AGTA Tutorial Sheet 3 Solutions

1 Question 1

Given the finite 2 player bimatrix game from below, we will use iterated elimination of *strictly* dominated strategies.

$$\begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{pmatrix} (5,2) & (22,4) & (4,9) & (7,6) \\ (16,4) & (18,5) & (1,10) & (10,2) \\ (15,12) & (16,9) & (18,10) & (11,3) \\ (9,15) & (23,9) & (11,5) & (5,13) \end{pmatrix} \end{array}$$

Player 1 is the row player and player 2 is the column player. We know that a pure strategy can be dominated by either a pure strategy or a *mixed* one. Observe that c_2 is strictly dominated by $(\frac{1}{2}c_1; \frac{1}{2}c_3)$ (easy to check). Therefore, we can eliminate strategy c_2 and obtain the following 4×3 residual game:

$$\begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & c_3 & c_4 \end{array} \\ \begin{pmatrix} (5,2) & (4,9) & (7,6) \\ (16,4) & (1,10) & (10,2) \\ (15,12) & (18,10) & (11,3) \\ (9,15) & (11,5) & (5,13) \end{pmatrix} \end{array}$$

Observe that r_1 and r_4 are now strictly dominated by the pure strategy r_3. After r_1 and r_4 are eliminated, in the resulting residual game, strategy c_4 is strictly dominated by c_3. Eliminating c_4 then results in the following 2×2 residual game:

$$\begin{array}{c} \\ r_2 \\ r_3 \end{array} \begin{array}{c} \begin{array}{cc} c_1 & c_3 \end{array} \\ \begin{pmatrix} (16,4) & (1,10) \\ (15,12) & (18,10) \end{pmatrix} \end{array}$$

There are no remaining strictly dominated strategies in this residual game. Furthermore, there are no pure NEs in this game, and every pure strategy for either player has a unique, pure, best response. This implies the only NE in this game must be "fully mixed" meaning both players must play both of their pure strategies with positive probability in any NE. Applying the same principle as in question 2 from Tutorial 2, we can set up simple equations to calculate these probabilities. In particular, letting p denote the probability of r_2 and $(1-p)$ denote the probability of r_3, we must have $4p + 12(1-p) = 10p + 10(1-p)$, and thus $p = \frac{1}{4}$ and $(1-p) = \frac{3}{4}$. We can set up a similar equation for player 2's mixed strategy, letting q denote the probability of c_1 and $(1-q)$ the probability of c_3, we must have $16q + (1-q) = 15q + 18(1-q)$, and thus $q = \frac{17}{18}$ and $(1-q) = \frac{1}{18}$. Thus the unique Nash equilibrium of this residual game is $((\frac{1}{4}r_2; \frac{3}{4}r_3); (\frac{17}{18}c_1; \frac{1}{18}c_3))$. Since we obtained the residual game by only iteratively eliminating strictly dominated strategies, the unique NE of the original bimatrix game is given by: $((0, \frac{1}{4}, \frac{3}{4}, 0), (\frac{17}{18}, 0, \frac{1}{18}, 0))$.

2 Question 2

The LCP system of constraints for this bimatrix game is as follows:

$$Mu + v = \mathbf{1}; \quad u \geq 0 \ , \quad v \geq 0 \ , \quad u^T v = 0.$$

where

$$M = \begin{pmatrix} 0 & 0 & 9 & 1 \\ 0 & 0 & 5 & 4 \\ 3 & 7 & 0 & 0 \\ 6 & 2 & 0 & 0 \end{pmatrix}$$

and $u = [x'_{1,1} \ x'_{1,2} \ x'_{2,1} \ x'_{2,2}]^T$ and $v = [y'_1 \ y'_2 \ z'_1 \ z'_2]^T$ are the vectors of 8 distinct variables (4 in each vector) used in the LCP.
We can write the constraints $Mu + v = \mathbf{1}$ as follows:

$$\begin{aligned} y'_1 &= 1 - 9x'_{2,1} - x'_{2,2} \\ y'_2 &= 1 - 5x'_{2,1} - 4x'_{2,2} \\ z'_1 &= 1 - 3x'_{1,1} - 7x'_{1,2} \\ z'_2 &= 1 - 6x'_{1,1} - 2x'_{1,2} \end{aligned}$$

Note that this is a *feasible dicationary*, with basis $B^0 = \{y'_1, y'_2, z'_1, z'_2\}$, for which the corresponding basic feasible solution (BFS) is $x'_{1,1} = x'_{1,2} = x'_{2,1} = x'_{2,2} = 0$ and $y'_1 = y'_2 = y'_3 = y'_4 = 1$.

This is also a *complementary* BFS in the sense that $u^T v = 0$, meaning that for every complementary pairing of variables, at least one of them is set to 0 in this BFS.

However, this is a "bogus" complementary BFS, because all $x'_{i,j}$ variables are set to 0, and hence can't be normalized to yield probability distributions and thus don't correspond to any NE. We are looking for a different complementary BFS, where $u \neq \mathbf{0}$.

The question asks us to, firstly, move from the above feasible dictionary with complementary basis B^0, via pivoting, to neighboring, 1-almost complementary, basis by moving the variable $x'_{1,1}$ into the basis, and moving the (uniquely determined) variable out of the basis that would retain *feasibility* of the dictionary.

Examining the above feasible dictionary, it is not hard to check that if we move $x'_{1,1}$ into the basis, the unique variable we must move out of the basis in order to retain feasibility of the dictionary is z'_2. In particular, we can rewrite the equality constraint $z'_2 = 1 - 6x'_{1,1} - 2x'_{1,2}$ as $x'_{1,1} = \frac{1}{6} - \frac{2}{6}x'_{i,2} - \frac{1}{6}z'_2$. If we then carry out this pivot, by plugging in the right hand side of this equality into every other occurence of $x'_{1,1}$, on the right hand sides of any of the rest of the dictionary equalities, we obtain a new feasible dictionary with the following equality constraints:

$$
\begin{aligned}
y'_1 &= 1 - 9x'_{2,1} - x'_{2,2} \\
y'_2 &= 1 - 5x'_{2,1} - 4x'_{2,2} \\
z'_1 &= \frac{1}{2} - 6x'_{1,2} + \frac{1}{2}z'_2 \\
x'_{1,1} &= \frac{1}{6} - \frac{2}{6}x'_{1,2} - \frac{1}{6}z'_2
\end{aligned}
$$

This is again a feasible dictionary, with basis $B^1 = \{y'_1, y'_2, z'_1, x'_{1,1}\}$, but it is no longer complementary, but only 1-almost complementary, because both of the 1st complementary pairing of variables, namely $\{x'_{1,1}, y'_1\}$, are in the basis.

So, according the Lemke-Howson algorithm, we need to pivot again, by bringing the variable paired with the one that was removed from the basis in the prior pivot step into the basis. In the prior pivot step, the variable z'_2 was removed from the basis, and its complementarily paired variable is $x'_{2,2}$, so at the next pivot step we need to move $x'_{2,2}$ into the basis. This time, it is not hard to check that if we pivot to move $x'_{2,2}$ into the basis, then the unique variable we must remove from the basis in order to maintain a feasible dictionary is y'_2. Specifically, we can rewrite the equation $y'_2 = 1 - 5x'_{2,1} - 4x'_{2,2}$ as $x'_{2,2} = \frac{1}{4} - \frac{5}{4}x'_{2,1} - \frac{1}{4}y'_2$, and using this and plugging in the RHS in place of all other occurences of $x'_{2,2}$ occuring on the RHS of other equality constraints, we obtain a new feasible dictionary with the following equality constraints:

$$
\begin{aligned}
y'_1 &= \frac{3}{4} - \frac{31}{4}x'_{2,1} + \frac{1}{4}y'_2 \\
z'_1 &= \frac{1}{2} - 6x'_{1,2} + \frac{1}{2}z'_2 \\
x'_{1,1} &= \frac{1}{6} - \frac{2}{6}x'_{1,2} - \frac{1}{6}z'_2 \\
x'_{2,2} &= \frac{1}{4} - \frac{5}{4}x'_{2,1} - \frac{1}{4}y'_2
\end{aligned}
$$

This is a feasible dictionary, with basis $B^2 = \{y'_1, z'_1, x'_{1,1}, x'_{2,2}\}$, but it is again not a complementary basis, only 1-almost complementary. So, we again need to pivot. This time, since y'_2 was removed from the basis in the previous pivot, we need to move its complementary paired variable $x'_{1,2}$ into the basis. It can be checked that the unique variable that needs to be removed from the basis when moving $x'_{1,2}$ into the basis, in order to retain feasibility, is z'_1. Specifically, we can rewrite the equation $z'_1 = \frac{1}{2} - 6x'_{1,2} + \frac{1}{2}z'_2$ as

Then plugging in the RHS of this into other occurences of $x'_{1,2}$ in other equations, we get a new feasible dictionary with the following equality constraints:

$$
\begin{aligned}
y'_1 &= \frac{3}{4} - \frac{31}{4}x'_{2,1} + \frac{1}{4}y'_2 \\
x'_{1,1} &= \frac{5}{36} + \frac{1}{18}z'_1 - \frac{7}{36}z'_2 \\
x'_{2,2} &= \frac{1}{4} - \frac{5}{4}x'_{2,1} - \frac{1}{4}y'_2 \\
x'_{1,2} &= \frac{1}{12} - \frac{1}{6}z'_1 + \frac{1}{12}z'_2.
\end{aligned}
$$

This is a feasible dictionary, with basis $B^3 = \{y'_1, x'_{1,1}, x'_{2,2}, x'_{1,2}\}$, but it is again not a complementary basis, only 1-almost complementary. So, we again need to pivot. The variable that was removed from the basis in the last pivot is z'_1, so we now need to move its complementary paired variable $x'_{2,1}$ into the basis. In doing so, it can be checked that the unique variable that can be removed from the basis while maintaining feasibility is y'_1. Specifically, we can rewrite the equality constraint $y'_1 = \frac{3}{4} - \frac{31}{4}x'_{2,1} + \frac{1}{4}y'_2$ as $x'_{2,1} = \frac{3}{31} - \frac{4}{31}y'_1 + \frac{1}{31}y'_2$. Substituting the RHS of this for all other occurences of $x'_{2,1}$, we obtain the new dictionary

$$
\begin{aligned}
x'_{1,1} &= \frac{5}{36} + \frac{1}{18}z'_1 - \frac{7}{36}z'_2 \\
x'_{2,2} &= \frac{4}{31} + \frac{5}{31}y'_1 - \frac{9}{31}y'_2 \\
x'_{1,2} &= \frac{1}{12} - \frac{1}{6}z'_1 + \frac{1}{12}z'_2 \\
x'_{2,1} &= \frac{3}{31} - \frac{4}{31}y'_1 + \frac{1}{31}y'_2
\end{aligned}
$$

This is a feasible dictionary, with basis $B^4 = \{x'_{1,1}, x'_{2,2}, x'_{1,2}, x'_{2,1}\}$, and moreover this is a genuinely *complementary* basis, and hence the corresponding BFS also satisfies the complementarity constraint $u^T v = 0$.

Hence, we have found a different complementary BFS solution to the LCP, given by $x'_{1,1} = \frac{5}{36}$, $x'_{1,2} = \frac{1}{12}$, $x'_{2,1} = \frac{3}{31}$, and $x'_{2,2} = \frac{4}{31}$ (and all other, non-basis, variables set to 0).

To obtain a Nash Equilibrium from this solution, all we need to do is to normalize the values of $x'_{i,j}$ variables to obtain probability distributions. Specifically, we need to normalize the variable pair $(x'_{1,1}, x'_{1,2})$ by multplying both by a suitable constant $w_2 > 0$ to obtain $x_{1,1} = w_2 \cdot x'_{1,1}$ and $x_{1,2} = w_2 \cdot x'_{1,2}$ such that $x_{1,1} + x_{1,2} = 1$. It is simple to check that the unique such normalizing value is $w_2 = \frac{36}{8} = \frac{9}{2}$. This yields $x_{1,1} = \frac{5}{8}$ and $x_{1,2} = \frac{3}{8}$. In the same way, the unique normalizing constant $w_1 > 0$ such that letting $x_{2,1} = w_1 \cdot x'_{2,1}$ and $x_{2,2} = w_1 \cdot x'_{2,2}$ will yield that $x_{2,1} + x_{2,2} = 1$ is $w_1 = \frac{31}{7}$, which yields $x_{2,1} = \frac{3}{7}$ and $x_{2,2} = \frac{4}{7}$.

Hence, the Nash Equilibrium computed by the Lemke-Howson algorithm for this bimatrix game is

$$((5/8, 3, 8), (3/7, 4/7))$$

and under this NE the expected payoff to player 1 is $w_1 = \frac{31}{7}$ and the expected payoff to player 2 is $w_2 = \frac{9}{2}$.

(In fact, as is easily checked via other methods, this is the unique NE of this simple 2×2 bimatrix game. In general, for arbitrary bimatrix games which may have multiple equilibria, the Lemke-Howson algorithm finds *one* NE for the game, but not necessarily a "good" one in any particular sense. In particular, the NE it finds need not be one that maximizes social welfare, or any other similar objective.)

Clearly, the Lemke-Howson algorithm is not the best way to find a NE for such simple 2×2 games. But for large bimatrix games Lemke-Howson provides a viable method to compute *some* Nash equilibrium (even though in the worst case it can require exponentially many pivots, as a function the number of pure strategies in the game).