AGTA Tutorial Sheet 3

Please read and attempt these questions before coming to the tutorial.

1. Use iterated elimination of strictly dominated strategies in order to find the unique Nash Equilibrium in this finite 2-player (bimatrix) game, by first reducing the game to a 2×2 game. (Recall that a pure strategy may be strictly dominated by a mixed strategy.)

$$\begin{bmatrix} (5,2) & (22,4) & (4,9) & (7,6) \\ (16,4) & (18,5) & (1,10) & (10,2) \\ (15,12) & (16,9) & (18,10) & (11,3) \\ (9,15) & (23,9) & (11,5) & (5,13) \end{bmatrix}$$

2. This question asks you to carry out the Lemke-Howson algorithm for computing a Nash equilibrium for a 2-player (bimatrix) game, namely the following very simple 2×2 bimatrix game:

$$\begin{bmatrix} (9,3) & (1,6) \\ (5,7) & (4,2) \end{bmatrix}$$

(Of course, already in Tutorial 2, as well as in question (1.) of this tutorial, we have learned a much simpler way to compute a Nash Equilbrium for such 2-player 2×2 games, without needing the Lemke-Howson algorithm. This question only aims to elucidate the steps involved in the Lemke-Howson algorithm on a simple example, to help you understand the algorithm.)

Consider the corresponding Linear Complementarity Problem (LCP) for finding a Nash Equilibrium in this bimatrix game, via the Lemke-Howson algorithm. Recall, the LCP has the form:

$$Mu + v = \mathbf{1}; \quad u \geq 0 \ , \quad v \geq 0 \ , \quad u^T v = 0.$$

where

$$M = \begin{pmatrix} 0 & 0 & 9 & 1 \\ 0 & 0 & 5 & 4 \\ 3 & 7 & 0 & 0 \\ 6 & 2 & 0 & 0 \end{pmatrix}$$

and $u = [x'_{1,1} \ x'_{1,2} \ x'_{2,1} \ x'_{2,2}]^T$ and $v = [y'_1 \ y'_2 \ z'_1 \ z'_2]^T$ are the vectors of 8 distinct variables (4 in each vector) used in the LCP.
The correponding "complementary pairing" of the variables in this LCP is as follows:

- the 1st pair is $\{x'_{1,1}, y'_1\}$,
- the 2nd pair is $\{x'_{1,2}, y'_2\}$,
- the 3rd pair is $\{x'_{2,1}, z'_1\}$,
- the 4th pair is $\{x'_{2,2}, z'_2\}$.

Since all these variables are constrained to be non-negative, the complementarity constraint $u^T v = 0$ tells us that in any solution to the LCP, for every complementary pairing, at least one of the two variables in the pairing must be assigned 0. Consider the trivial "complementary basis" for this LCP, consisting of all variables in v, namely the basis $B^0 = \{y'_1, y'_2, z'_1, z'_2\}$, and the corresponding complementary, but "bogus", basic feasible solution (BFS) for this LCP, corresponding to the trivial complementary basis B^0, namely $u = \mathbf{0}$ and $v = \mathbf{1}$. Recall that this is a bogus solution that does not correspond to a Nash Equilibrium, because the non-basis variables $x'_{i,j}$ are all set to zero, and hence can't be normalized to yield probability distributions (mixed strategies) for the two players.

Now, carry out the first step of the Lemke-Howson algorithm, under the (arbitrary) choice of letting $i := 1$, i.e., allowing the 1st (and *only* the 1st) complementary pair $\{x'_{1,1}, y'_1\}$ to possibly violate the complementarity constraint (meaning for all $i \in \{2, 3, 4\}$ we must still have at least one of the two variables in the ith complementary pairing be outside the basis, and hence set to 0 in the corresponding BFS, but not necessarily for $i = 1$). Use a single pivot step to move from the trivial complementary basis, B^0, to a neighboring "i-almost" complementary basis, for $i := 1$, by moving the variable $x'_{1,1}$ into the basis, and removing a (uniquely determined) variable out of the basis so as to obtain a resulting *feasible* dictionary.

Note that in this first pivoting step we could have chosen to move to an i-almost complementary basis for any $i \in \{1, \ldots, 4\}$. That choice was arbitrary. But having chosen $i = 1$, thereafter, in subsequent pivoting steps, we no longer have a choice: the basis remains i-almost complementary, for $i = 1$, until we eventually reach a different actual complementary basis, from which we can extract a Nash Equilibrium of the game by normalizing the resulting variable values in the final BFS corresponding to the new complementary basis, namely the values of variable pairs $(x'_{1,1}\ ,\ x'_{1,2})$ and $(x'_{2,1}\ ,\ x'_{2,2})$ are both normalized, by multiplying them by positive values $w_2 > 0$ and $w_1 > 0$, respectively, so as to yield new value pairs $(x_{1,1} = w_2 \cdot x'_{1,1}\ ,\ x_{1,2} = w_2 \cdot x'_{1,2})$ and $(x_{2,1} = w_1 \cdot x'_{2,1}\ ,\ x_{2,2} = w_1 \cdot x'_{2,2})$, such that both pairs of values are non-negative and sum to 1. These describe the Nash Equilibrium of the game that is computed by the Lemke-Howson algorithm, and w_1 and w_2 provide the expected payoffs to player 1 and player 2, respectively, in that NE.

Carry out the subsequent iterations of the Lemke-Howson algorithm, which proceed as follows: unless you have already arrived at a new genuinely complementary basis (in which case we can stop and compute the NE by normalization), in order to determine which variable to bring into the basis next, first observe from which complementary pairing of variables one of its two variables was removed from the basis in the previous pivot iteration. Suppose this removed variable was some variable named q. We must next bring *the other* variable, call it q', in the same pairing as q into the basis, while maintaining a feasible dictionary. In this setting again (due to the non-degeneracy of this LCP), once the choice of which variable q' to move into the basis has been made, there will be a *unique* variable (whose constraint is the "most strictly bounding" for q') that must be removed from the basis in order to move to a new and *feasible* dictionary, and obtain a new corresponding 1-almost complementary basis. Carry out this pivot. Then repeat the same process, carrying out such pivoting steps repeatedly until you arrive at a complementary basis. At that point use the final BFS to compute a corresponding Nash Equilibrium by normalizing the respective variables as described above. (Note that the respective normalizing values $w_2 > 0$ and $w_1 > 0$ that we multiply by are precisely the expected payoffs for player 2 and player 1, respectively, in the resulting Nash Equilibrium. These are necessarily positive because this bimatrix game contains only positive payoffs.)