AGTA Tutorial Sheet 2 solutions

1 Question 1

Consider the finite 2 player zero sum game given by the following payoff matrix A, for player 1 (row player):

$$A = \begin{pmatrix} 4 & 2 & 9 & 2 & 5 \\ 6 & 3 & 5 & 9 & 7 \\ 1 & 4 & 8 & 5 & 7 \\ 5 & 1 & 3 & 5 & 6 \end{pmatrix}$$

Observe that the last row is strictly dominated by the second row. Also, the second column strictly dominates the 5th column (since $-2 > -5$, $-3 > -7$, $-4 > -7$) obtaining in the residual game matrix:

$$\begin{pmatrix} 4 & 2 & 9 & 2 \\ 6 & 3 & 5 & 9 \\ 1 & 4 & 8 & 5 \end{pmatrix}$$

Note that the second column strictly dominates the 3rd column, since $-2 > -9$,$-3 > -5$, $-4 > -8$. Now, row 1 is strictly dominated by row 2, and in the residual game, column 2 strictly dominates column 4, since $-3 > -9$,$-4 > -5$, leaving us with the final residual game:

$$A' = \begin{pmatrix} 6 & 3 \\ 1 & 4 \end{pmatrix}$$

This is a 2×2 game. The answer to the next question will show that there is in fact a very simple way to "solve" a 2×2 game without using LP, even when it is not zero-sum. However, just to illustrate the use of LP for solving zero-sum games, and to illustrate the use of Fourier-Motzkin (FM) elimination to solve LPs, we will use LP, and FM, to solve this 2×2 zero-sum game.

To compute the minmaximizer strategy for player 1, as well as the value of the game v, we can set up the following LP: Let $x^T = [p_1 \;\; p_2]$ to represent the minmaximizer strategy for player 1. The LP is:

Maximize v

Subject to:

$6p_1 + p_2 \geq v$

$3p_1 + 4p_2 \geq v$

$p_1 + p_2 = 1$

$p_1, p_2 \geq 0.$
Using $p_2 = 1 - p_1$, and substituting $(1 - p_1)$ for p_2, the system becomes:
Maximize v
Subject to:
$p_1 \geq \frac{v-1}{5}$
$p_1 \leq 4 - v$
$p_1 \geq 0$, $p_1 \leq 1$
Using FM to eliminate the variable p_1, we have:
$\frac{v-1}{5} \leq 4 - v$, or equivalently $v \leq \frac{21}{6} = 3.5$
$\frac{v-1}{5} \leq 1$, or equivalently $v \leq 6$,
and
$4 - v \geq 0$, or equivalently, $v \leq 4$.

Note that to maximize v we can match the upper bound constraint that is most stringent, in this case $v \leq 3.5$. Therefore, maximizing v, we have that $v = 3.5$. Therefore the optimal solution is $v = 3.5$, and plugging this back into the eliminated inequalities, we get that $p_1 = p_2 = 0.5$.

Taking $y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ to represent the maxminimizer strategy for player 2, the dual of the LP presented above is:
Minimize u
subject to:
$6q_1 + 3q_2 \leq u$
$q_1 + 4q_2 \leq u$
$q_1 + q_2 = 1$
$q_1, q_2 \geq 0$
Using $q_2 = 1 - q_1$, the system becomes
Minimize u
subject to
$q_1 \leq \frac{u-3}{3}$
$q_1 \geq \frac{4-u}{3}$
$q_1 \geq 0$
$q_1 \leq 1$
By eliminating q_1, we have:
$\frac{4-u}{3} \leq \frac{u-3}{3}$, or equivalently $u \geq \frac{7}{2} = 3.5$
$\frac{4-u}{3} \leq 1$, or equivalently $u \geq 1$
$\frac{u-3}{3} \geq 0$, or equivalently $u \geq 3$.

Since we minimize u, we conclude that the minimum value which satisfies the constraints, based on the most stringent lower bound constraint, is $u = 3.5$, and therefore plugging this back into the eliminated constraints, we get $q_1 = \frac{1}{6}$ and $q_2 = \frac{5}{6}$. Note that, as expected, $u = v$, as indeed it should because this follows by the minimax theorem.

2 Question 2

$$G = \begin{pmatrix} (7,3) & (6,4) & (5,5) & (4,7) \\ (4,2) & (7,9) & (8,6) & (8,8) \\ (6,1) & (9,7) & (2,4) & (6,9) \end{pmatrix}$$

Note that column 1 is strictly dominated by column 2, for player 2 (the column player). Also, in the residual game bimatrix, row 1 is strictly dominated by row 2 (for player 1, the row player). Also, column 3 is strictly dominated by column 4. We are now left with the following residual game:

$$G' = \begin{pmatrix} (7,9) & (8,8) \\ (9,7) & (6,9) \end{pmatrix}$$

It is easy to check that there are no pure Nash Equilibria. This is because if either player plays a pure strategy, then it is clear by inspection of the game that the unique best response of the other player is a pure strategy, and it is also clear by inspection of the game that no pair of pure strategies constitutes a NE. In every NE of this risidual game, it must be the case that both players use both of their strategies with positive probability. We use the corollary to Nash's theorem to compute the unique NE in the residual game as follows: Suppose player 1 plays strategy 1 with probability p and strategy 2 with probability (1-p) in some N.E, where $0 < p < 1$. Suppose player 2 plays strategy 1 with probability q and strategy 2 with probability (1-q) in some N.E, where $0 < q < 1$.

Using the corollary of the Nash theorem, if player 2 is playing against player 1's mixed strategy, both of player 2's pure strategies must be a best response to player 1. The same argument applies for player 1.

Therefore,

$$7q + 8(1-q) = 9q + 6(1-q)$$

$$9p + 7(1-p) = 8p + 9(1-p)$$

By doing the arithmetic, we find $p = \frac{2}{3}$ and $q = \frac{1}{2}$.

So, a NE for this game is: $[(0, \frac{2}{3}, \frac{1}{3}, 0); (0, \frac{1}{2}, 0, \frac{1}{2})]$. The expected payoff for player 1 under this strategy profile is 7.5, whereas the expected payoff for player 2 is 8.33.

A strategy that is strictly dominated can not be played with non-zero probability in any NE, and therefore we don't eliminate NE's by eliminating strictly dominated strategies. Also, p and q are both uniquely determined so it must imply that there is only one NE in this game.

Final answer: $[(0, \frac{2}{3}, \frac{1}{3}); (0, \frac{1}{2}, 0, \frac{1}{2})]$

3 Question 3.

Claim: $A = -A^T$ implies $x^T Ay = -y^T Ax$ for all vectors x, y of the right length.

Proof. $x^T Ay = x^T(-A^T)y = -(x^T A^T y) = -(x^T A^T y)^T = -y^T Ax$, where the second to last step uses the fact that $B^T = B$ for all 1×1-matrices, and the last step uses the facts that $(B^T)^T = B$ and $(BC)^T = C^T B^T$. (One could of course prove the claim by e.g. direct calculation) $\square$

In particular, the claim implies that $x^T Ax = -x^T Ax$, which gives $x^T Ax = 0$. This means that whenever both players play with the same mixed strategy x, they both have an expected payoff of zero. Thus in any strategy profile (x, y), if one of the players has a negative expected payoff, they can improve by copying the other players strategy. Thus no strategy profile giving non-zero expected payoffs can be a Nash equilibrium of the game.

4 Question 4

Using the recipe from page 12 of the slides for lecture 4, we get the linear program

Maximize v
Subject to:
$(x^T A)_j \geq v \;\;,\;\; j = 1, 2, 3$
$\sum_{i=1}^{2} x_i = 1$
$x_1 \geq 0 \;,\;\; x_2 \geq 0$

Writing this out explicitly, we get the linear program

Maximize v
Subject to:
$2x_1 + 7x_2 \geq v$
$9x_1 + 0x_2 \geq v$
$4x_1 + 3x_2 \geq v$
$x_1 + x_2 = 1$
$x_1 \geq 0, x_2 \geq 0$

which is equivalent to the linear program that was given in the question. To compute the dual using the general recipe, we first need to express this LP in the general form of the "primal" in the "general recipe" (slide 8, of Lecture 7 on LP duality), namely:

Maximize $c^T x$
Subject to:
$(Bx)_i \leq b_i \;\;,\;\; i = 1, \ldots, d$
$(Bx)_j = b_j \;\;,\;\; i = d + 1, \ldots, m$

$x_i \geq 0 \;\;, \;\; i = 1, \ldots, r$

Using the given linear program we can get to the desired form by setting $b^T = (0,0,0,1)$, $c^T = (0,0,1)$, $x^T = (x_1, x_2, v)$, and

$$B = \begin{bmatrix} -2 & -7 & 1 \\ -9 & 0 & 1 \\ -4 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

To be clear with the indices, our linear program is then
Maximize $c^T x$
Subject to:
$(Bx)_i \leq b_i$ for $i = 1, 2, 3$
$(Bx)_4 = b_4$
$x_i \geq 0$ for $i = 1, 2$

Now, using the general recipe the dual is

Minimize $b^T y$
Subject to:
$(B^T y)_i \geq c_i$ for $i = 1, 2$
$(B^T y)_3 = c_3$
$y_i \geq 0$ for $i = 1, 2, 3,$

which, when setting $y = (y_1, y_2, y_3, v)$ translates to

Mimimize v
Subject to:
$-2y_1 - 9y_2 - 4y_3 + v \geq 0$
$-7y_1 + 0y_2 - 3y_3 + v \geq 0$
$y_1 + y_2 + y_3 = 1$
$y_i \geq 0$ for $i = 1, 2, 3$

which is easily seen to be equivalent to the LP for computing a maxminimizer strategy, y, for player 2 together with the (minimax) value, v, of the game.