Algorithmic Game Theory and Applications

Congestion Games

Routing Traffic

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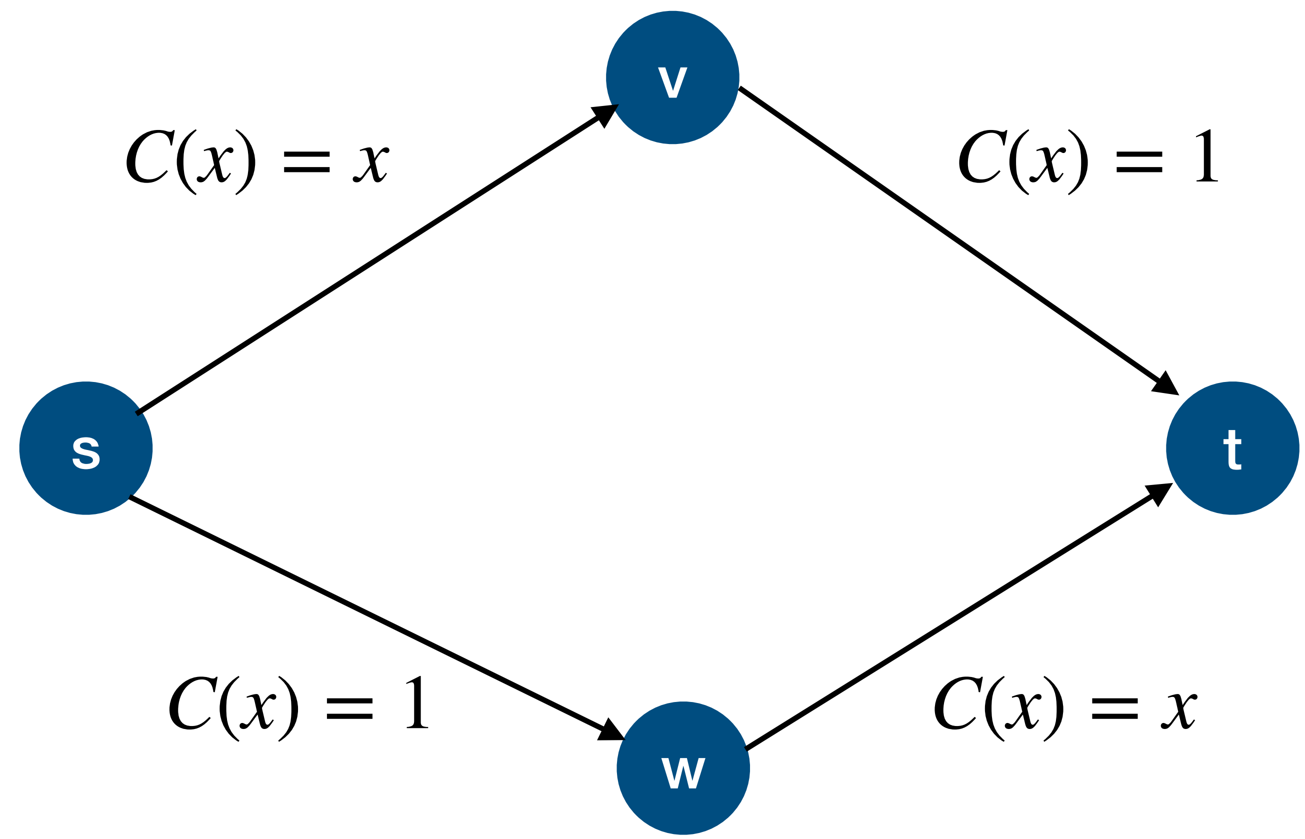
There are two ways to get there, each contains a fast route and a slow route.

Every player controls a small part of the traffic, e.g., 1/100 of the whole unit.



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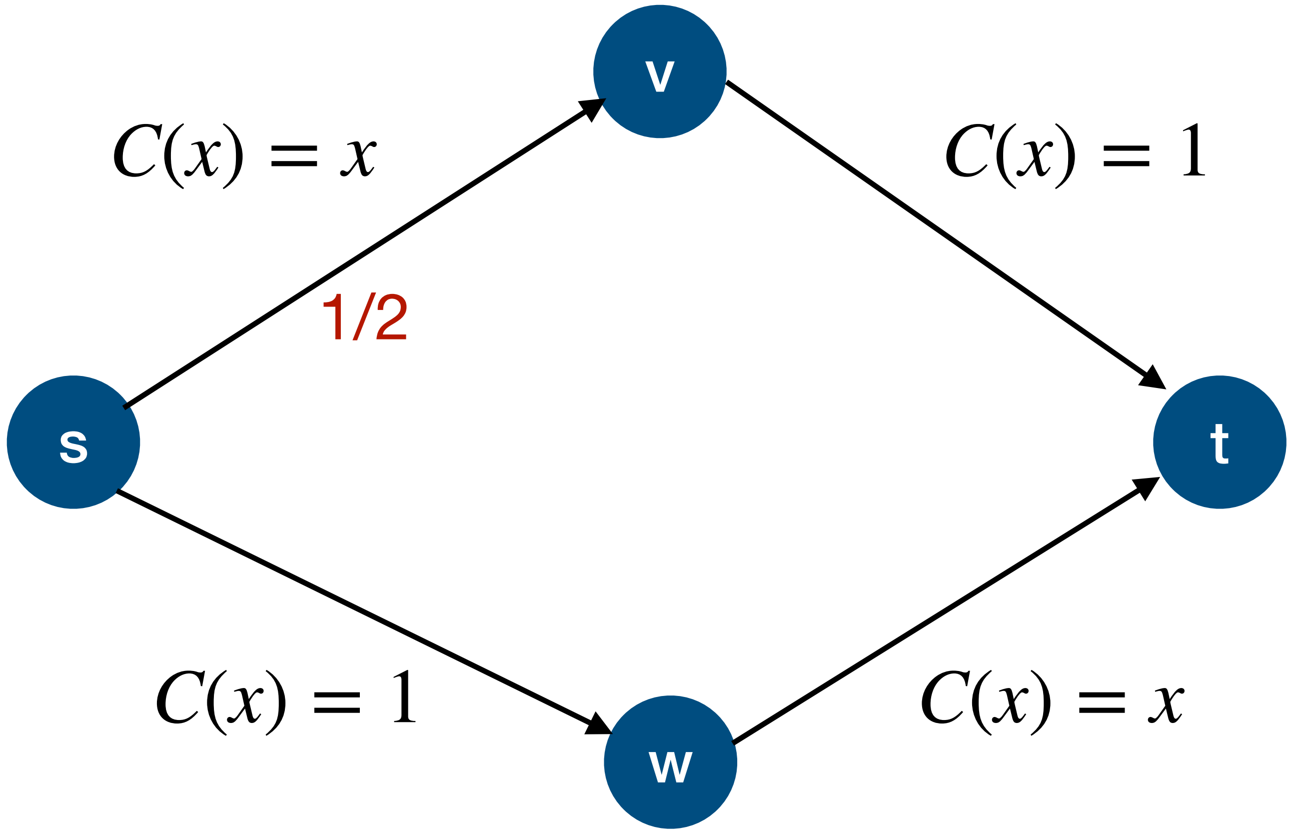

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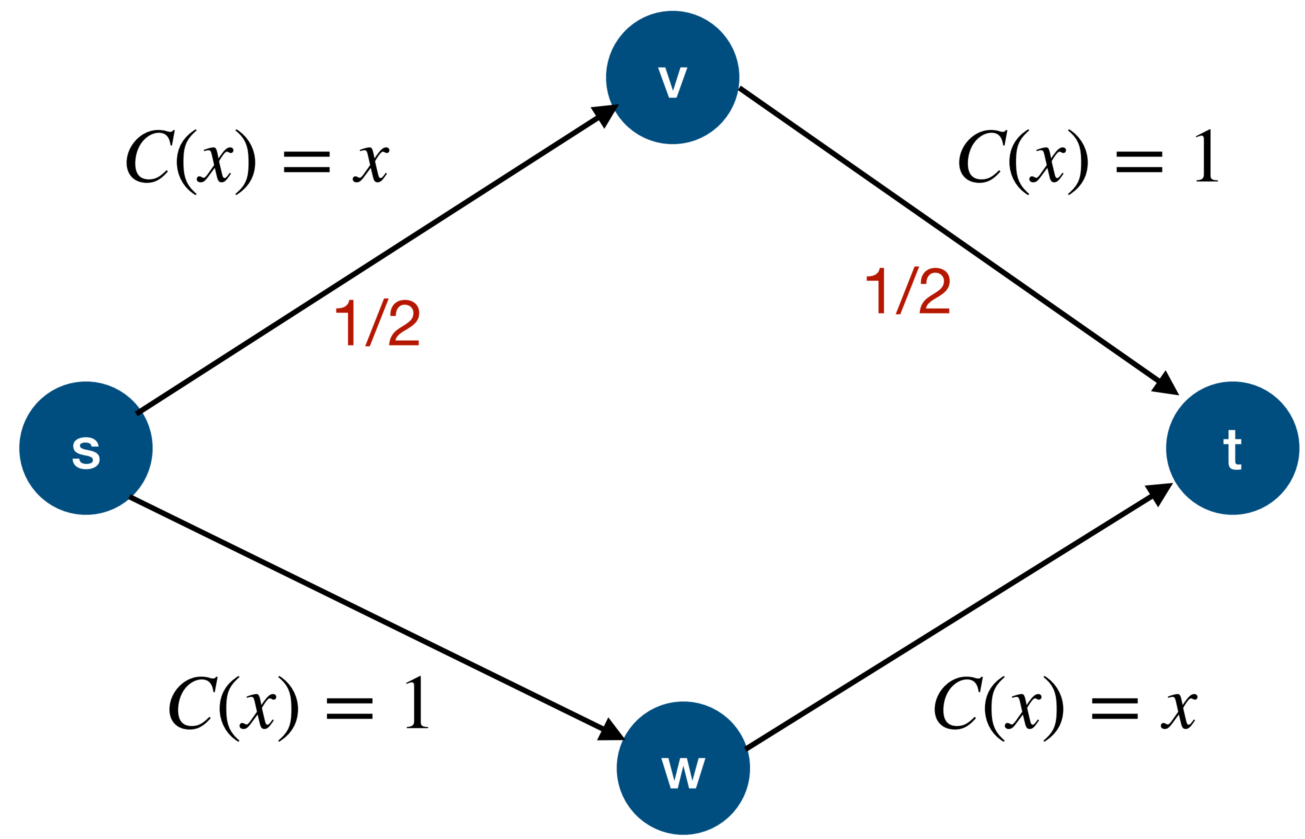

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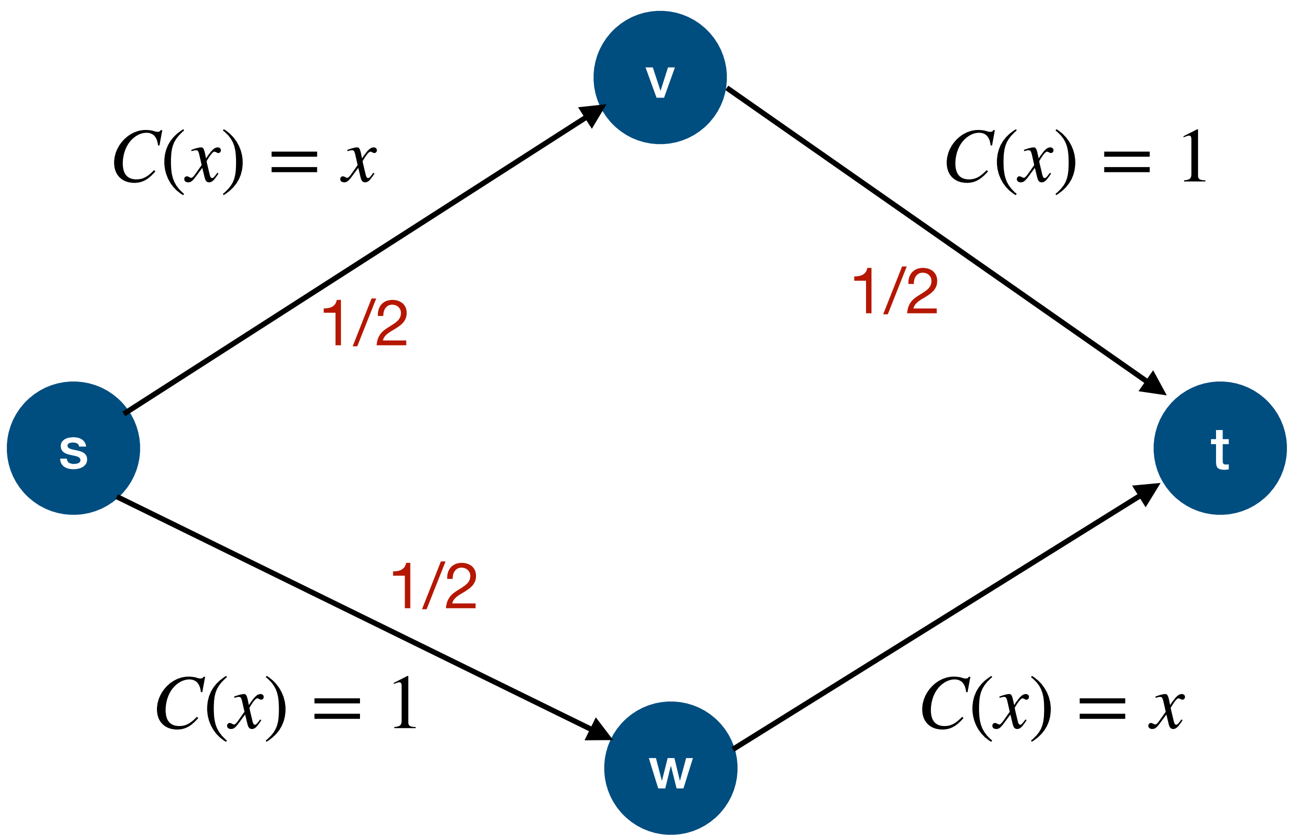

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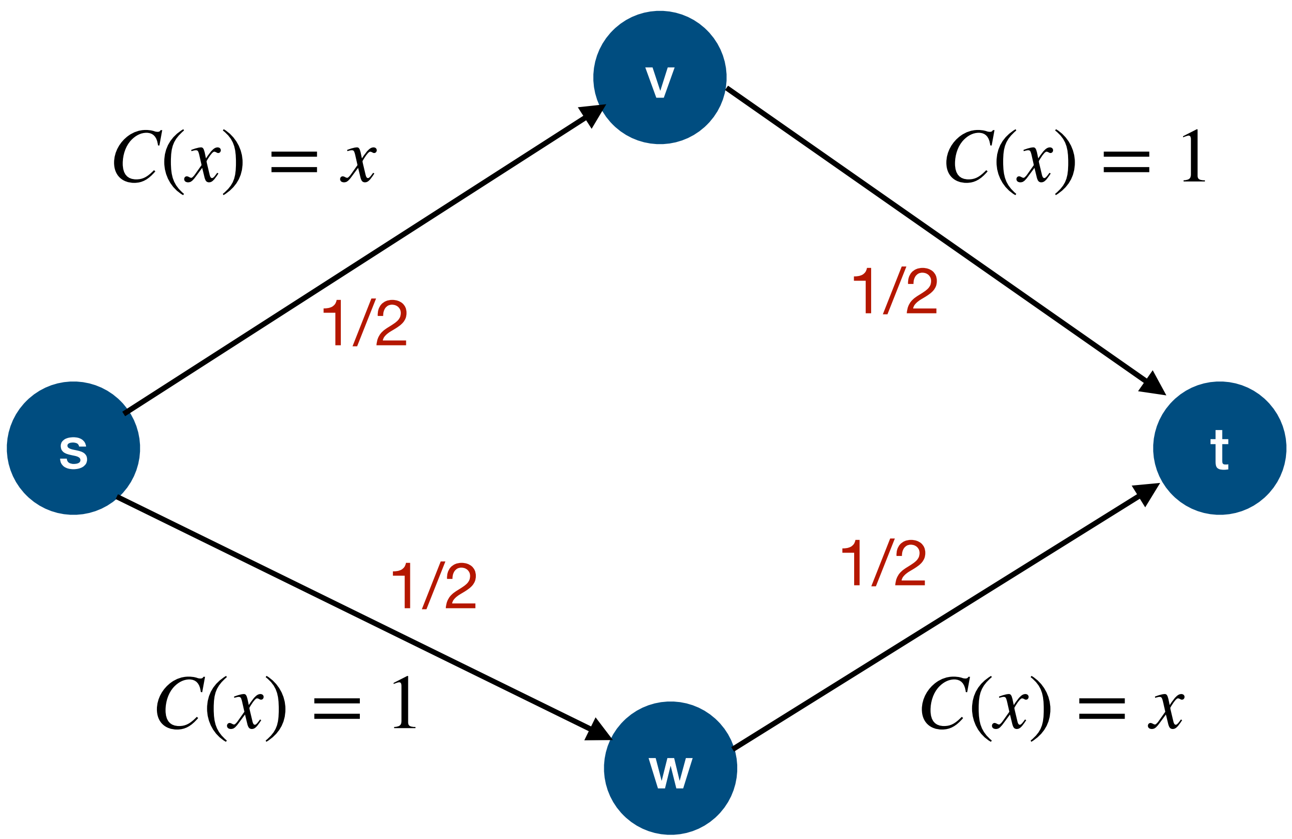

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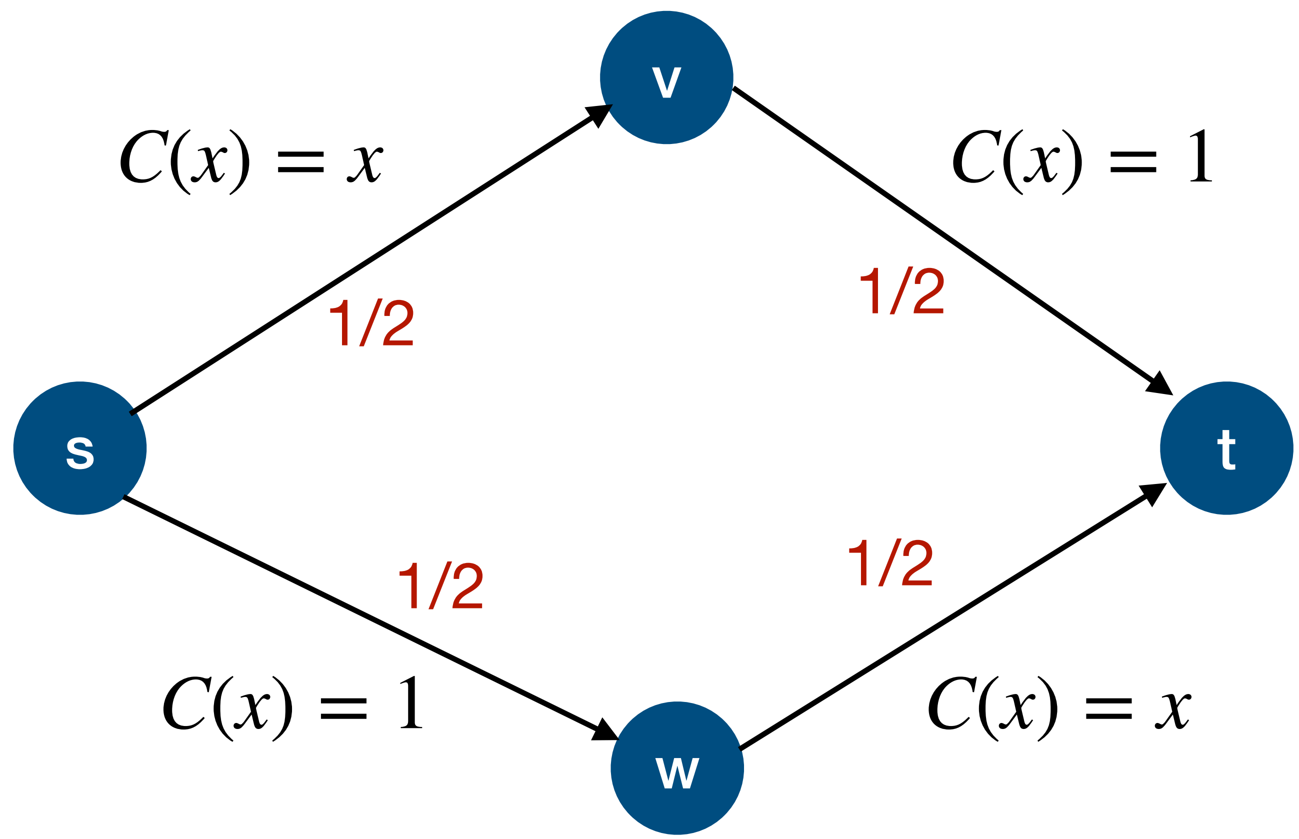

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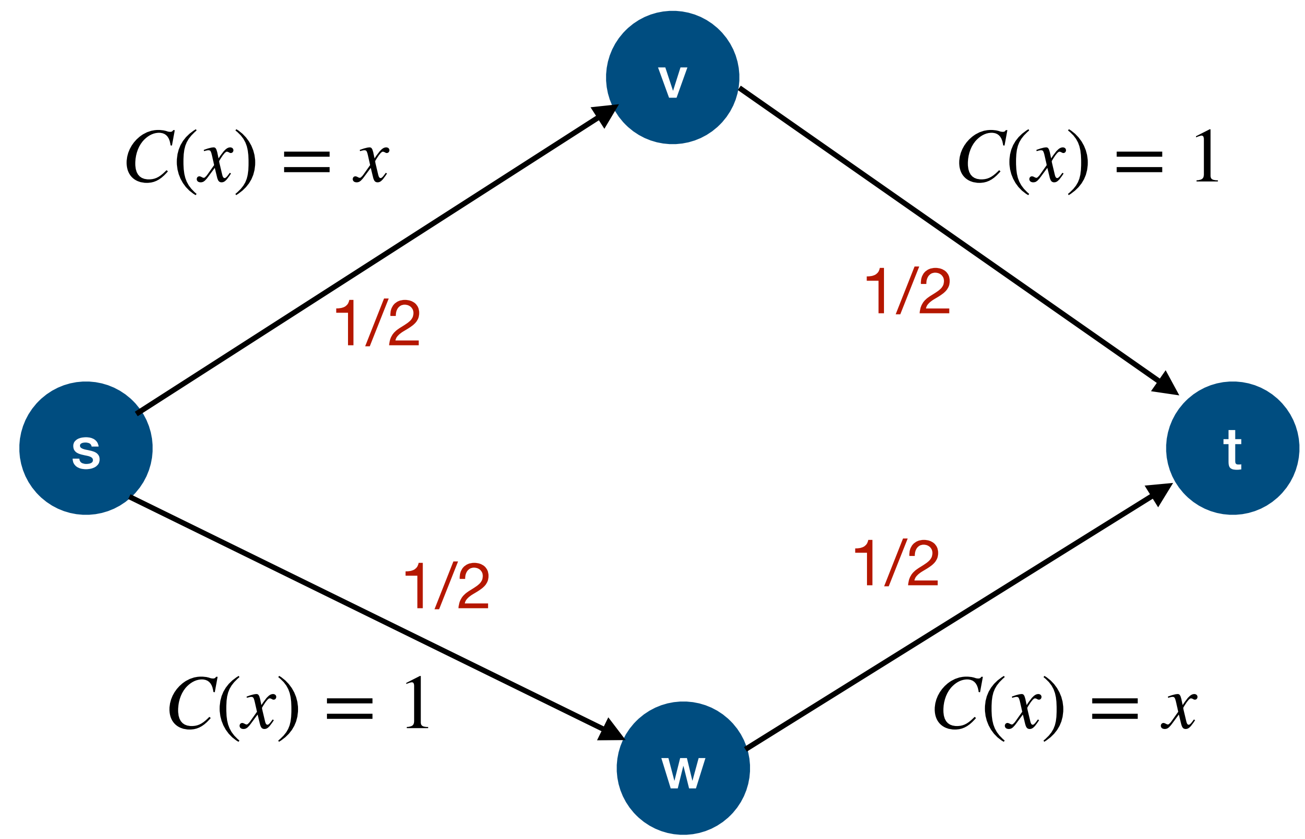

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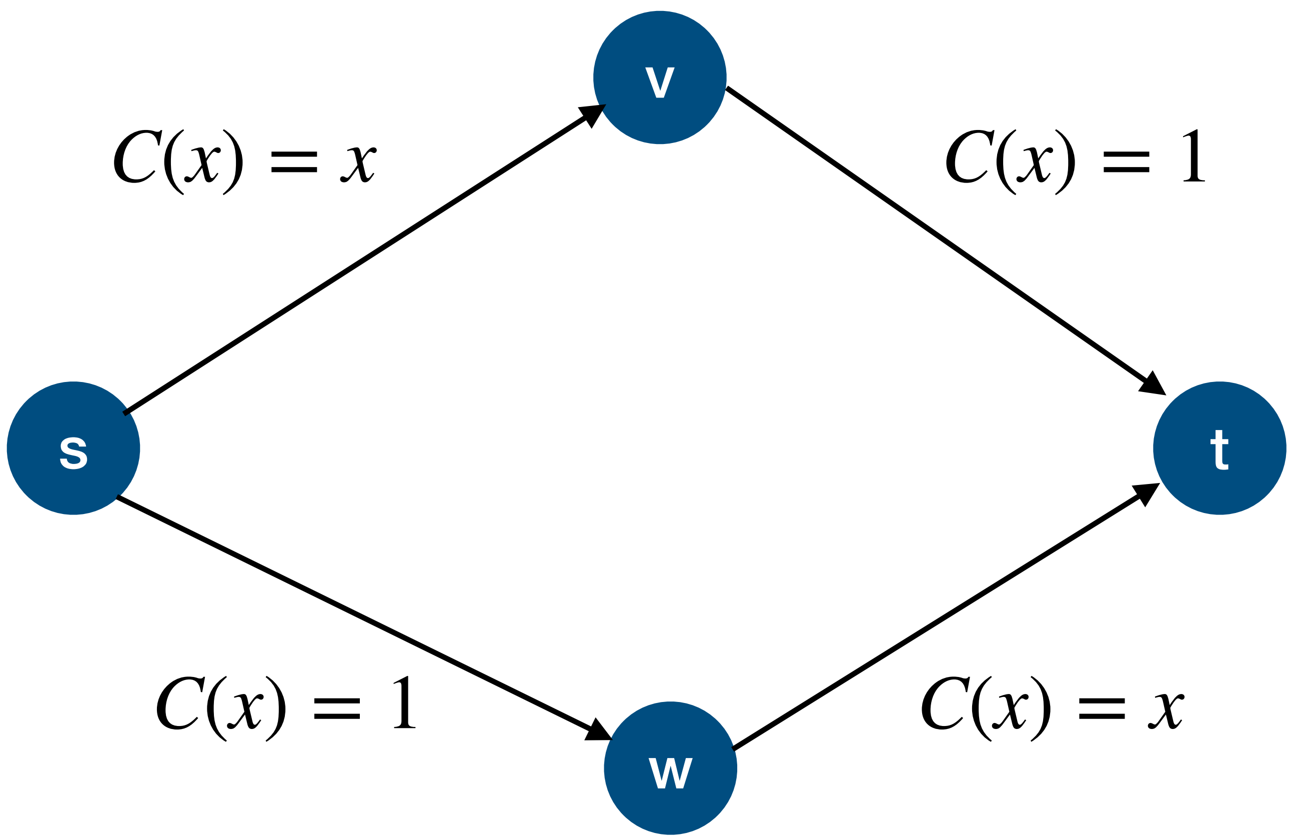

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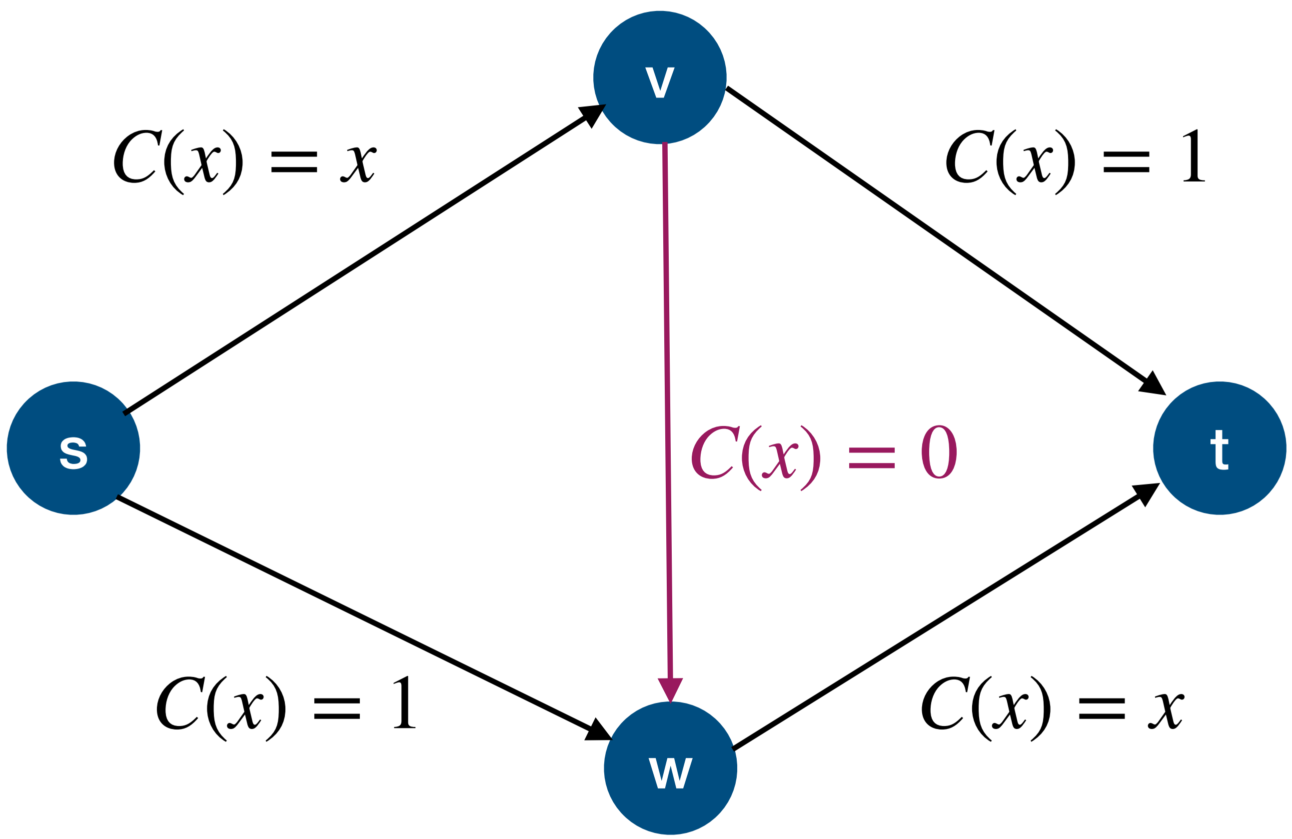

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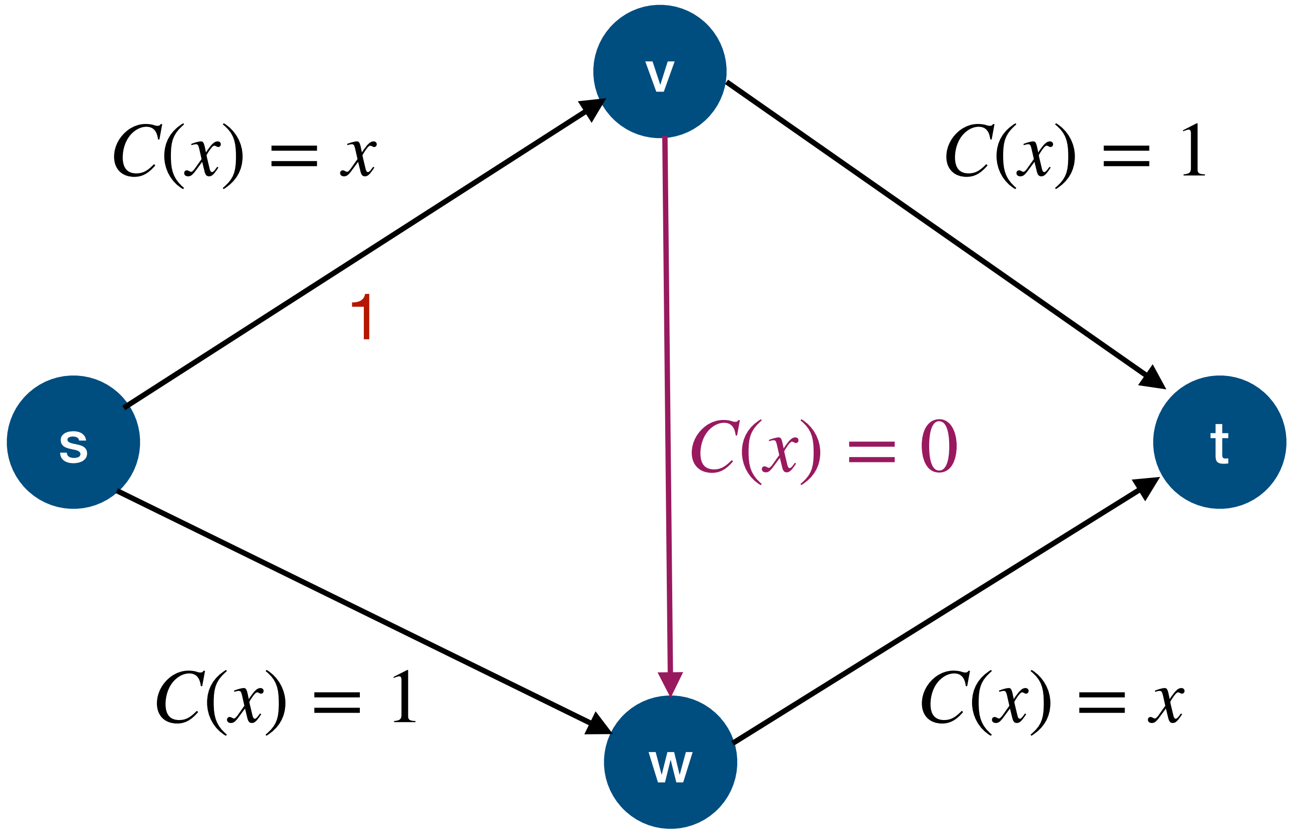

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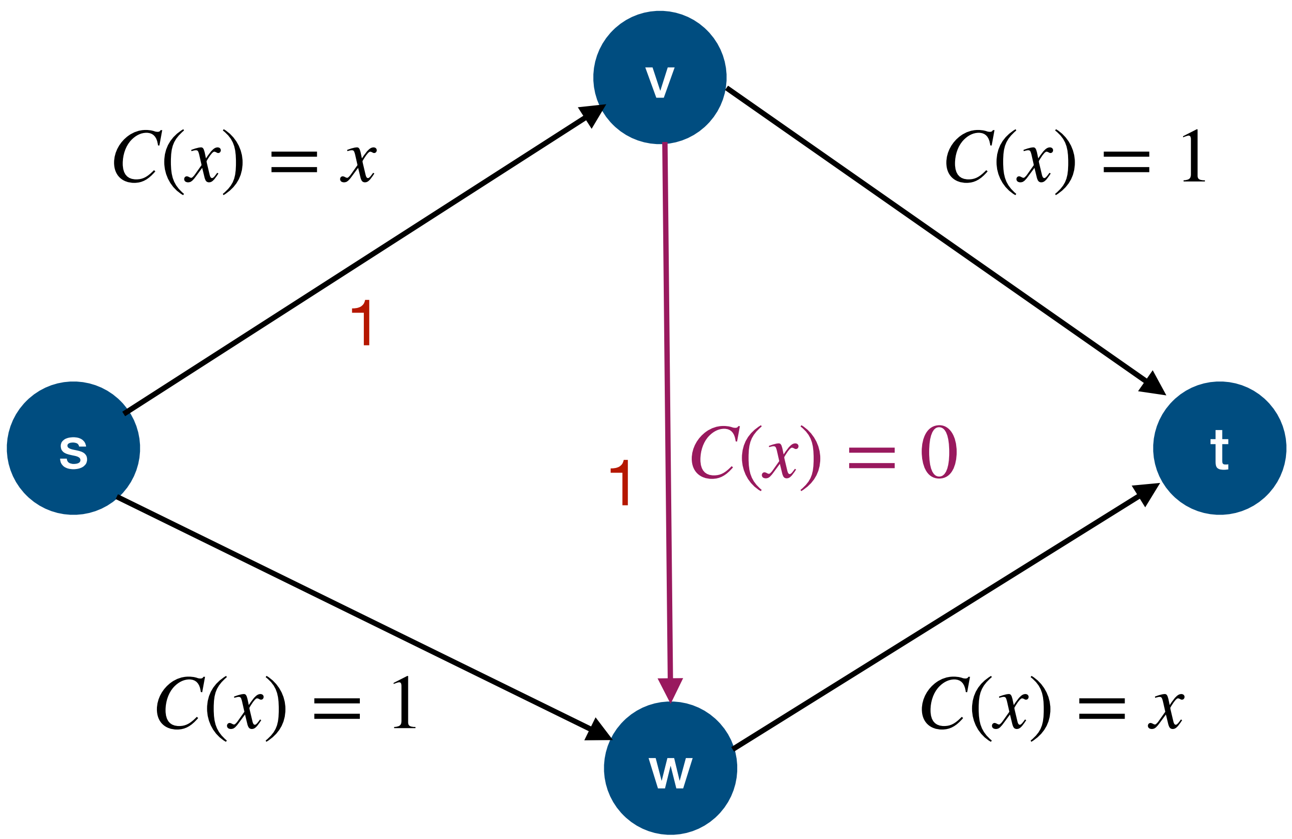

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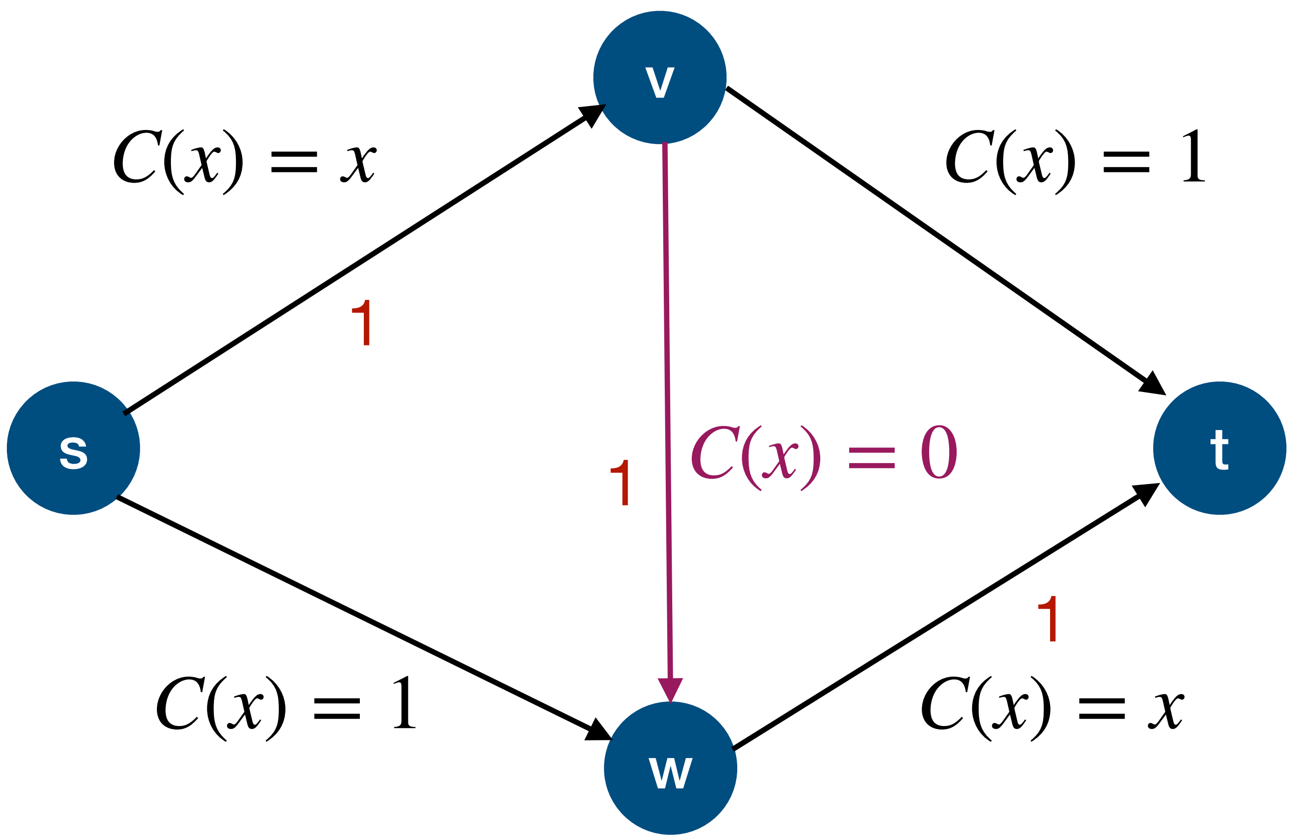

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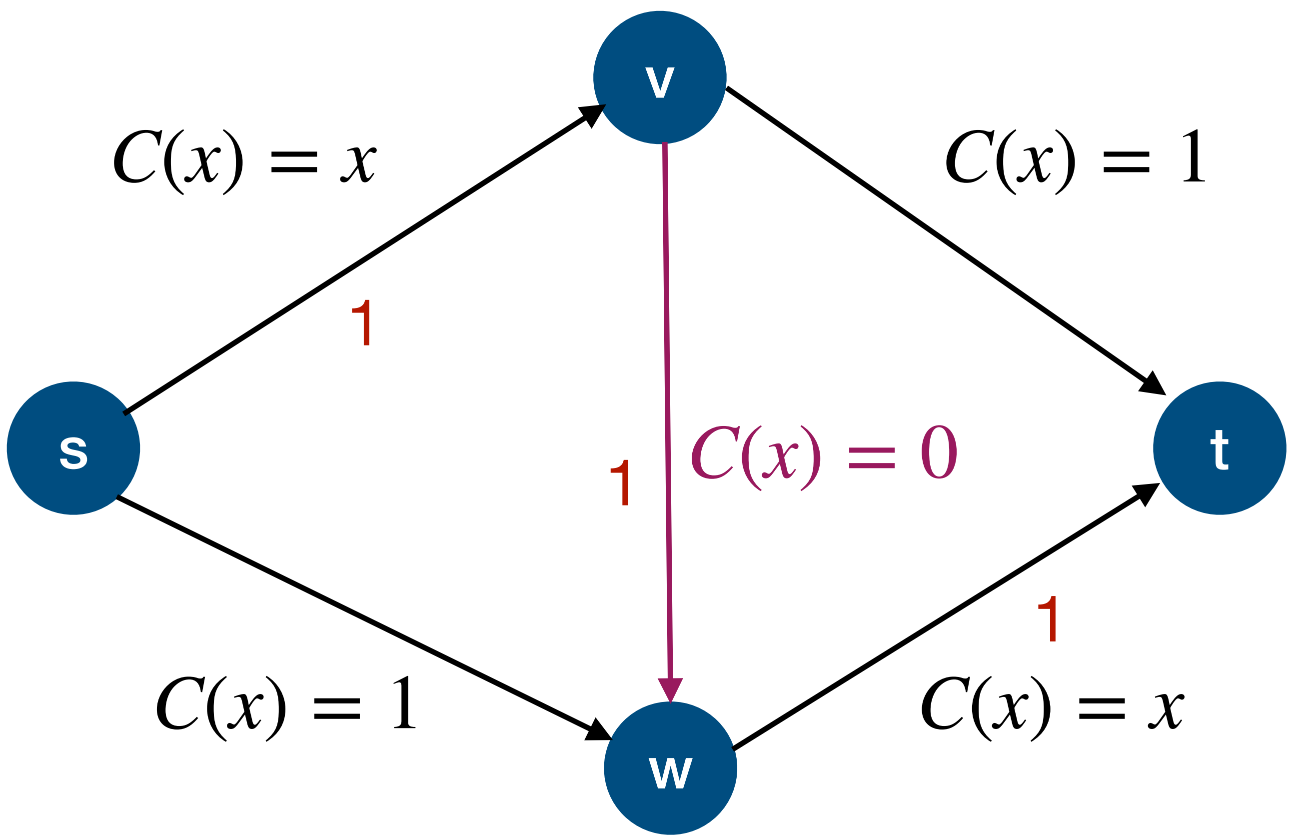

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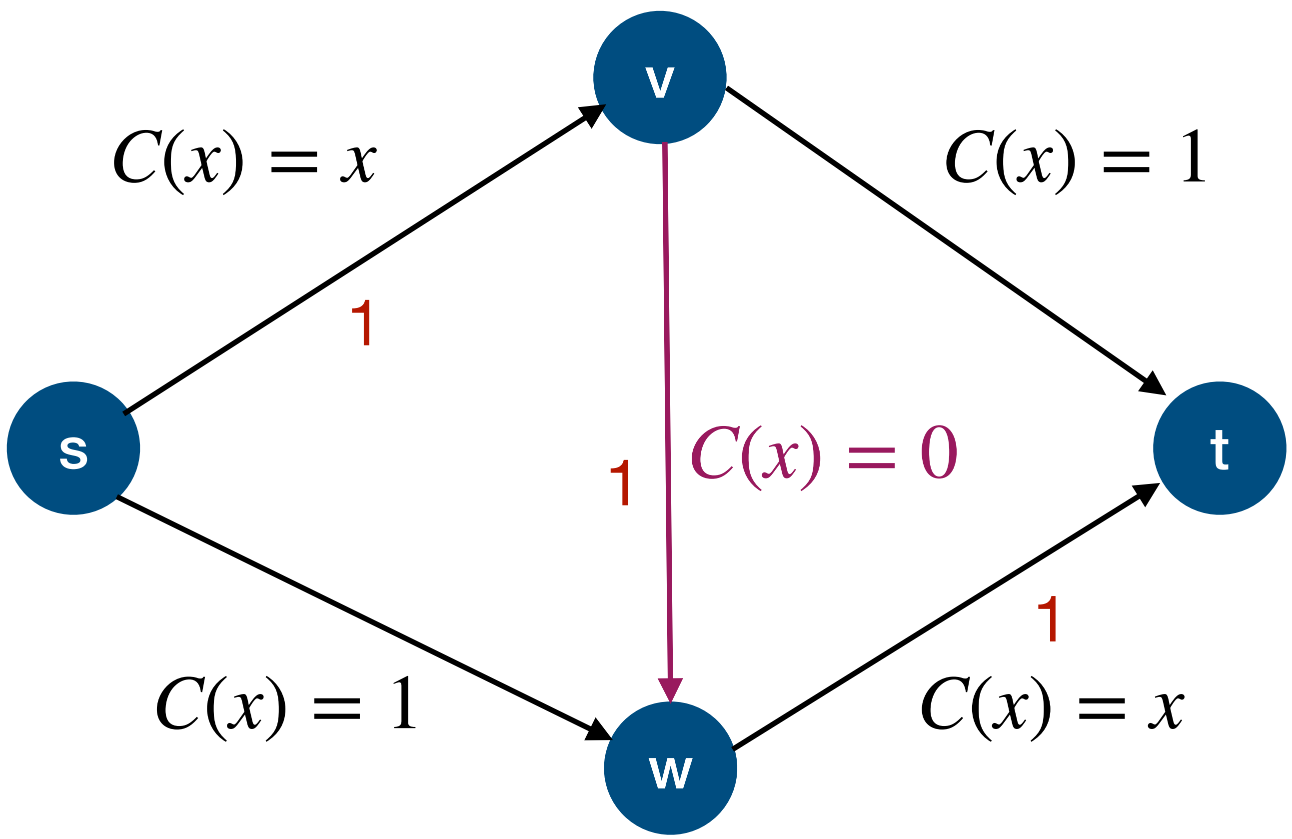

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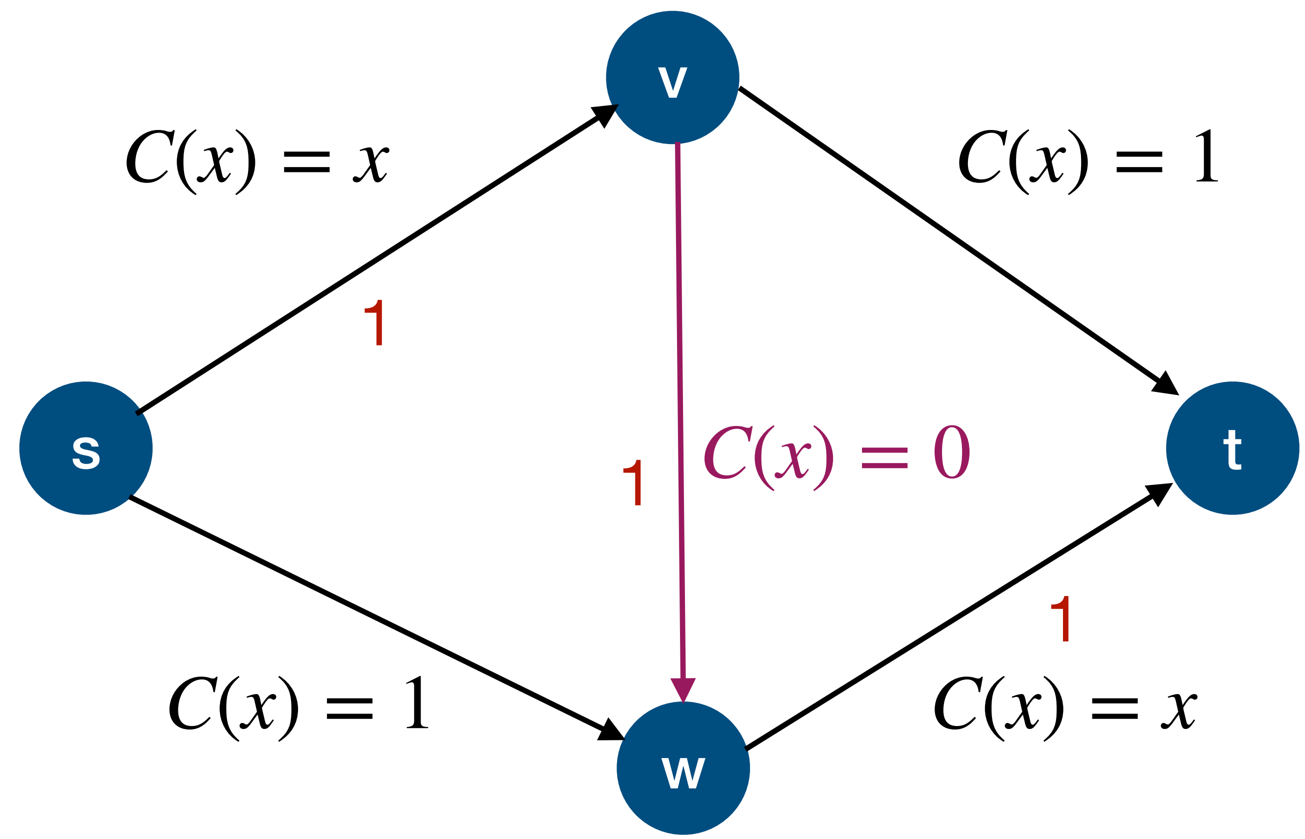

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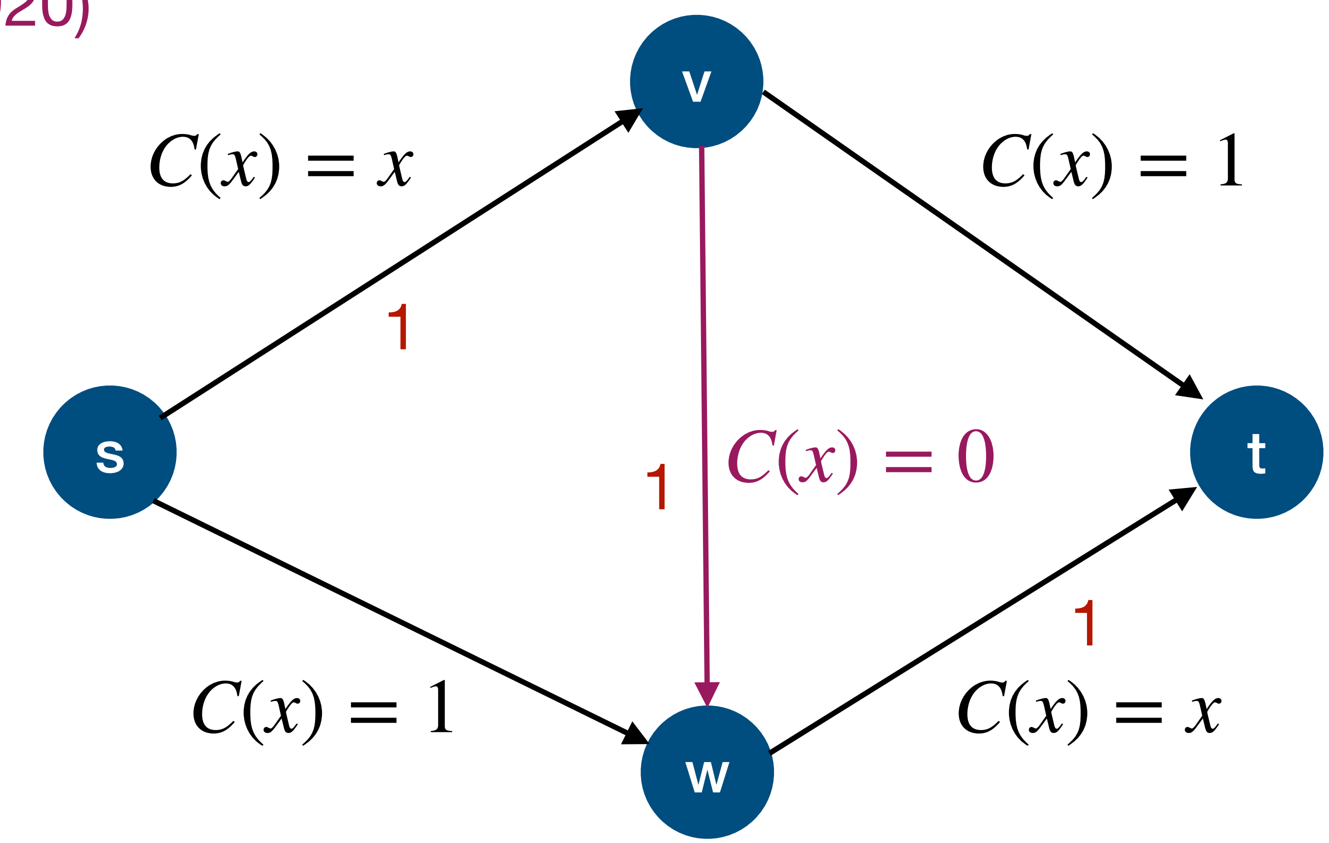

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But it is not unreasonable to not have this in some cases, e.g., the *El Farol Bar problem*.

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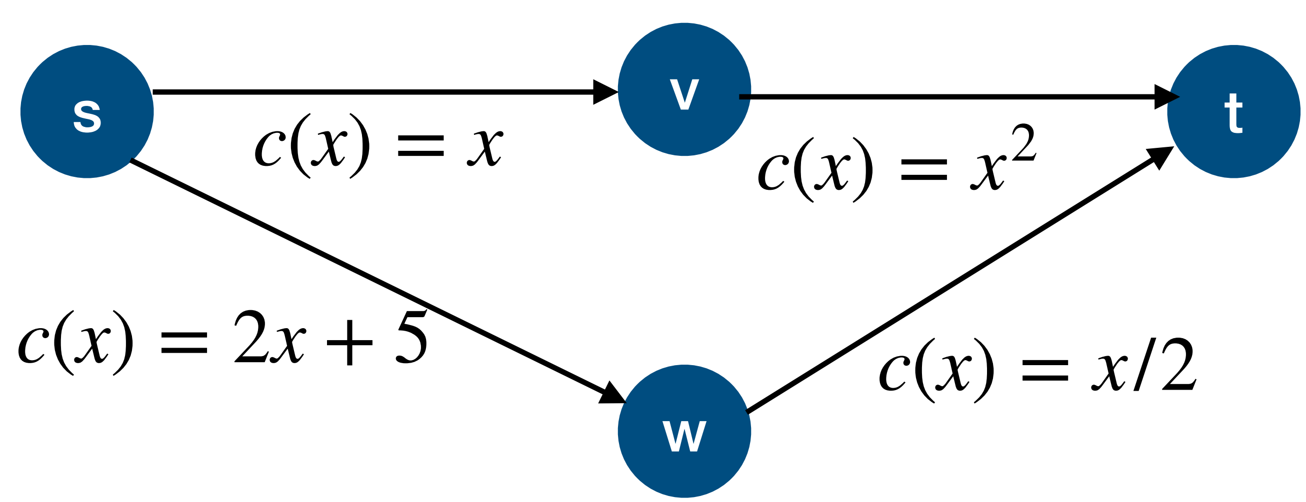

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For example: $c_e(x)$ could be a linear function

$$c_e(x) = \alpha_e x + \beta_e$$

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The theorem also gives us an *algorithm* to find a PNE:

- start from any arbitrary strategy profile,
- run the best response dynamics until we reach a PNE.

Potential Games

Definition: A game is an (exact) potential game if there exists a potential function $\Phi : S_1 \times \ldots \times S_n \to \mathbb{R}$ such that for all $i \in N$, all $s_{-i} \in S_{-i}$ and $s_i, s_i' \in S_i$, we have that

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In particular, this also holds for $s' = (s'_i, s^*_{-i})$. Since the game is a potential game, this means that $\text{cost}_i(s^*) \leq \text{cost}_i(s')$, and hence s^* is a pure Nash equilibrium.

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Best Response Dynamics in Congestion Games

Theorem (Rosenthal 1973): In any congestion game, the best response dynamics always converges to a pure Nash equilibrium.

In particular, this implies that every congestion game has a pure Nash equilibrium.

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- The cost functions for each agent can be represented in space $O(m \cdot n \cdot \log \max_j r_j(n))$, where we represent the function $r_j(\cdot)$ using a binary representation.

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Intuition: If the cost functions are represented with fairly small numbers, then it is a fast algorithm.

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An algorithm for finding PNE of congestion games

The theorem also gives us an *algorithm* to find a PNE:

- start from any arbitrary strategy profile,
- run the best response dynamics until we reach a PNE.

By the potential argument, the algorithm will terminate. But will it terminate in polynomial time?

<u>We would have to show two things:</u>

- The best response of a player can be computed in polynomial time. ✔
- The equilibrium will be found within a polynomial number of best response steps. ✘

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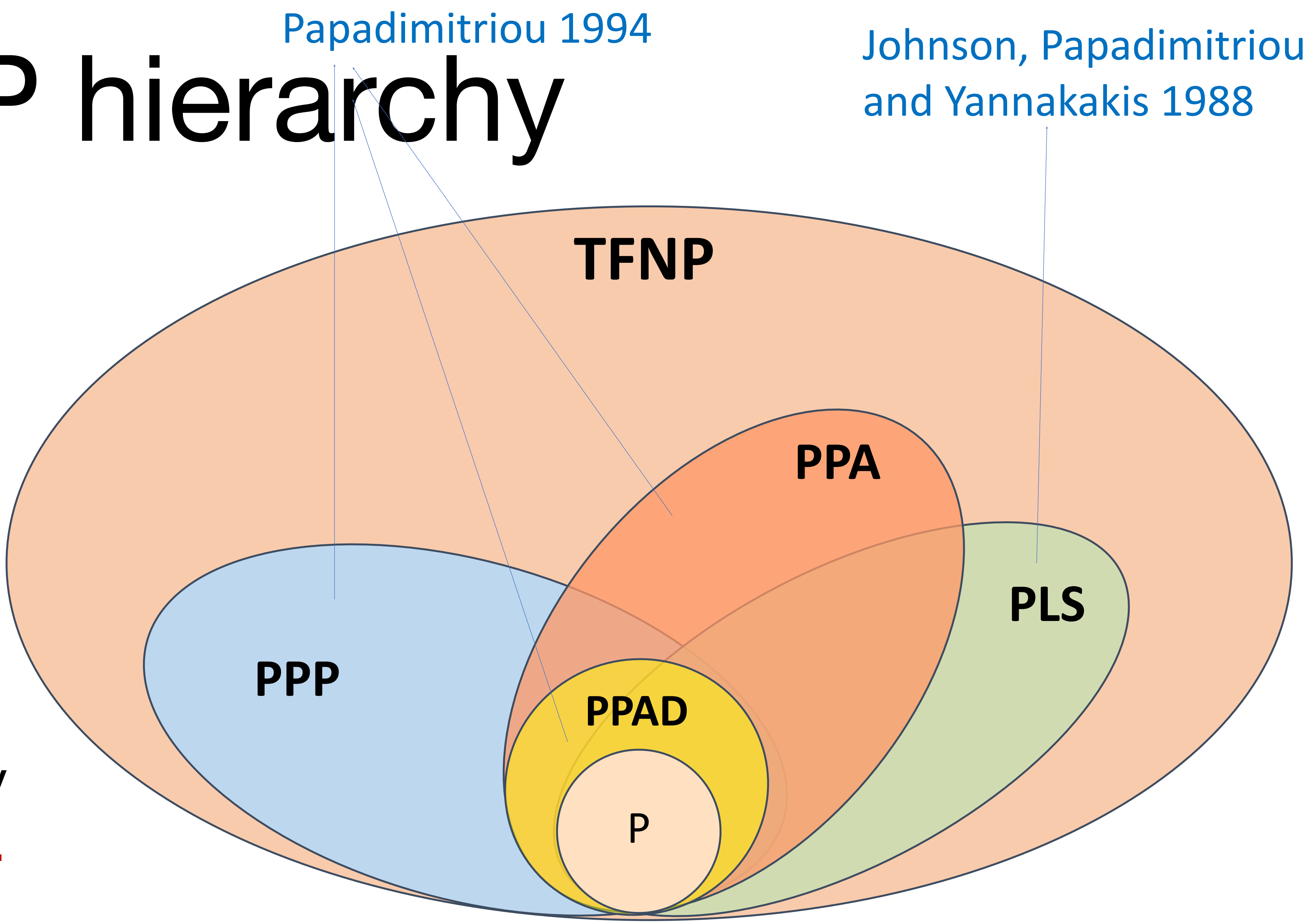
Theorem (Johnson et al. 1988, Fabrikant et al. 2004): Computing a PNE of a congestion game is PLS-complete.

The TFNP hierarchy

Define several subclasses of TFNP.

Show completeness results for those classes.

Approach initiated by (Papadimitriou 1994).



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- They exist by Nash (1951), since congestion games are finite normal form games.

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Theorem (Babichenko and Rubinstein 2021): Computing a MNE of a congestion game is PPAD ∩ PLS - complete.

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