

# Algorithmic Game Theory and Applications

## Lecture 4: 2-Player Zero-Sum Games and the Minimax Theorem

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## 2-person zero-sum games

A finite 2-person zero-sum (2p-zs) strategic game  $\Gamma$ , is a strategic game where:

- ▶ For players  $i \in \{1, 2\}$ , the *payoff functions*  $u_i : S \mapsto \mathbb{R}$  are such that for all  $s = (s_1, s_2) \in S$ ,

$$u_1(s) + u_2(s) = 0$$

i.e.,  $u_1(s) = -u_2(s)$ .

$u_i(s_1, s_2)$  can conveniently be viewed as a  $m_1 \times m_2$  payoff matrix  $A_i$ , where:

$$A_1 = \begin{bmatrix} u_1(1, 1) & \dots & u_1(1, m_2) \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ u_1(m_1, 1) & \dots & u_1(m_1, m_2) \end{bmatrix}$$

Note,  $A_2 = -A_1$ . Thus we may assume only one function  $u(s_1, s_2)$  is given, as one matrix,  $A = A_1$ .

## 2-player zero-sum game matrix

Thus, a 2-player zero-sum game can be described by a single  $m_1 \times m_2$  matrix:

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{1,m_2} \\ \vdots & & \vdots \\ \vdots & a_{i,j} & \vdots \\ \vdots & & \vdots \\ a_{m_1,1} & \dots & a_{m_1,m_2} \end{bmatrix}$$

where  $a_{i,j} = u_1(i, j)$ .

Player 1 (the row player) wants to maximize  $u(i, j)$ , whereas Player 2 (the column player) wants to minimize it (i.e., to maximize its negative).

# review of matrix and vector notations

For any  $(n_1 \times n_2)$ -matrix  $A$  we'll either use  $a_{i,j}$  or  $(A)_{i,j}$  to denote the entry in the  $i$ 'th row and  $j$ 'th column of  $A$ .

For  $(n_1 \times n_2)$  matrices  $A$  and  $B$ , let

$$A \geq B$$

denotes that for all  $i, j$ ,  $a_{i,j} \geq b_{i,j}$ .

Let

$$A > B$$

denotes that for all  $i, j$ ,  $a_{i,j} > b_{i,j}$ .

For a matrix  $A$ , let  $A \geq 0$  denote that every entry is  $\geq 0$ . Likewise, let  $A > 0$  mean every entry is  $> 0$ .

## more review of matrices and vectors

Recall matrix multiplication: given  $(n_1 \times n_2)$ -matrix  $A$  and  $(n_2 \times n_3)$ -matrix  $B$ , the product  $AB$  is an  $(n_1 \times n_3)$ -matrix  $C$ , where

$$c_{i,j} = \sum_{k=1}^{n_2} a_{i,k} \cdot b_{k,j}$$

Fact: matrix multiplication is “associative”: i.e.,

$$(AB)C = A(BC)$$

(Note: for the multiplications to be defined, the dimensions of the matrices  $A$ ,  $B$ , and  $C$  need to be “consistent”:  $(n_1 \times n_2)$ ,  $(n_2 \times n_3)$ , and  $(n_3 \times n_4)$ , respectively.)

Fact: For matrices  $A$ ,  $B$ ,  $C$ , of appropriate dimensions, if  $A \geq B$ , and  $C \geq 0$ , then

$$AC \geq BC, \text{ and likewise, } CA \geq CB.$$

## more review of matrix and vector notation

For a  $(n_1 \times n_2)$  matrix  $B$ , let  $B^T$  denote the  $(n_2 \times n_1)$  **transpose** matrix, where  $(B^T)_{i,j} := (B)_{j,i}$ .

We can view a column vector,  $y = \begin{bmatrix} y(1) \\ \vdots \\ \vdots \\ y(m) \end{bmatrix}$ , as a

$(m \times 1)$ -matrix. Then,  $y^T$  would be a  $(1 \times m)$ -matrix, i.e., a row vector.

Typically, we think of “vectors” as column vectors. We’ll call a length  $m$  vector an  $m$ -vector.

Multiplying a  $(n_1 \times n_2)$ -matrix  $A$  by a  $n_2$ -vector  $y$  is just a special case of matrix multiplication:  $Ay$  is a  $n_1$ -vector.

Likewise,  $y^T A$  is a  $n_2$ -row vector.

For a column (row) vector  $y$ , we use  $(y)_j$  to denote its  $j$ ’th entry.

## A matrix view of zero-sum games

Suppose we have a 2p-zs game given by a  $(m_1 \times m_2)$ -matrix,  $A$ .

Suppose Player 1 chooses a mixed strategy  $x_1$ , and Player 2 chooses mixed strategy  $x_2$  (assume  $x_1$  and  $x_2$  are given by column vectors).

$$x_1^T A x_2 = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} (x_1(i) \cdot x_2(j)) \cdot a_{i,j}$$

But note that  $(x_1(i) \cdot x_2(j))$  is precisely the probability of the pure combination  $s = (i, j)$ . Thus, for the mixed profile  $x = (x_1, x_2)$

$$x_1^T A x_2 = U_1(x) = -U_2(x)$$

where  $U_1(x)$  is the expected payoff (which Player 1 is trying to maximize, and Player 2 is trying to minimize).

## “minmaximizing” strategies

Suppose Player 1 chooses a mixed strategy  $x_1^* \in X_1$ , by trying to maximize the “worst that can happen”. The worst that can happen would be for Player 2 to choose  $x_2$  which minimizes  $(x_1^*)^T A x_2$ .

**Definition:**  $x_1^* \in X_1$  is a **minmaximizer** for Player 1 if

$$\min_{x_2 \in X_2} (x_1^*)^T A x_2 = \max_{x_1 \in X_1} \min_{x_2 \in X_2} (x_1)^T A x_2$$

Similarly,  $x_2^* \in X_2$  is a **maximizer** for Player 2 if

$$\max_{x_1 \in X_1} (x_1)^T A x_2^* = \min_{x_2 \in X_2} \max_{x_1 \in X_1} x_1^T A x_2$$

Note that

$$\min_{x_2 \in X_2} (x_1^*)^T A x_2 \leq (x_1^*)^T A x_2^* \leq \max_{x_1 \in X_1} x_1^T A x_2^*$$

Amazingly, von Neumann (1928) showed equality holds!



# The Minimax Theorem

**Theorem**(von Neumann) Let a 2p-zs game  $\Gamma$  be given by an  $(m_1 \times m_2)$ -matrix  $A$  of real numbers. There exists a unique value  $v^* \in \mathbb{R}$ , such that there exists  $x^* = (x_1^*, x_2^*) \in X$  such that

1.  $((x_1^*)^T A)_j \geq v^*$ , for  $j = 1, \dots, m_2$ .
2.  $(Ax_2^*)_j \leq v^*$ , for  $j = 1, \dots, m_1$ .
3. And (thus)  $v^* = (x_1^*)^T Ax_2^*$  and

$$\max_{x_1 \in X_1} \min_{x_2 \in X_2} (x_1)^T Ax_2 = v^* = \min_{x_2 \in X_2} \max_{x_1 \in X_1} x_1^T Ax_2$$

4. In fact, the above conditions all hold precisely when  $x^* = (x_1^*, x_2^*)$  is any Nash Equilibrium.  
Equivalently, they hold precisely when  $x_1^*$  is any minmaximizer and  $x_2^*$  is any maxminimizer.

## some remarks

Note:

(1.) says  $x_1^*$  guarantees Player 1 at least expected profit  $v^*$ ,  
and

(2.) says  $x_2^*$  guarantees Player 2 at most expected “loss”  $v^*$ .

We call any such  $x^* = (x_1^*, x_2^*)$  a **minimax profile**.

We call the unique  $v^*$  the **minimax value** of game  $\Gamma$ .

It is obvious that the maximum profit that Player 1 can guarantee for itself should be  $\leq$  the minimum loss that Player 2 can guarantee for itself, i.e., that

$$\max_{x_1 \in X_1} \min_{x_2 \in X_2} (x_1)^T A x_2 \leq \min_{x_2 \in X_2} \max_{x_1 \in X_1} x_1^T A x_2$$

What is not obvious at all is why these two values should be the same!

# Proof of the Minimax Theorem

The Minimax Theorem follows directly from Nash's Theorem (but historically, it predates Nash).

**Proof:** Let  $x^* = (x_1^*, x_2^*) \in X$  be a NE of the 2-player zero-sum game  $\Gamma$ , with matrix  $A$ .

Let  $v^* := (x_1^*)^T A x_2^* = U_1(x^*) = -U_2(x^*)$ .

Since  $x_1^*$  and  $x_2^*$  are “best responses” to each other, we know that for  $i \in \{1, 2\}$

$$U_i(x_{-i}^*; \pi_{i,j}) \leq U_i(x^*).$$

But

1.  $U_1(x_{-1}^*; \pi_{1,j}) = (A x_2^*)_j$ . Thus,

$$(A x_2^*)_j \leq v^* = U_1(x^*)$$

for all  $j = 1, \dots, m_1$ .

2.  $U_2(x_{-2}^*; \pi_{2,j}) = -((x_1^*)^T A)_j$ . Thus,

$$((x_1^*)^T A)_j \geq v^* = -U_2(x^*)$$

for all  $j = 1, \dots, m_2$ .

3.  $\max_{x_1 \in X_1} (x_1)^T A x_2^* \leq v^*$  because  $(x_1)^T A x_2^*$  is a “weighted average” of  $(A x_2^*)_j$ ’s.

Similarly,  $v^* \leq \min_{x_2 \in X_2} (x_1^*)^T A x_2$  because  $(x_1^*)^T A x_2$  is a “weighted average” of  $((x_1^*)^T A)_j$ ’s. Thus

$$\max_{x_1 \in X_1} (x_1)^T A x_2^* \leq v^* \leq \min_{x_2 \in X_2} (x_1^*)^T A x_2$$

We earlier noted the opposite inequalities, so,

$$\min_{x_2 \in X_2} \max_{x_1 \in X_1} x_1^T A x_2 = v^* = \max_{x_1 \in X_1} \min_{x_2 \in X_2} (x_1)^T A x_2$$

4. We didn’t assume anything about the particular Nash Equilibrium we chose. So, for every NE,  $x^*$ , letting  $v' = (x_1^*)^T A x_2^*$ ,

$$\max_{x_1 \in X_1} \min_{x_2 \in X_2} (x_1)^T A x_2 = v' = v^* = \min_{x_2 \in X_2} \max_{x_1 \in X_1} x_1^T A x_2$$

Moreover, if  $x^* = (x_1^*, x_2^*)$  satisfies conditions (1.) and (2.) for some  $v^*$ , then  $x^*$  must be a Nash Equilibrium.

**Q.E.D. (Minimax Theorem)**

# remarks and food for thought

- ▶ Thus, for 2-player zero-sum games, Nash Equilibria and Minimax profiles are the same thing.

- ▶ Let us note here

**Useful Corollary for Minimax:** In a minimax profile

$$x^* = (x_1^*, x_2^*),$$

1. if  $x_2^*(j) > 0$  then  $((x_1^*)^T A)_j = (x_1^*)^T A x_2^* = v^*$ .
2. if  $x_1^*(j) > 0$  then  $(A x_2^*)_j = (x_1^*)^T A x_2^* = v^*$ .

This is an immediate consequence of the Useful Corollary for Nash Equilibria.

- ▶ If you were playing a 2-player zero-sum game (say, as player 1) would you always play a minmaximizer strategy?
- ▶ What if you were convinced your opponent is an idiot?
- ▶ Notice, we have no clue yet how to compute the minimax value and a minimax profile.

That is about to change.

# minimax as an optimization problem

Consider the following “optimization problem”:

**Maximize**  $v$

**Subject to constraints:**

$$(x_1^T A)_j \geq v \text{ for } j = 1, \dots, m_2,$$

$$x_1(1) + \dots + x_1(m_1) = 1,$$

$$x_1(j) \geq 0 \text{ for } j = 1, \dots, m_1$$

It follows from the minimax theorem that an optimal solution  $(x_1^*, v^*)$  would give precisely the minimax value  $v^*$ , and a minmaximizer  $x_1^*$  for Player 1.

We are optimizing a “linear objective”,  
under “linear constraints” (or “linear inequalities”).

That’s what Linear Programming is.

Fortunately, we have good algorithms for it.

Next time, we start Linear Programming.