Algorithms and Data Structures

Longest Common Subsequence, Subset Sum

Dynamic Programming

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Solve the subproblems from the smallest to the largest. When you solve a subproblem, store the solution (e.g., in an array) and use it to solve the larger subproblems.

Longest Common Subsequence

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Motivation: A strand of DNA consists of a string of bases, with possible values Adenine, Cytosine, Guanine, and Thymine.

We can express a strand of DNA as a string, e.g.,

$S_1 =$ ACCGGTCGAGTGCGCGGAAGCCGGCCAA

$S_2 =$ GTCGTTCGGAATGCCGTTGCTCTGTAAA

We would like to compare two strands to see how similar they are (in order to see how similar two organisms are).

Various way to do that. Here: Find the longest possible strand S_3 which is a subsequence of both S_1 and S_2.

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$Z = \langle z_1, z_2, \ldots, z_k \rangle$ is a *subsequence* of $X = \langle x_1, x_2, \cdots, x_m \rangle$ if there exists a strictly increasing sequence $\langle i_1, i_2, \ldots, i_k \rangle$ of indices of X such that for all $j = 1,2,\ldots, k$ we have $x_{i_j} = z_j$.

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Example: $Z = \langle B, C, D, B \rangle$ is a subsequence of $X = \langle A,\ B, C, B, D, A, B \rangle$.

Brute Force

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Exponential time.

Optimal Substructure

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Given a sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$, the ith *prefix* of X (for $i = 1,2,\ldots, m$) is defined as $X_i = \langle x_1, x_2, \ldots, x_i \rangle$.

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Example: $X = \langle A, B, C, B, D, A, B \rangle$, $X_4 = \langle A, B, C, B \rangle$ and $X_0 = \langle \rangle$.

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Theorem (Optimal Substructure):
Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be two sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y. Then.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$ and $z_k \neq x_m$, then Z is an LCS of X_{m-1} and Y.
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Assume by contradiction that there exists a longer common subsequence W of X_{m-1} and Y_{n-1} that has length at least k.

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If we append $x_m = y_n$ to W we obtain a common subsequence of X and Y of length $k + 1$ contradicting the fact that Z is a LCS.

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(2) Since $z_k \neq x_m$, Z is a common subsequence of X_{m-1} and Y. Assume by contradiction that it is no a LCS.

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But W is a common subsequence of X and Y, contradicting the fact that Z is a LCS.

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  - Find a LCS of X and Y_{n-1}.
  - Choose the one that is longer.

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$$C[i, j] = \begin{cases} 0, & \textbf{if } i = 0 \textbf{ or } j = 0, \\ C[i - 1,j - 1] + 1, & \textbf{if } i, j > 0 \textbf{ and } x_i = y_j, \\ \max\{C[i, j - 1] \,.\, C[i - 1,j]\} & \textbf{if } i, j > 0 \textbf{ and } x_i \neq y_j \,. \end{cases}$$

A Dynamic Programming Algorithm

```
LCS-LENGTH(X, Y, m, n)
 1  let b[1:m, 1:n] and c[0:m, 0:n] be new tables
 2  for i = 1 to m
 3      c[i, 0] = 0
 4  for j = 0 to n
 5      c[0, j] = 0
 6  for i = 1 to m          // compute table entries in row-major order
 7      for j = 1 to n
 8          if x_i == y_j
 9              c[i, j] = c[i − 1, j − 1] + 1
10              b[i, j] = "↖"
11          elseif c[i − 1, j] ≥ c[i, j − 1]
12              c[i, j] = c[i − 1, j]
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16  return c and b

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0	0	0	0

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0	★	★	★
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0	★	★	★
0	★	★	★
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0 ←	★	★	★
0	★	★	★
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0			

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0	0	0	0
0	★	★	★
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Example

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```

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0						
B	0						
C	0						
B	0						
D	0						
A	0						
B	0						

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 8          if x_i == y_j
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16  return c and b
```

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0						
C	0						
B	0						
D	0						
A	0						
B	0						

$C[0,1] = 0$ $C[0,2] = 0$

$C[1,0] = 0$ $C[1,1] = 0$

Example

```
LCS-LENGTH(X, Y, m, n)
 1  let b[1:m, 1:n] and c[0:m, 0:n] be new tables
 2  for i = 1 to m
 3      c[i, 0] = 0
 4  for j = 0 to n
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15              b[i, j] = "←"
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```

		B	*D*	*C*	*A*	*B*	*A*
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1				
B	0						
D	0						
A	0						
B	0						

Example

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A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1				
B	0						
D	0						
A	0						
B	0						

Next to compute: $C[3,3]$

Example

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A	0	0	0	0	1	1	1
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C	0	1	1				
B	0						
D	0						
A	0						
B	0						

Next to compute: $C[3,3]$

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Is $x_3 = y_3$?

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	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1				
B	0						
D	0						
A	0						
B	0						

Next to compute: $C[3,3]$

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A	0	0	0	0	1	1	1
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B	0						
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A	0						
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Next to compute: $C[3,3]$

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```

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1	2			
B	0						
D	0						
A	0						
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Next to compute: $C[3,3]$

Example

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```

		B	D	C	A	B	A
	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1
B	0	1	1	1	1	2	2
C	0	1	1	2	2	2	2
B	0	1	1	2	2	3	3
D	0	1	2	2	2	3	3
A	0	1	2	2	3	3	4
B	0	1	2	2	3	4	4

Example

```
PRINT-LCS(b, X, i, j)
1  if i == 0 or j == 0
2       return              // the LCS has length 0
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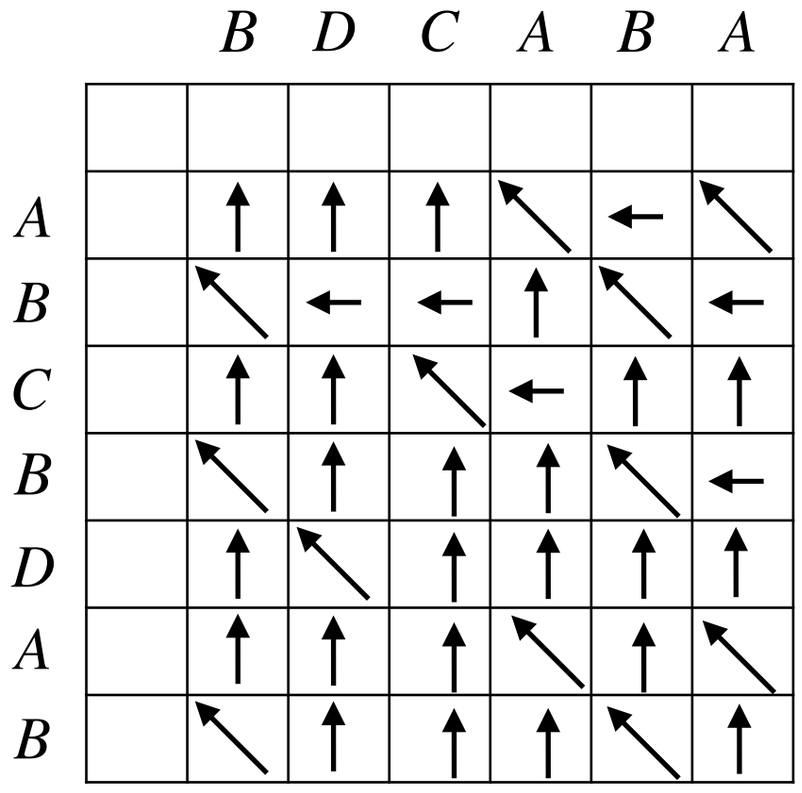

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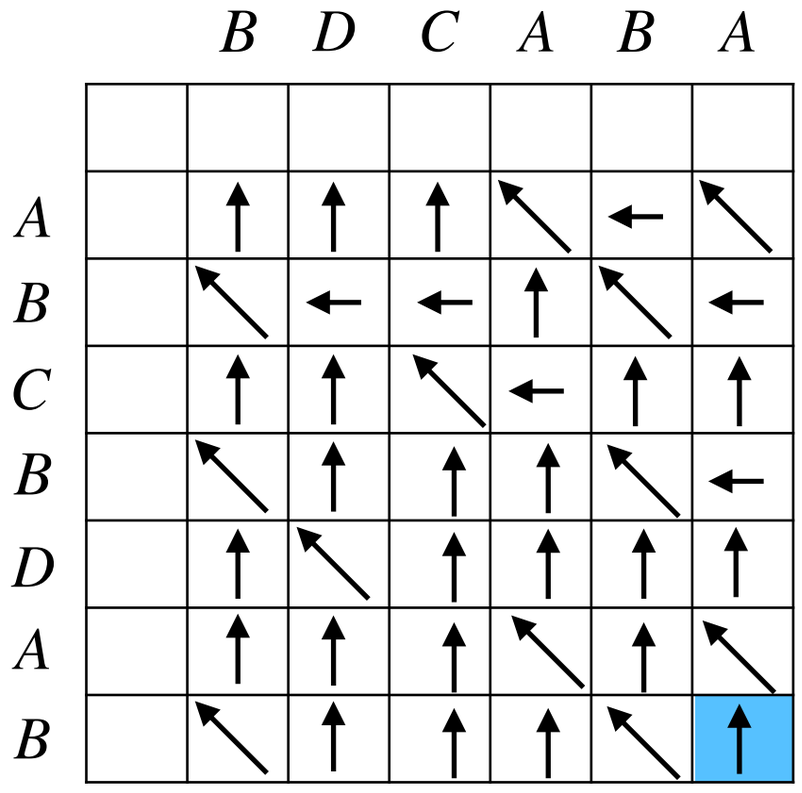

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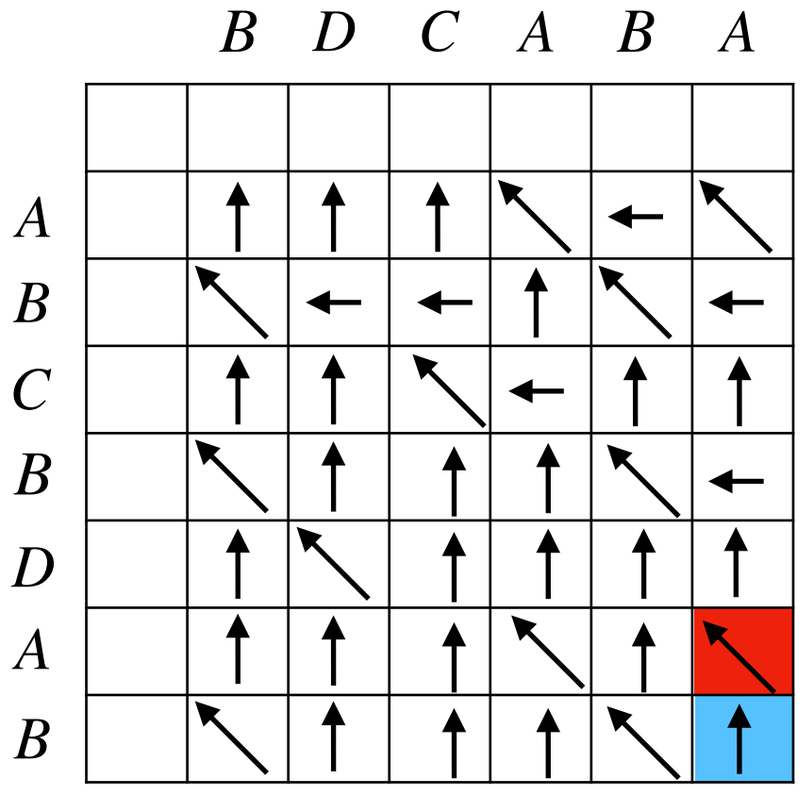

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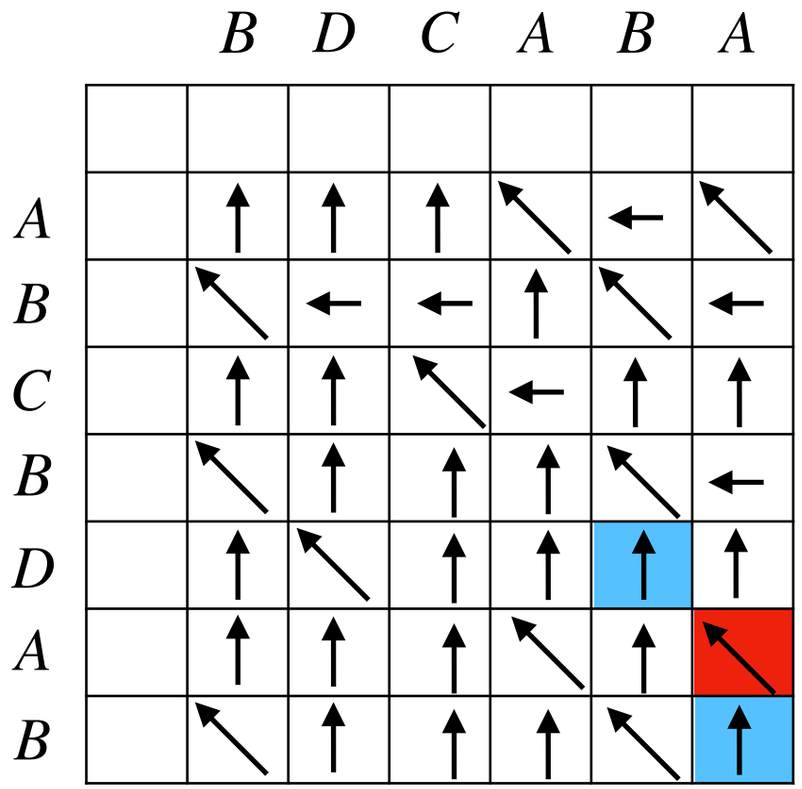

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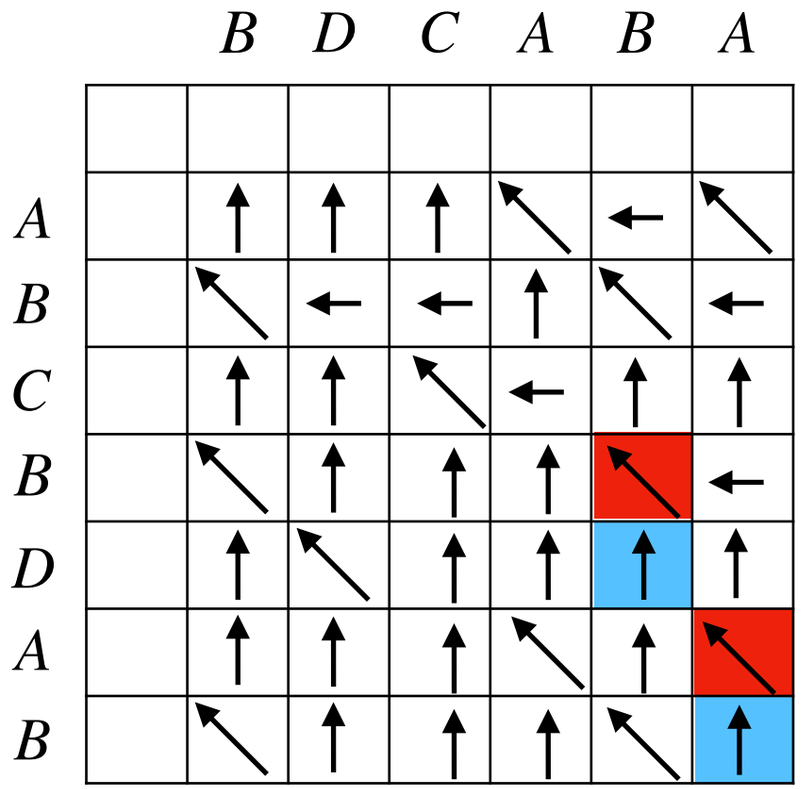

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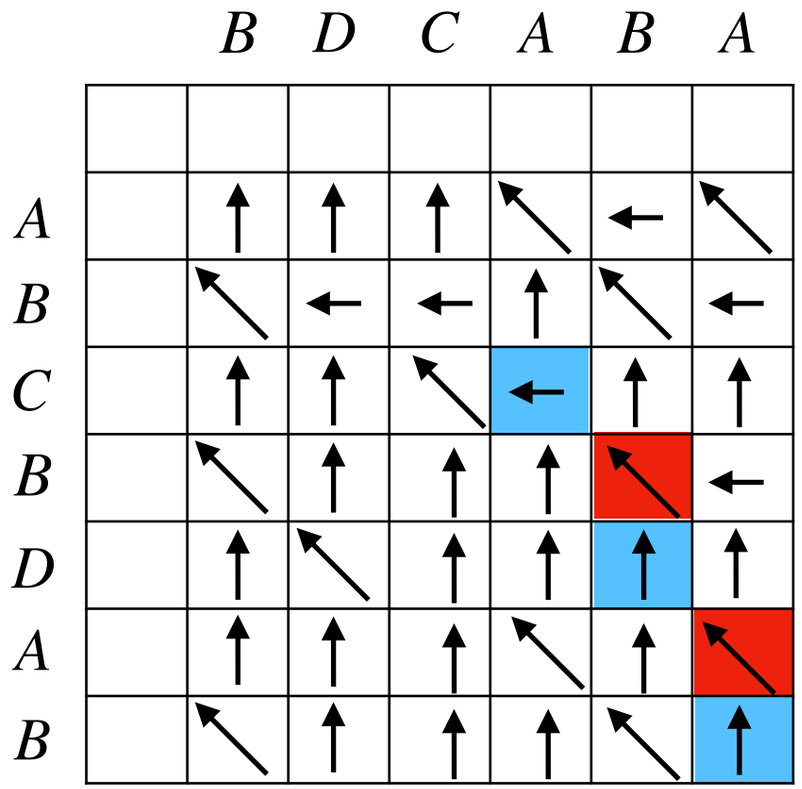

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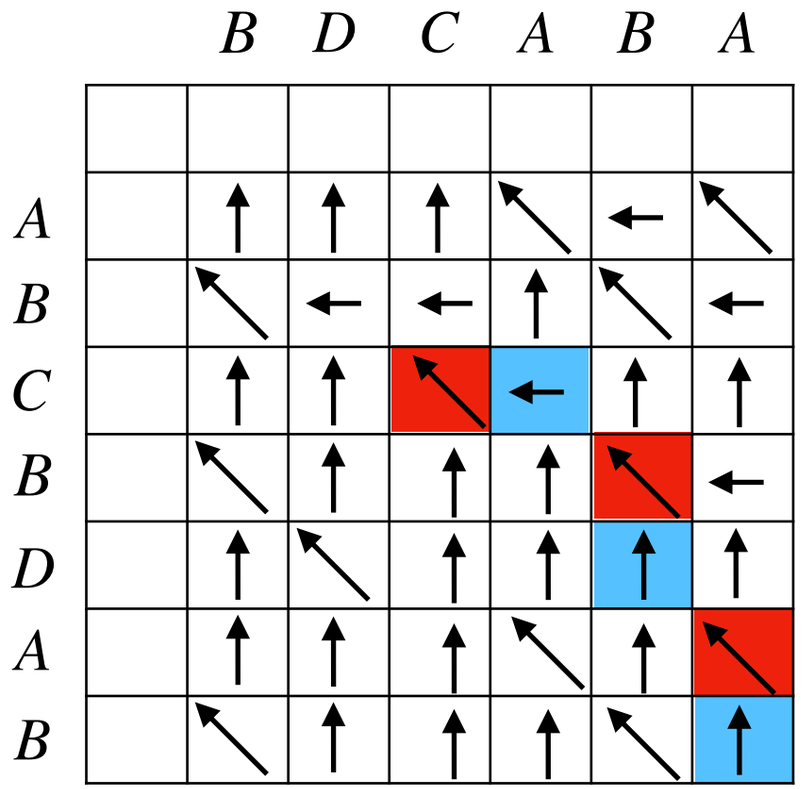

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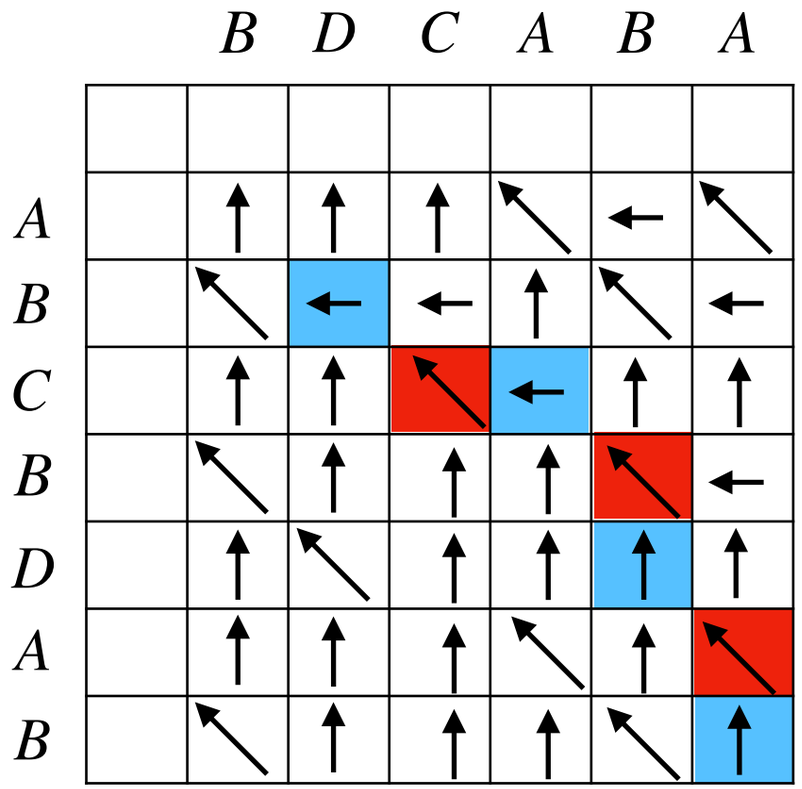

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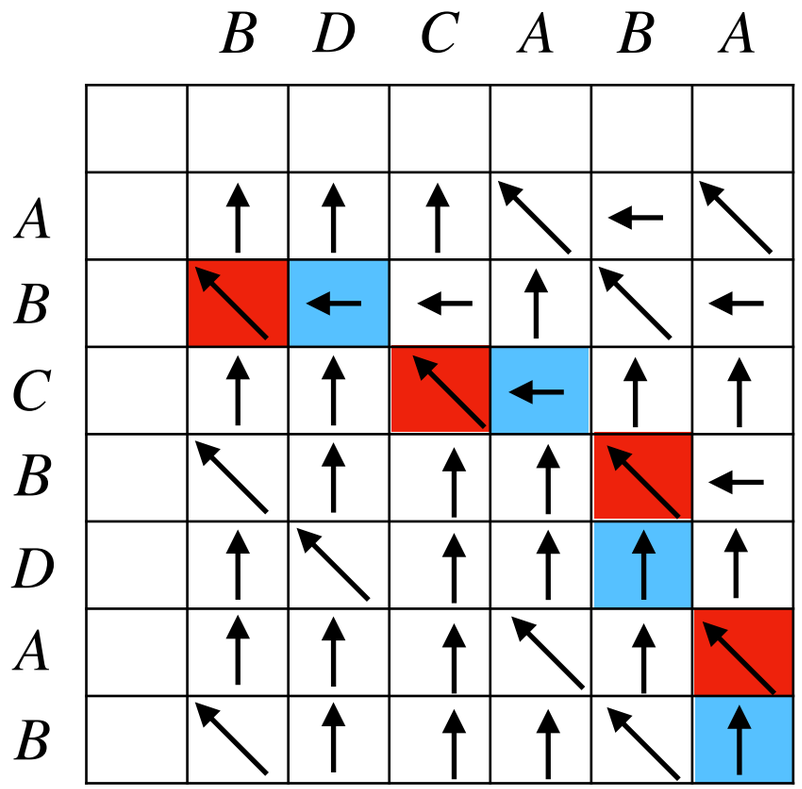

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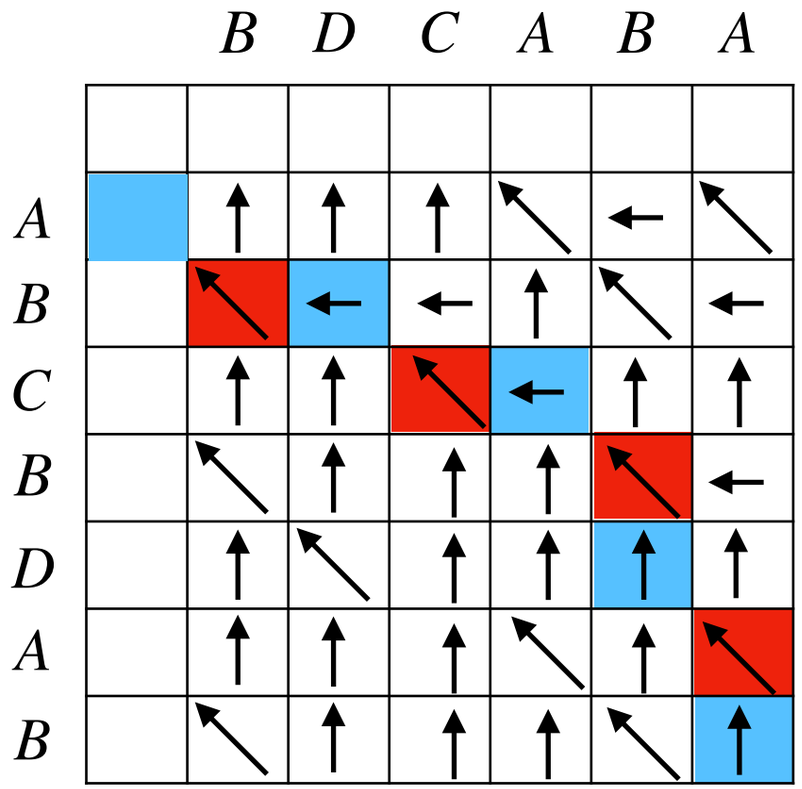

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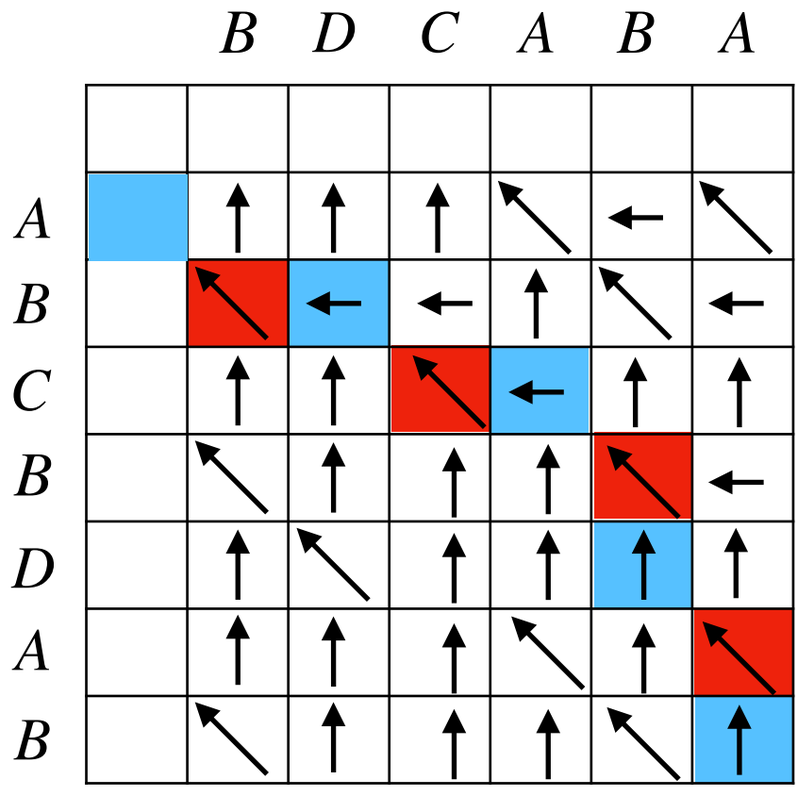

A

Example

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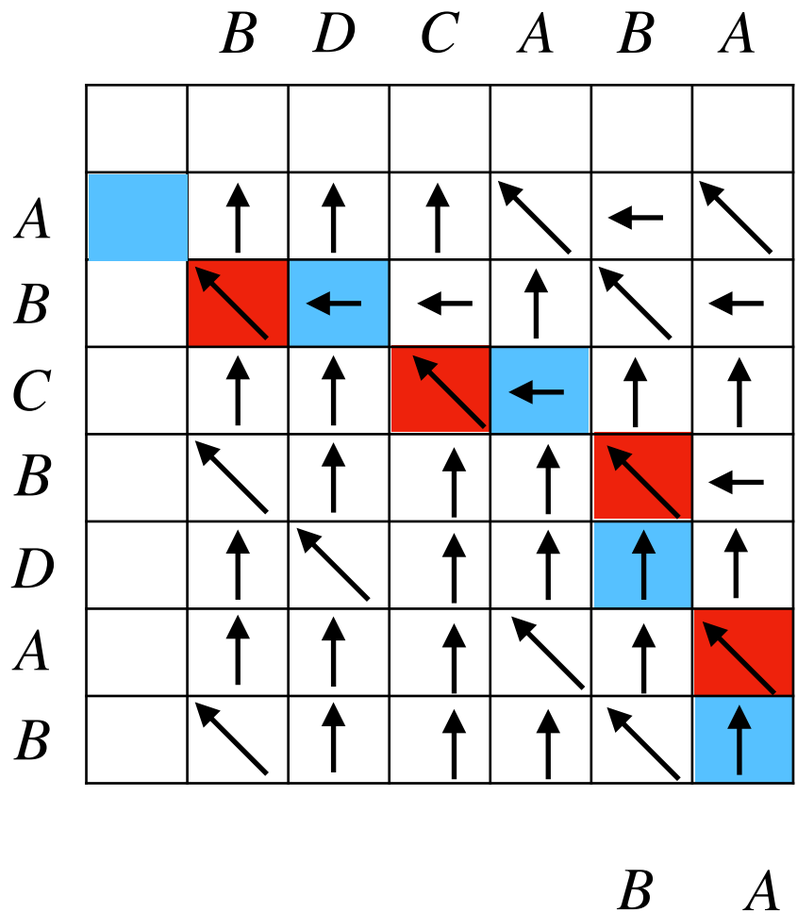

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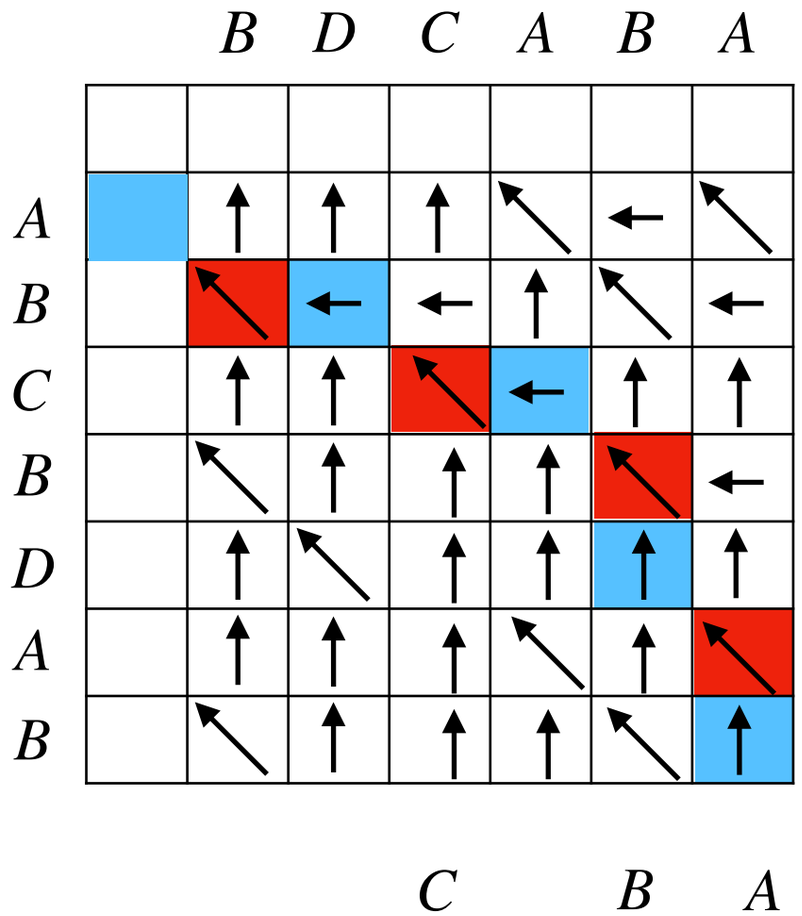

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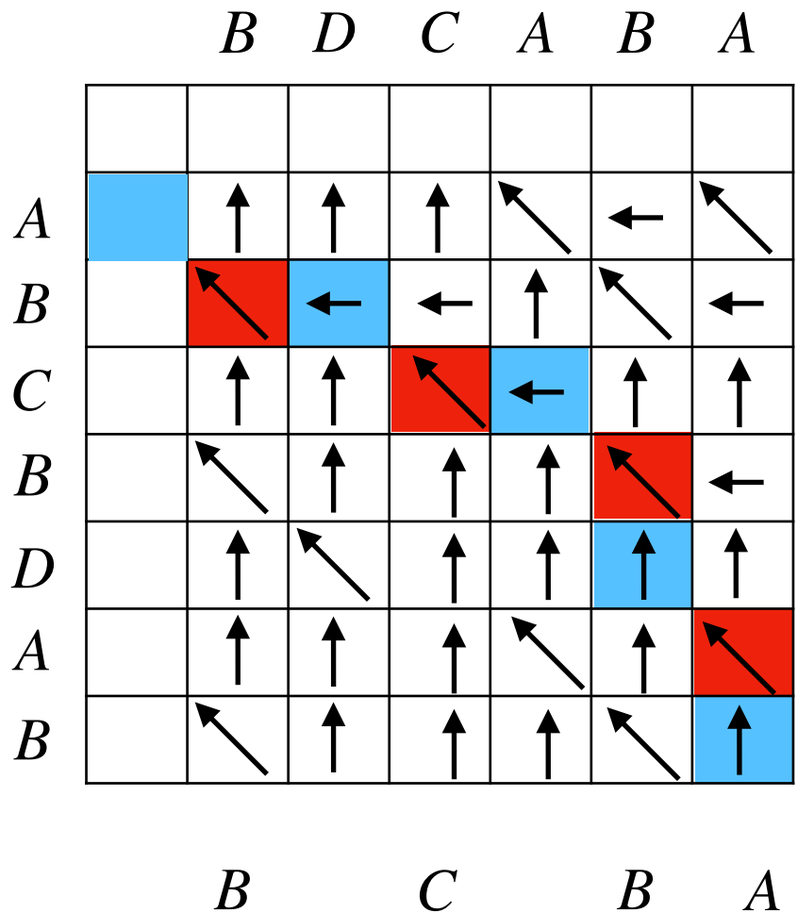

A Dynamic Programming Algorithm

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LCS-LENGTH(X, Y, m, n)
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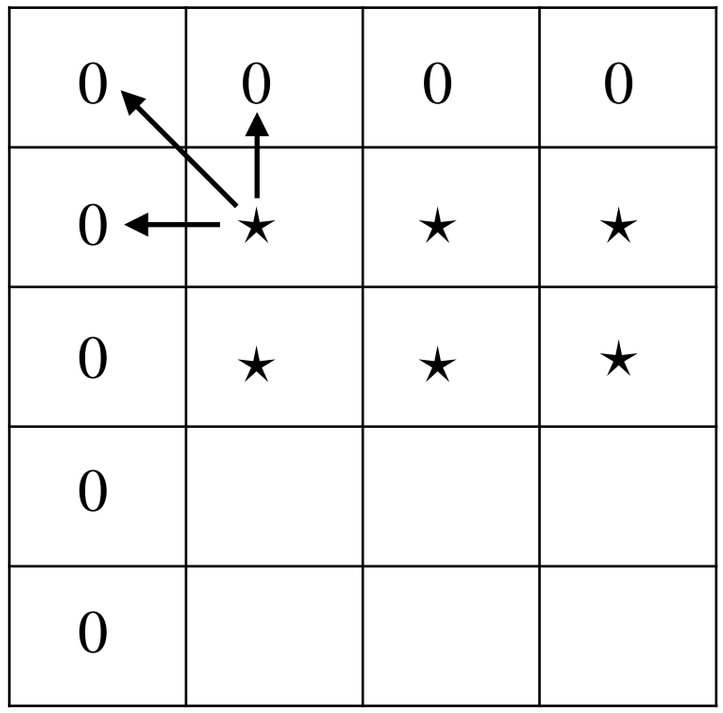

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8  else PRINT-LCS(b, X, i, j − 1)
```

do we need this?

0	0	0	0
0	★	★	★
0	★	★	★
0			
0			

A Dynamic Programming Algorithm

```
LCS-LENGTH(X, Y, m, n)
 1  let b[1:m, 1:n] and c[0:m, 0:n] be new tables
 2  for i = 1 to m
 3      c[i, 0] = 0
 4  for j = 0 to n
 5      c[0, j] = 0
 6  for i = 1 to m          // compute table entries in row-major order
 7      for j = 1 to n
 8          if x_i == y_j
 9              c[i, j] = c[i − 1, j − 1] + 1
10              b[i, j] = "↖"
11          elseif c[i − 1, j] ≥ c[i, j − 1]
12              c[i, j] = c[i − 1, j]
13              b[i, j] = "↑"
14          else c[i, j] = c[i, j − 1]
15              b[i, j] = "←"
16  return c and b
```

can we use less space here?

0	0	0	0
0	★	★	★
0	★	★	★
0			
0			

```
PRINT-LCS(b, X, i, j)
1  if i == 0 or j == 0
2      return              // the LCS has length 0
3  if b[i, j] == "↖"
4      PRINT-LCS(b, X, i − 1, j − 1)
5      print x_i           // same as y_j
6  elseif b[i, j] == "↑"
7      PRINT-LCS(b, X, i − 1, j)
8  else PRINT-LCS(b, X, i, j − 1)
```

do we need this?

The subset sum problem

We are given a set of n items {*1*, *2*, … , *n*}.

Each item *i* has a non-negative weight w_i.

We are given a bound W.

Goal: Select a subset S of the items such that

$\sum_{i \in N} w_i \leq W$ and $\sum_{i \in N} w_i$ is maximised.

Dynamic Programming

Dynamic Programming

We need to identify the appropriate subproblems to use in order to solve the main problem.

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Let O_i be the optimal solution to the subset be the optimal solution to the subset sum problem, using a subset of {*1*, *2*, … , *i*}, and let OPT(*i*) be its value.

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If yes, ?

If n is in O

If n is in O

What information do we get about the other items?

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There is no reason to a-priori exclude any remaining item, unless adding it would exceed the weight.

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There is no reason to a-priori exclude any remaining item, unless adding it would exceed the weight.

The only information that we really get is that we now have weight W - w_n left.

What we really need

What we really need

To find the optimal value OPT(n), we need

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The optimal value OPT(n-1) if n is not in O.

What we really need

To find the optimal value OPT(*n*), we need

The optimal value OPT(*n-1*) if *n* is not in O.

The optimal value of the solution on input {*1*, *2* , …, *n-1*} and w = W - w_n .

What we really need

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How many subproblems do we need?

What we really need

To find the optimal value OPT(*n*), we need

The optimal value OPT(*n-1*) if *n* is not in O.

The optimal value of the solution on input {*1*, *2* , …, *n-1*} and w = W - w_n .

How many subproblems do we need?

One for each initial set {*1*, *2* , …, *i*} of items and each possible value for the remaining weight w.

Subproblems

Subproblems

Assumptions:

Subproblems

Assumptions:

W is an integer.

Subproblems

Assumptions:

W is an integer.

Every w_i is an integer.

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We will have one subproblem for each $i=0,1,\ldots,n$ and each integer $0 \le w \le W$.

Subproblems

Assumptions:

W is an integer.

Every w_i is an integer.

We will have one subproblem for each $i=0,1,\ldots,n$ and each integer $0 \le w \le W$.

Let OPT(i,w) be the value of the optimal solution on subset $\{1, 2, \ldots, i\}$ and maximum allowed weight w.

Subproblems

Subproblems

Using this notation, what are we looking for?

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OPT(*n*,W)

Subproblems

Using this notation, what are we looking for?

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OPT(*n*,W)

Should item *n* be in the optimal solution O or not?

If no, then OPT(*n*,W) = OPT(*n-1*,W).

Subproblems

Using this notation, what are we looking for?

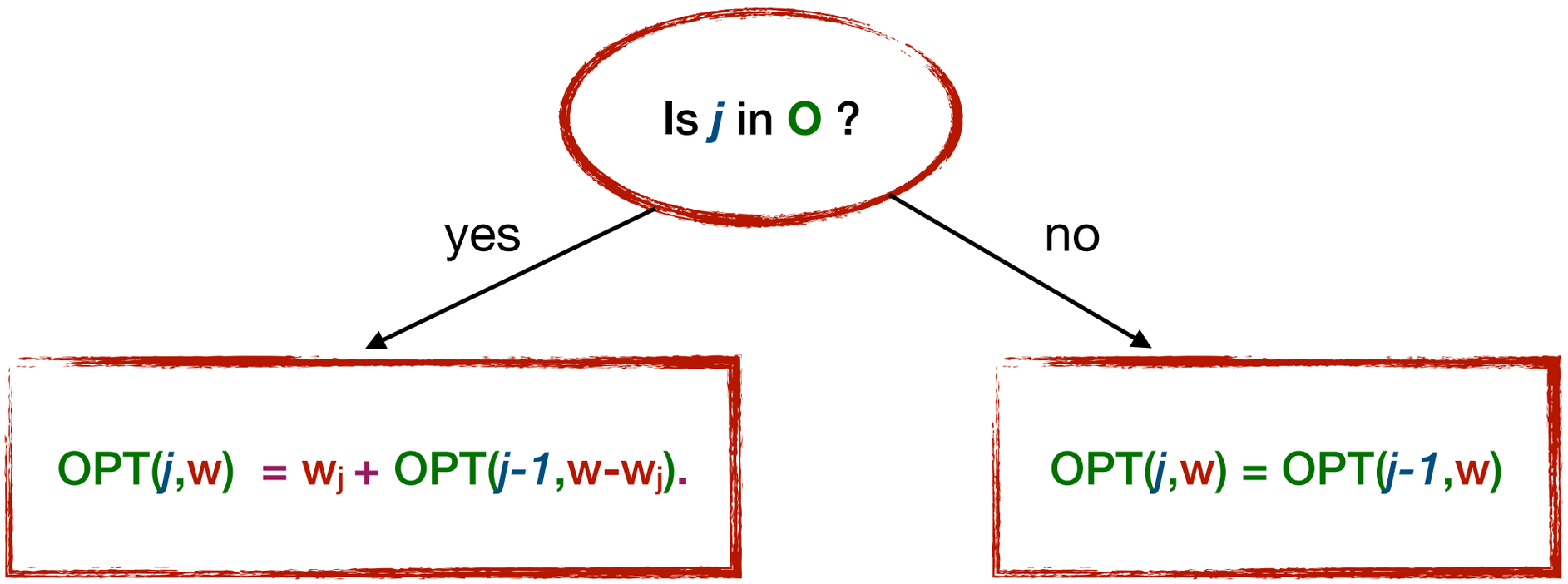
OPT(*n*,W)

Should item *n* be in the optimal solution O or not?

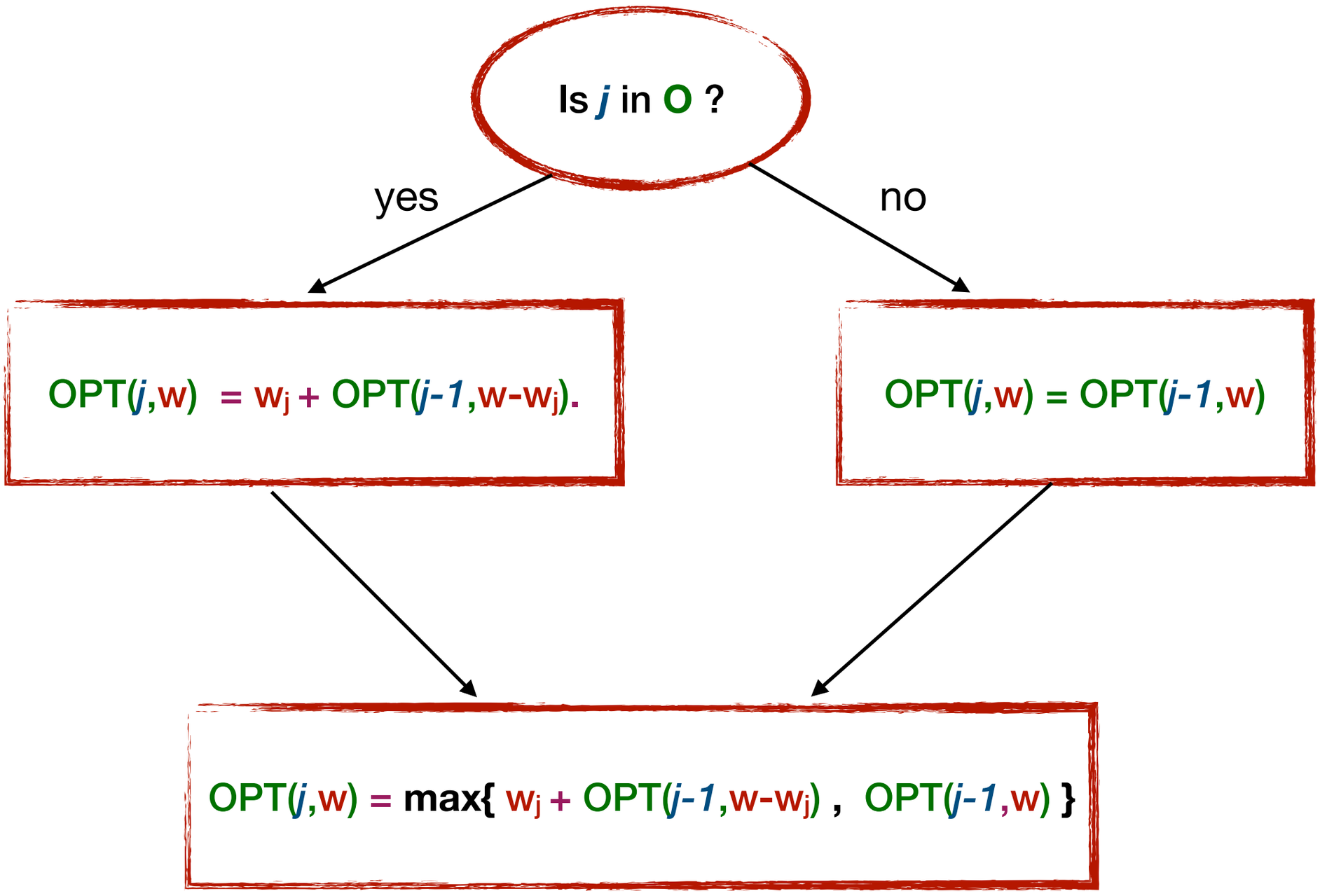
If no, then OPT(*n*,W) = OPT(*n-1*,W).

If yes, then OPT(*n*,W) = w_n + OPT(*n-1*,W-w_n).

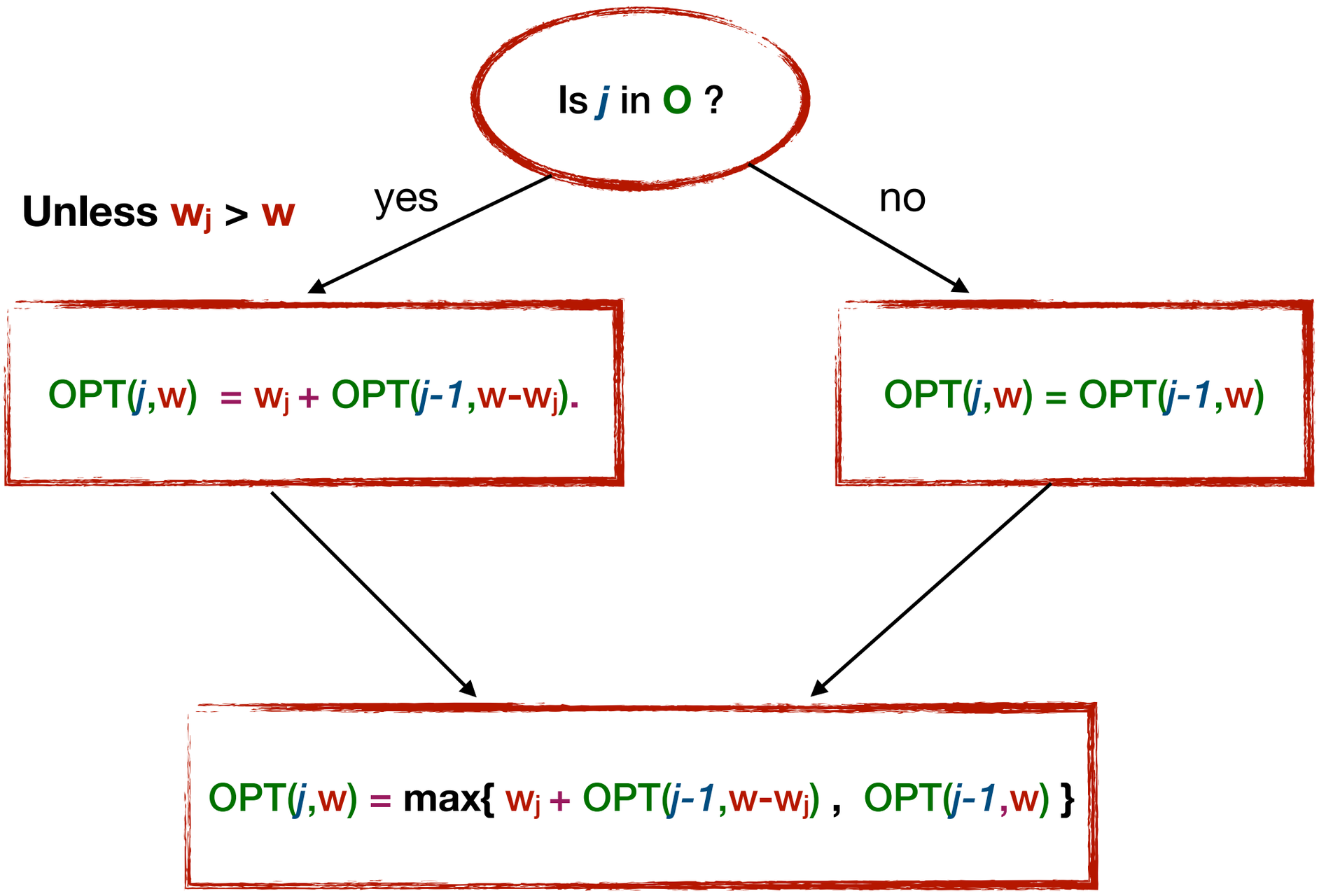
Subproblems



Subproblems



Subproblems



Algorithm

Algorithm **SubsetSum**(*n*,W)

Array M=[*0 … n*, 0 … W]
Initialise M[*0*, w] = 0, for each w = 0, 1 , … , W

For i = *1*, *2*, … , *n*
For w = 0 , … , W
If (w_i > w) * *If the item does not fit* *\
M[*i*, w] = M[*i-1*, w]
Else
M[*i*, w] = **max{**M[*i-1*, w] , w_i + M[*i-1*, w-w_i]**}**
EndIf

Return M[*n*, W]

Two dimensional array

n	0										
n-1	0										
…	0										
…	0										
i	0										
i-1	0										
…	0										
2	0										
1	0										
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w				W

Two dimensional array

											OPT(*n*,W)
n	0										
n-1	0										
…	0										
…	0										
i	0										
i-1	0										
…	0										
2	0										
1	0										
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w				W

Two dimensional array

n	0										
n-1	0										
…	0										
…	0										
i	0										
i-1	0										
…	0										
2	0										
1	0										
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w				W

OPT(*n*,W)

Two dimensional array

n	0										
n-1	0										
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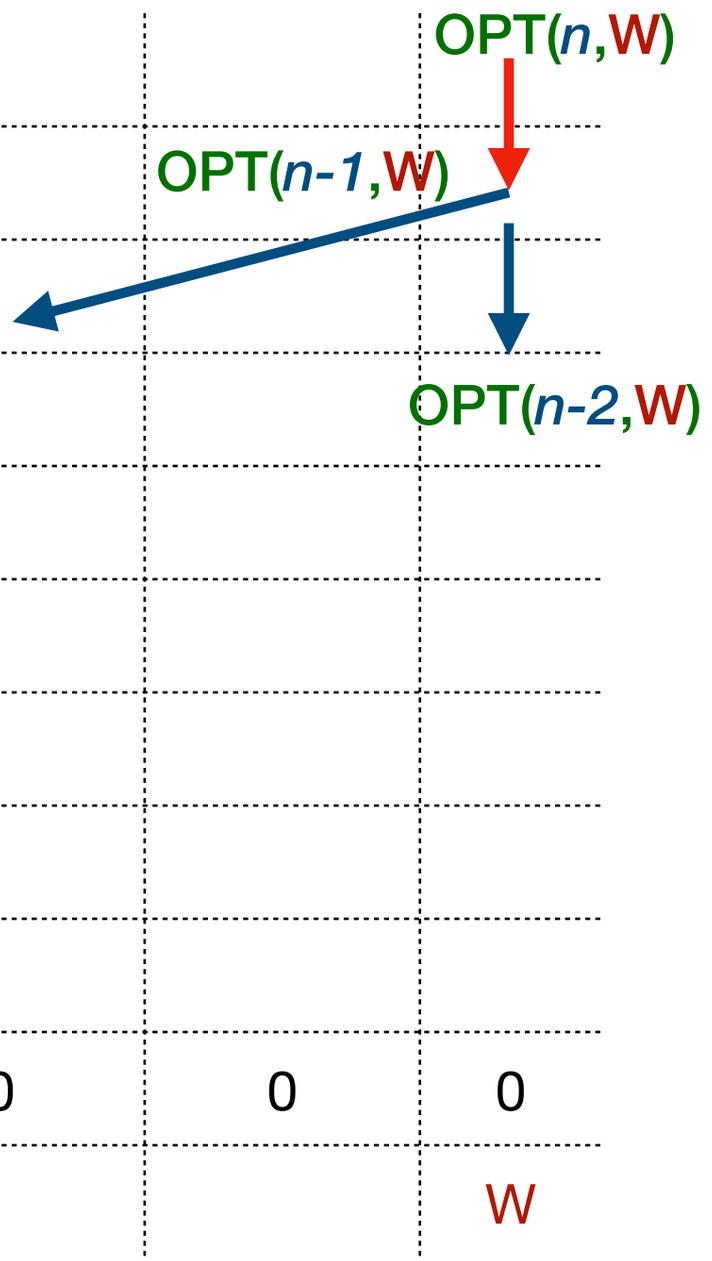
…	0										
i	0										
i-1	0										
…	0										
2	0										
1	0										
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w				W

OPT(*n*,W)

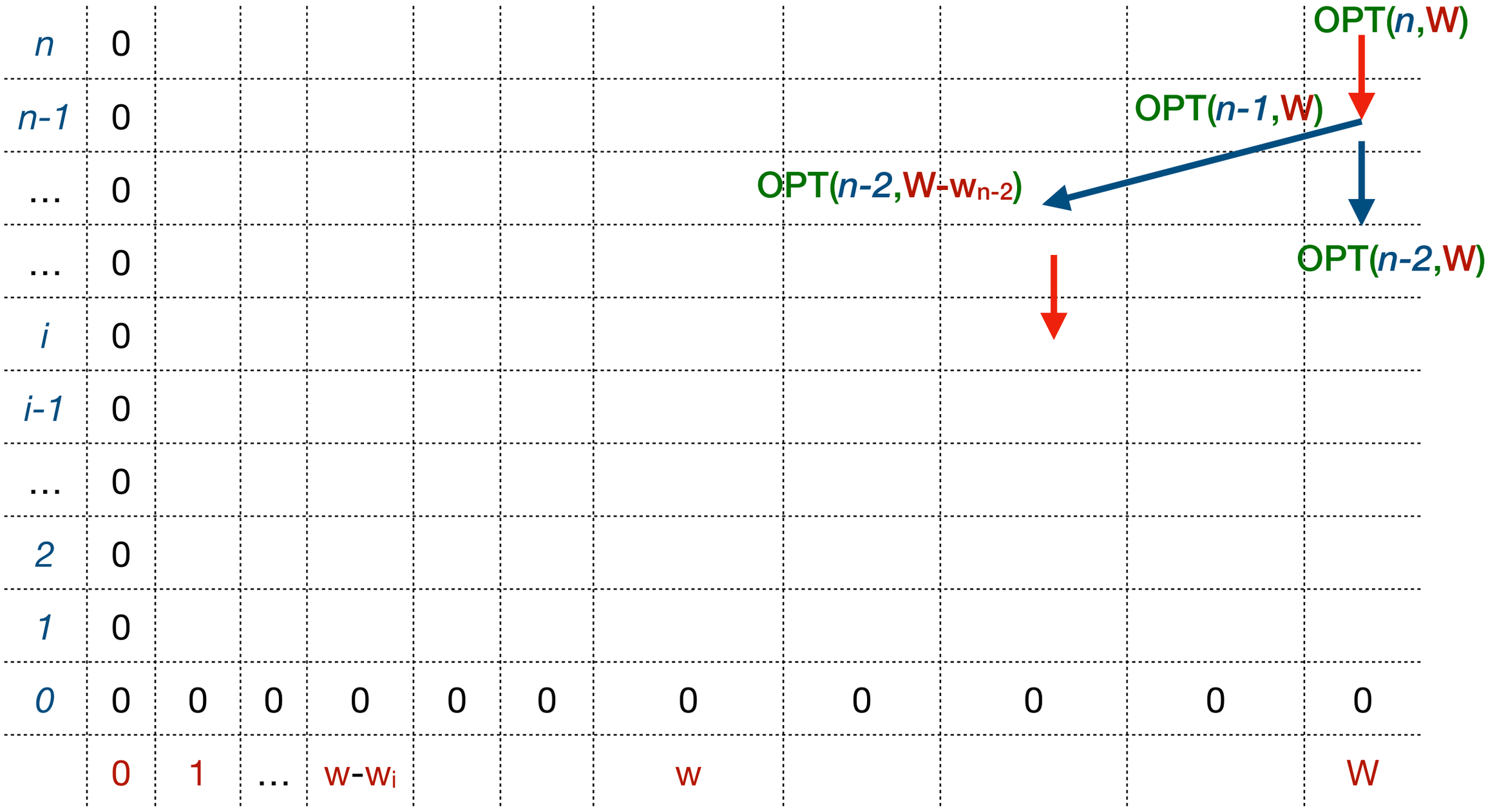
OPT(*n-1*,W)

Two dimensional array

n	0											
n-1	0											
…	0											
…	0											
i	0											
i-1	0											
…	0											
2	0											
1	0											
0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w					W



Two dimensional array



Two dimensional array

n	0										
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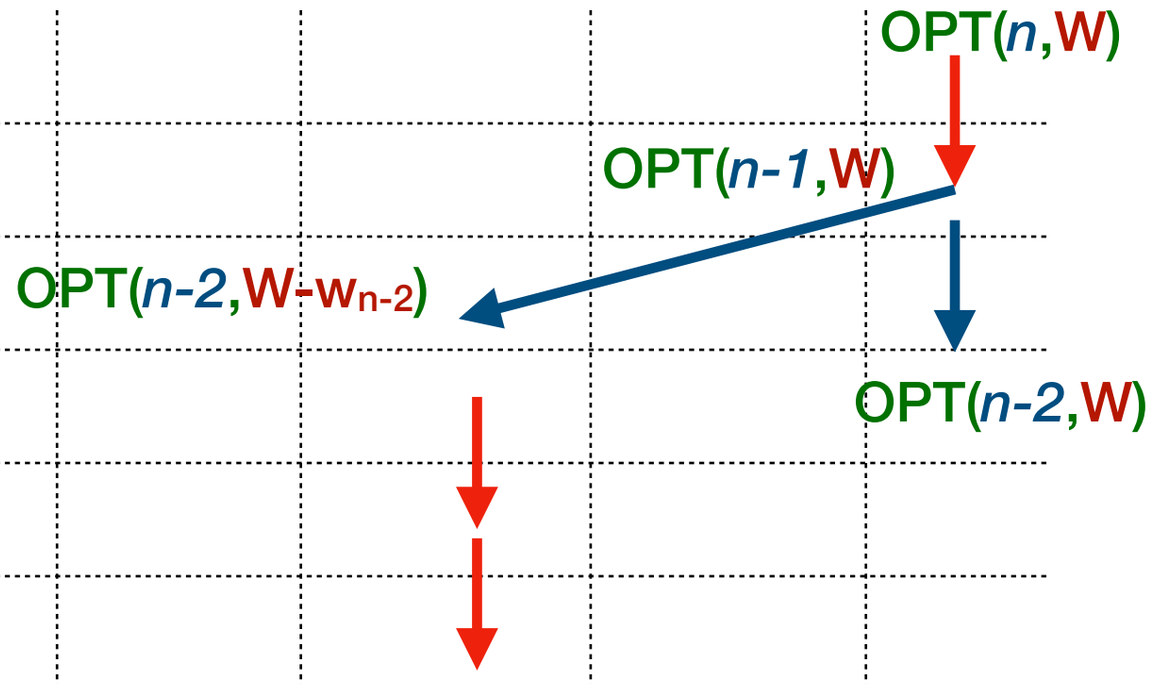
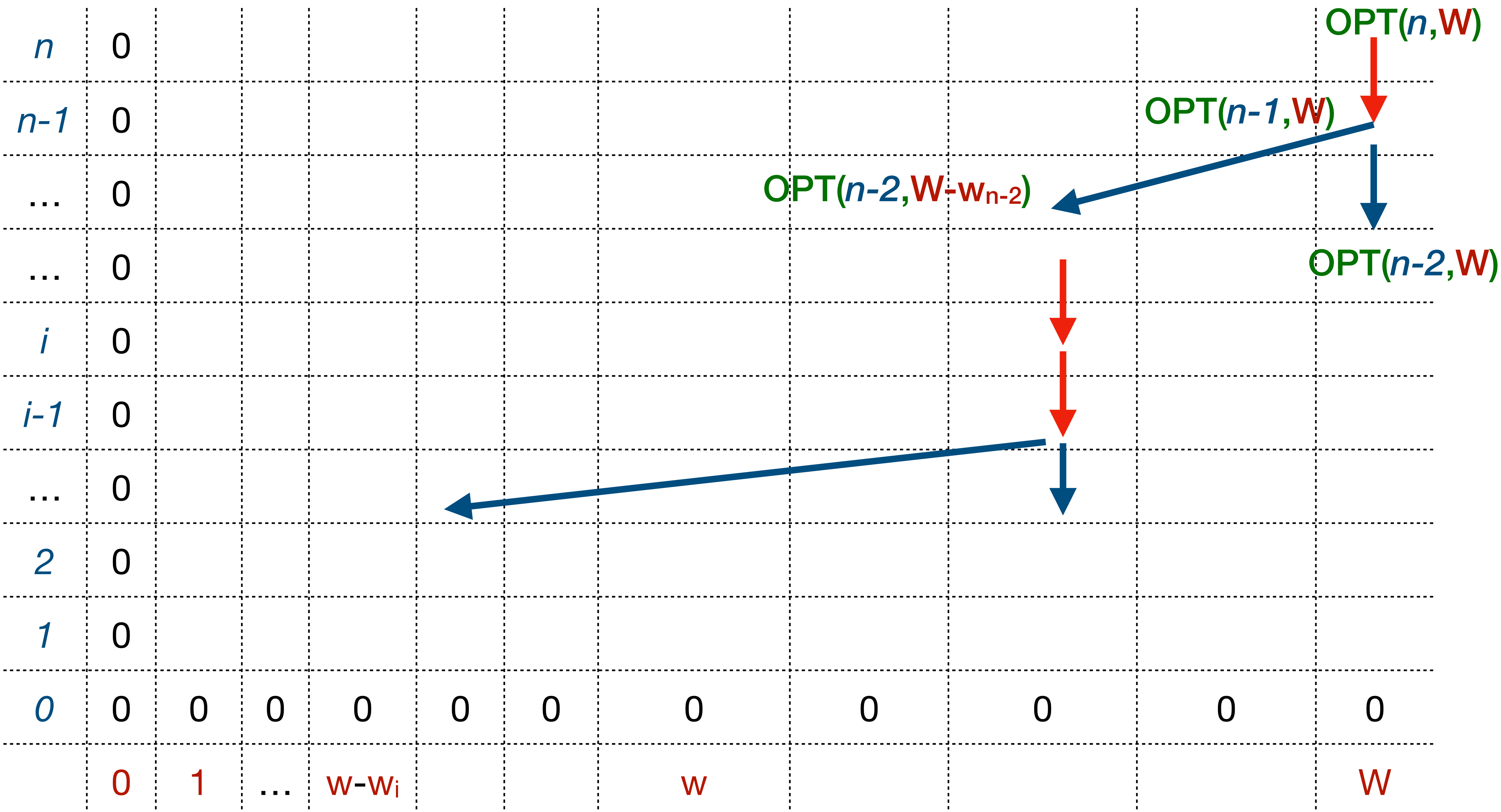
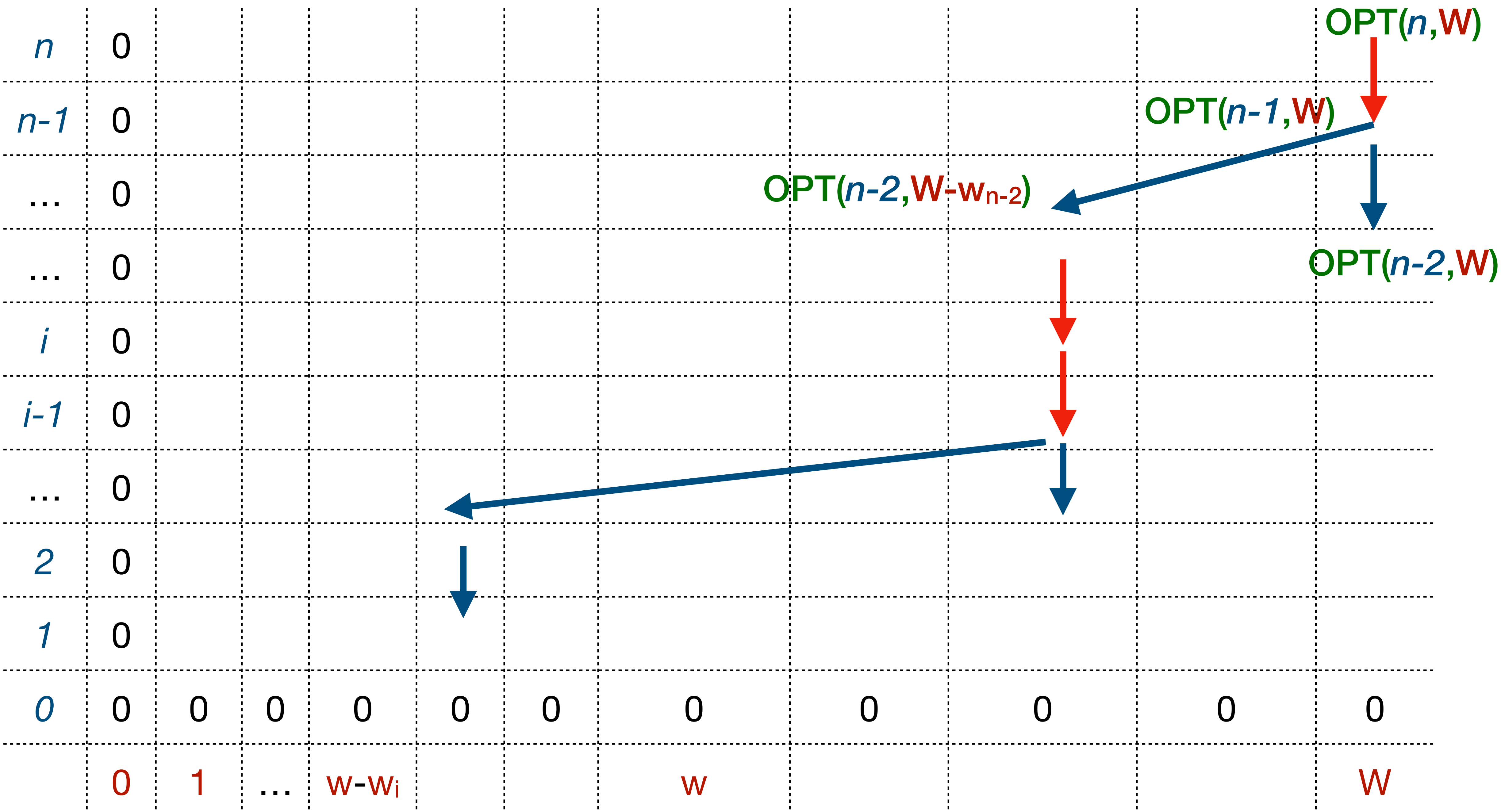
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w				W



Two dimensional array



Two dimensional array



Example

n=3, W=6, w_1 = w_2 = 2 and w_3 = 3.

3							
2							
1							
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

Array M=[*0* … *n*, 0 … W]

Initialise M[*0*, w] = 0, for each w = 0, 1 , … , W

Example

n=3, W=6, w_1 = w_2 = 2 and w_3 = 3.

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For w = 0 , … , W
    If (w_i > w)
        M[*i*, w] = M[*i-1*, w]
    Else
        M[*i*, w] = **max{**M[*i-1*, w] , w_i + M[*i-1*, w-w_i]**}**
    EndIf

Example

n=3, W=6, w_1 = w_2 = 2 and w_3 = 3.

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2							
1							
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

For w = 0 , … , W

If (w_i > w) **doesn't fit**

M[*i*, w] = M[*i-1*, w]

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n=3, W=6, w_1 = w_2 = 2 and w_3 = 3.

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i \ w	0	1	2	3	4	5	6
3							
2							
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0	0	0	0	0	0	0	0

For w = 0 , … , W
  If (w_i > w) **fits**
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Example

n=3, W=6, w_1 = w_2 = 2 and w_3 = 3.

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2	0	0	2	2			
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

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Example

n=3, W=6, w_1 = w_2 = 2 and w_3 = 3.

3							
2	0	0	2	2	4		
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

For w = 0 , … , W
    If (w_i > w)
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2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

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3							
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
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    EndIf

Example

n=3, W=6, w_1 = w_2 = 2 and w_3 = 3.

i							
3	0	0	2	2	4	4	4
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

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    If (w_i > w)
        M[*i*, w] = M[*i-1*, w]
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Example

n=3, W=6, w_1 = w_2 = 2 and w_3 = 3.

3	0	0	2				
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

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Example

n=3, W=6, w_1 = w_2 = 2 and w_3 = 3.

3	0	0	2	3			
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

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        M[*i*, w] = **max{**M[*i-1*, w] , w_i + M[*i-1*, w-w_i]**}**
    EndIf

Example

n=3, W=6, w_1 = w_2 = 2 and w_3 = 3.

3	0	0	2	3	4		
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

For w = 0 , … , W
    If (w_i > w)
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Example

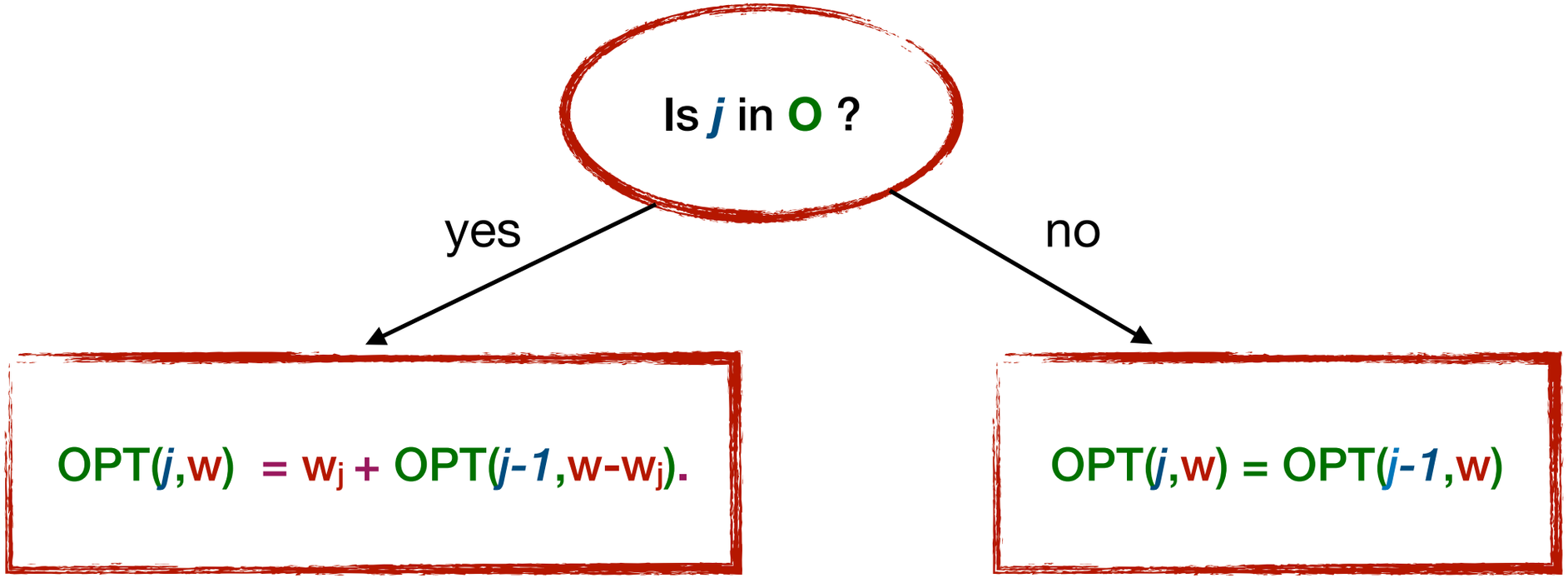
n=3, W=6, $w_1 = w_2 = 2$ and $w_3 = 3$.

Optimal value

3	0	0	2	3	4	5	5
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

For w = 0 , … , W
    If (w_i > w)
        M[*i*, w] = M[*i-1*, w]
    Else
        M[*i*, w] = **max{**M[*i-1*, w] , w_i + M[*i-1*, w-w_i]**}**
    EndIf

From values to solutions



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What is the running time overall?

How many entries does the table M have?

Running Time

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SubsetSum(n,W) runs in time O(nW).

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Is this a polynomial time algorithm?

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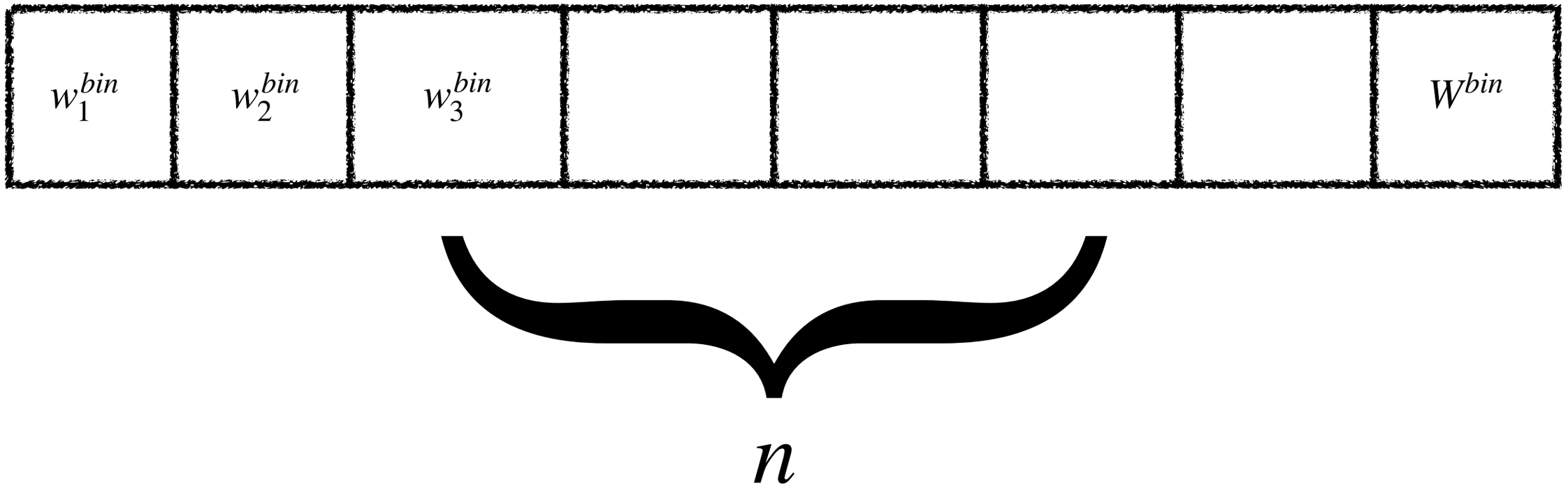
SubsetSum(n,W) runs in time O(nW).

Is this a polynomial time algorithm?

Not quite, because it depends on W.

Our input



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Is this a polynomial time algorithm?

It is *pseudopolynomial*, as it runs in time polynomial in n and the unary representation of W.

It is fairly efficient, if in the numbers involved in the input are reasonably small.

Should we be happy?

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But why not polynomial?

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Hard enough to justify a reward of 1 million dollars!

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More about that later on in the course.