i:k} A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$  // remember this cost
12                  $s[i, j] = k$  // remember this index
13  return  $m$  and  $s$ 
```

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$  } chains of length 1 // chain length 1
4  for  $l = 2$  to  $n$  chains of length  $l$  //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$  // chain begins at  $A_i$   $i = 2$ 
6           $j = i + l - 1$  // chain ends at  $A_j$   $j = 5$ 
7           $m[i, j] = \infty$  // We are now computing  $M[2,5]$ 
8          for  $k = i$  to  $j - 1$  // try  $A_{i:k}A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$  // remember this cost
12                  $s[i, j] = k$  // remember this index
13  return  $m$  and  $s$ 
```

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```
1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$  } chains of length 1 // chain length 1
4  for  $l = 2$  to  $n$  chains of length  $l$  //  $l$  is the chain length
5      for  $i = 1$  to  $n - l + 1$  // chain begins at  $A_i$   $i = 2$ 
6           $j = i + l - 1$  // chain ends at  $A_j$   $j = 5$ 
7           $m[i, j] = \infty$  // We are now computing  $M[2,5]$ 
8          for  $k = i$  to  $j - 1$  // try  $A_{i:k}A_{k+1:j}$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$  // remember this cost
12                  $s[i, j] = k$  // remember this index
13  return  $m$  and  $s$ 
```

$k = 2$

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- // chain length 1* (next to line 2)
- chains of length 1* (next to line 3)
- // l is the chain length* (next to line 4)
- // chain begins at A_i* (next to line 5, $i = 2$)
- // chain ends at A_j* (next to line 6, $j = 5$)
- We are now computing $M[2,5]$* (next to line 7)
- // try $A_{i:k}A_{k+1:j}$* (next to line 8)
- // remember this cost* (next to line 11)
- // remember this index* (next to line 12)

$$k = 2 \quad M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$$

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- // chain length 1* (next to line 2)
- chains of length 1* (next to line 3)
- // l is the chain length* (next to line 4)
- // chain begins at A_i* (next to line 5, $i = 2$)
- // chain ends at A_j* (next to line 6, $j = 5$)
- We are now computing $M[2,5]$* (next to line 7)
- // try $A_{i:k}A_{k+1:j}$* (next to line 8)
- // remember this cost* (next to line 11)
- // remember this index* (next to line 12)

$$k = 2 \quad M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$$

$$k = 3$$

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- // chain length 1* (next to line 2)
- chains of length 1* (next to line 3)
- // l is the chain length* (next to line 4)
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- // try $A_{i:k}A_{k+1:j}$* (next to line 8)
- // remember this cost* (next to line 11)
- // remember this index* (next to line 12)

$$k = 2 \quad M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$$

$$k = 3 \quad M[2,3] + M[4,5] + p_1p_3p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$$

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- // chain length 1* (next to line 2)
- chains of length 1* (next to line 3)
- chains of length 4* (next to line 4)
- // l is the chain length* (next to line 4)
- // chain begins at A_i* (next to line 5, $i = 2$)
- // chain ends at A_j* (next to line 6, $j = 5$)
- We are now computing $M[2,5]$* (next to line 7)
- // try $A_{i:k}A_{k+1:j}$* (next to line 8)
- // remember this cost* (next to line 11)
- // remember this index* (next to line 12)

$$k = 2 \quad M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$$

$$k = 3 \quad M[2,3] + M[4,5] + p_1p_3p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$$

$$k = 4$$

A Dynamic Programming algorithm

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- chain length 1** (for $l=1$)
- chains of length 1** (for $i=j$)
- chains of length 4** (for $l=4$)
- l is the chain length**
- chain begins at A_i** ($i=2$)
- chain ends at A_j** ($j=5$)
- We are now computing $M[2,5]$**
- try $A_{i:k}A_{k+1:j}$**
- remember this cost**
- remember this index**

$$k = 2 \quad M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$$

$$k = 3 \quad M[2,3] + M[4,5] + p_1p_3p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$$

$$k = 4 \quad M[2,4] + M[5,5] + p_1p_4p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$$

The table M

0	15750	7875	9375		
	0	2625	4375	7125	
		0	750	2500	
			0	1000	3500
				0	5000
					0

The table M

0	15750	7875	9375	11875	15125
	0	2625	4375	7125	10500
		0	750	2500	5375
			0	1000	3500
				0	5000
					0

The table M

0	15750	7875	9375	11875	15125	optimal cost
	0	2625	4375	7125	10500	
		0	750	2500	5375	
			0	1000	3500	
				0	5000	
					0	

Computing the optimal solution

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- // chain length 1* (next to line 2)
- chains of length 1* (next to line 3)
- chains of length 4* (next to line 4)
- // l is the chain length* (next to line 4)
- // chain begins at A_i* (next to line 5, $i = 2$)
- // chain ends at A_j* (next to line 6, $j = 5$)
- We are now computing $M[2,5]$* (next to line 7)
- // try $A_{i:k}A_{k+1:j}$* (next to line 8)
- // remember this cost* (next to line 11)
- // remember this index* (next to line 12)

$$k = 2 \quad M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$$

$$k = 3 \quad M[2,3] + M[4,5] + p_1p_3p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125$$

$$k = 4 \quad M[2,4] + M[5,5] + p_1p_4p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$$

Computing the optimal solution

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- Line 2: **for $i = 1$ to n** } chains of length 1
- Line 3: $m[i, i] = 0$ // chain length 1
- Line 4: **for $l = 2$ to n** // l is the chain length
- Line 5: **for $i = 1$ to $n - l + 1$** // chain begins at A_i $i = 2$
- Line 6: $j = i + l - 1$ // chain ends at A_j $j = 5$
- Line 7: $m[i, j] = \infty$ We are now computing $M[2,5]$
- Line 8: **for $k = i$ to $j - 1$** // try $A_{i:k}A_{k+1:j}$
- Line 9: $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$
- Line 10: **if $q < m[i, j]$**
- Line 11: $m[i, j] = q$ // remember this cost
- Line 12: $s[i, j] = k$ // remember this index

$k = 2$

$$M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$$

$k = 3$

$$M[2,3] + M[4,5] + p_1p_3p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125 \quad \text{optimal split}$$

$k = 4$

$$M[2,4] + M[5,5] + p_1p_4p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$$

Computing the optimal solution

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- // chain length 1* (next to line 2)
- chains of length 1* (next to line 3)
- chains of length 4* (next to line 4)
- // l is the chain length* (next to line 4)
- // chain begins at A_i* (next to line 5, $i = 2$)
- // chain ends at A_j* (next to line 6, $j = 5$)
- We are now computing $M[2,5]$* (next to line 7)
- // try $A_{i:k}A_{k+1:j}$* (next to line 8)
- // remember this cost* (next to line 11)
- // remember this index* (next to line 12)

$k = 2$

$$M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$$

$k = 3$

$$M[2,3] + M[4,5] + p_1p_3p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125 \quad \text{optimal split}$$

$k = 4$

$$M[2,4] + M[5,5] + p_1p_4p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$$

Computing the optimal solution

MATRIX-CHAIN-ORDER(p, n)

```

1  let  $m[1:n, 1:n]$  and  $s[1:n-1, 2:n]$  be new tables
2  for  $i = 1$  to  $n$ 
3       $m[i, i] = 0$ 
4  for  $l = 2$  to  $n$ 
5      for  $i = 1$  to  $n - l + 1$ 
6           $j = i + l - 1$ 
7           $m[i, j] = \infty$ 
8          for  $k = i$  to  $j - 1$ 
9               $q = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10             if  $q < m[i, j]$ 
11                  $m[i, j] = q$ 
12                  $s[i, j] = k$ 
13  return  $m$  and  $s$ 

```

Annotations:

- Line 2: // chain length 1
- Line 3: } chains of length 1
- Line 4: // l is the chain length
- Line 5: // chain begins at A_i $i = 2$
- Line 6: // chain ends at A_j $j = 5$
- Line 7: We are now computing $M[2,5]$
- Line 8: // try $A_{i:k}A_{k+1:j}$
- Line 11: // remember this cost
- Line 12: // remember this index $S[2,5] = 2$

$k = 2$

$$M[2,2] + M[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13000$$

$k = 3$

$$M[2,3] + M[4,5] + p_1p_3p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125 \quad \text{optimal split}$$

$k = 4$

$$M[2,4] + M[5,5] + p_1p_4p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11375$$

The table S

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

The table S

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$\rightarrow A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$

The table $\mathcal{S}$

$$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$$

The table $\mathcal{S}$

$$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$$

The diagram shows a 5x5 table $\mathcal{S}$ with rows and columns indexed from 1 to 5. The table contains the following values:

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

Red circles highlight the elements at (1,3), (1,6), and (4,6). Red arrows indicate the following relationships:

- An arrow from the equation $A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$ points to the element at (1,3).
- An arrow points from the element at (1,6) to the equation $A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$.
- An arrow points from the element at (4,6) to the equation $A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$.

The table $\mathcal{S}$

$$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$$



The diagram shows a 5x5 table $\mathcal{S}$ with rows and columns indexed from 1 to 5. The table contains numerical values. Three specific elements are highlighted with red circles: the value 1 at row 1, column 3; the value 3 at row 1, column 6; and the value 5 at row 4, column 6. Red arrows point from these circled elements to external equations. An arrow points from the circled 1 to the equation $A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$. Another arrow points from the circled 3 to the equation $A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$. A third arrow points from the circled 5 to the equation $A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$.

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$$

$$A_{1:6} = (A_{1:1} \cdot (A_{2:3})) \cdot ((A_{4:5}) \cdot A_{6:6})$$

$$A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$$

The table $\mathcal{S}$

$$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$$

$$A_{2:3} = (A_{2:2}) \cdot (A_{3:3})$$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$$

$$A_{1:6} = (A_{1:1} \cdot (A_{2:3})) \cdot ((A_{4:5}) \cdot A_{6:6})$$

$$A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$$

The table $\mathcal{S}$

$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$

$A_{2:3} = (A_{2:2}) \cdot (A_{3:3})$

$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$

$A_{1:6} = (A_{1:1} \cdot (A_{2:3})) \cdot ((A_{4:5}) \cdot A_{6:6})$

$A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$

$A_{4:5} = (A_{4:4}) \cdot (A_{5:5})$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

The table $\mathcal{S}$

$A_{1:3} = (A_{1:1}) \cdot (A_{2:3})$

$A_{2:3} = (A_{2:2}) \cdot (A_{3:3})$

$A_{1:6} = (A_{1:3}) \cdot (A_{4:6})$

$A_{1:6} = (A_{1:1} \cdot (A_{2:3})) \cdot ((A_{4:5}) \cdot A_{6:6})$

$A_{4:6} = (A_{4:5}) \cdot (A_{6:6})$

$A_{4:5} = (A_{4:4}) \cdot (A_{5:5})$

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

$$A_{1:6} = (A_{1:1} \cdot (A_{2:2} \cdot A_{3:3})) \cdot ((A_{4:4} \cdot A_{5:5}) \cdot A_{6:6}) = (A_1 \cdot (A_2 \cdot A_3)) \cdot ((A_4 \cdot A_5) \cdot A_6)$$

Computing a solution

We have the following relation:

$$M[i, j] = \begin{cases} 0, & \text{if } i = j, \\ \min_{k \in [i, j)} \{M[i, k] + M[k + 1, j] + p_{i-1}p_kp_j\} & \text{if } i < j. \end{cases}$$

We could now define a recursive algorithm straightforwardly using this.

```
RECURSIVE-MATRIX-CHAIN( $p, i, j$ )
1  if  $i == j$ 
2      return 0
3   $m[i, j] = \infty$ 
4  for  $k = i$  to  $j - 1$ 
5       $q = \text{RECURSIVE-MATRIX-CHAIN}(p, i, k)$ 
            $+ \text{RECURSIVE-MATRIX-CHAIN}(p, k + 1, j)$ 
            $+ p_{i-1}p_kp_j$ 
6      if  $q < m[i, j]$ 
7           $m[i, j] = q$ 
8  return  $m[i, j]$ 
```

Memoization

We write the procedure recursively in a natural manner, but we modify it to save the result of each subproblem in a table.

The procedure first checks if we have computed the solution already in our table.

If we have, it uses it without calling itself recursively.

If we have not, then it recurses to compute it.

A Dynamic Programming algorithm (top-down)

MEMOIZED-MATRIX-CHAIN(p, n)

```
1  let  $m[1:n, 1:n]$  be a new table
2  for  $i = 1$  to  $n$ 
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LOOKUP-CHAIN(m, p, i, j)

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A Dynamic Programming algorithm (top-down)

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} Use recursion to compute q

Dynamic Programming vs Divide and Conquer

DP is an optimisation technique and is only applicable to problems with optimal substructure.

DP splits the problem into parts, finds solutions to the parts and joins them.

The parts are not significantly smaller and are overlapping.

In DP, the subproblem dependency can be represented by a DAG.

DQ is not normally used for optimisation problems.

DQ splits the problem into parts, finds solutions to the parts and joins them.

The parts are significantly smaller and do not normally overlap.

In DQ, the subproblem dependency can be represented by a tree.