Algorithmic Game Theory and Applications

Inefficiency of Equilibria

Prisoner’s Dilemma

Player 1 \ Player 2	Confess	Silent
Confess	5, 5	9,0
Silent	0, 9	8, 8

Both players confessing is the *logical* outcome of this game.

	Confess	Silent
Confess	5	9
Silent	0	8

Player 1 (row player)

For Player 1, confessing is better regardless of the strategy of Player 2

	Confess	Silent
Confess	5	0
Silent	9	8

Player 2 (column player)

For Player 2, confessing is better regardless of the strategy of Player 1

Notions of Efficiency

Social Welfare (in a game with utilities): The (expected) social welfare of a strategy profile $x = (x_1, \ldots, x_n)$ is the sum of utilities of all the players, i.e.,

$$\mathrm{SW}(x) = \sum_{i \in N} u_i(x).$$

Social Cost (in a game with costs): The (expected) social cost of a strategy profile $x = (x_1, \ldots, x_n)$ is the sum of utilities of all the players, i.e.,

$$\mathrm{SC}(x) = \sum_{i \in N} \mathrm{cost}_i(x).$$

Equilibrium Inefficiency

Imagine that we had an *omnipotent entity* that could “force” the player to play any strategy the entity wishes.

Or, imagine that we lived in a world where the players were not selfish, and their goal was to to do what’s best for society.

Then, the entity could select a strategy profile x that maximises the social welfare.

In fact, in most cases we can assume that this strategy profile is pure, therefore s.

Equilibrium Inefficiency

Then, the entity could select a strategy profile s that maximises the social welfare.

There is no guarantee that this profile s is going to be a Nash equilibrium.

Then, the natural question becomes:

“How much worse is the social welfare of the Nash equilibrium compared to the maximum social welfare in any strategy profile?

Price of Anarchy (Koutsoupias and Papadimitriou 1999)

Equilibrium Inefficiency

Price of Anarchy (Koutsoupias and Papadimitriou 1999)

Informally:

$$\frac{\text{The maximum social welfare of any strategy profile}}{\text{The social welfare of the equilibrium}}$$

Prisoner's Dilemma

Player 1 \ Player 2	Confess	Silent
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Both players confessing is the *logical* outcome of this game.

What is the maximum SW?

What is the SW of the equilibrium?

	Confess	Silent
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Player 2 (column player)

For Player 1, confessing is better regardless of the strategy of Player 2

For Player 2, confessing is better regardless of the strategy of Player 1

What is the Price of Anarchy of the game?

Chicken

Player 1 \ Player 2	Swerve	Straight
Swerve	6, 6	2,7
Straight	7, 2	0, 0

MNE: (2/3,1/3), (2/3,1/3)

Expected social welfare: 9.3

What is the maximum SW?

What is the SW of the equilibrium?

Which equilibrium?

Equilibrium Inefficiency

Price of Anarchy (Koutsoupias and Papadimitriou 1999)

Informally:

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Equilibrium Inefficiency

Price of Anarchy (Koutsoupias and Papadimitriou 1999)

Informally:

$$\frac{\text{The maximum social welfare of any strategy profile}}{\text{The social welfare of the worst equilibrium}}$$

Price of Anarchy of a game

$$\text{PoA}\,(G) = \frac{\text{SW}(x^*)}{\min_{x \in \text{MNE}(G)} \text{SW}(x)},$$

where $x^* \in \arg\max_x \text{SW}(x)$ and $\text{MNE}(G)$ is the set of mixed Nash equilibria of the game G.

Since we are considering all MNE, we refer to this as the "mixed Price of Anarchy".

We can also have the "pure Price of Anarchy" with only referring to PNE above.

We can have this actually for any solution concept, e.g., "correlated Price of Anarchy" for *correlated equilibria* (*not covered*).

Chicken

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MNE: (2/3,1/3), (2/3,1/3)

Expected social welfare: 9.3

What is the maximum SW?

What is the SW of the equilibrium?

Which equilibrium?

What is the pure Price of Anarchy of the game?

What is the mixed Price of Anarchy of the game?

Price of Anarchy of a class of games

$$\text{PoA}\,(\mathcal{G}) = \max_{G \in \mathcal{G}} \frac{\text{SW}(x^*)}{\min_{x \in \text{MNE}(G)} \text{SW}(x)},$$

where $x^* \in \arg\max_{x} \text{SW}(x)$ and $\text{MNE}(G)$ is the set of mixed Nash equilibria of the game G.

For cost minimisation games

$$\text{PoA}\ (\mathcal{G}) = \max_{G \in \mathcal{G}} \frac{\max_{x \in \text{MNE}(G)} \text{SC}(x)}{\text{SC}(x^*)},$$

where $x^* \in \arg\min_x \text{SC}(x)$ and $\text{MNE}(G)$ is the set of mixed Nash equilibria of the game G.

We flip the ratio to maintain the convention that $\text{PoA} \geq 1$ always.

Pessimist or Optimist

The Price of Anarchy is truly a worst-case guarantee.

It says that even if the players end up at the worst possible equilibrium, the multiplicative difference in social welfare (or social cost) will be bounded by the Price of Anarchy.

Maybe we can be more optimistic: what if we consider the best possible equilibrium instead?

Price of Stability (Anshelevich et al. 2006).

Definitions Lookup

<u>utilities, social welfare</u>

$$\text{PoA}\ (\mathcal{G}) = \max_{G\in\mathcal{G}} \frac{\text{SW}(x^*)}{\min_{x\in \text{NE}(G)} \text{SW}(x)}$$

$$\text{PoS}\ (\mathcal{G}) = \max_{G\in\mathcal{G}} \frac{\text{SW}(x^*)}{\max_{x\in \text{NE}(G)} \text{SW}(x)}$$

<u>costs, social cost</u>

$$\text{PoA}\ (\mathcal{G}) = \max_{G\in\mathcal{G}} \frac{\max_{x\in \text{NE}(G)} \text{SC}(x)}{\text{SC}(x^*)}$$

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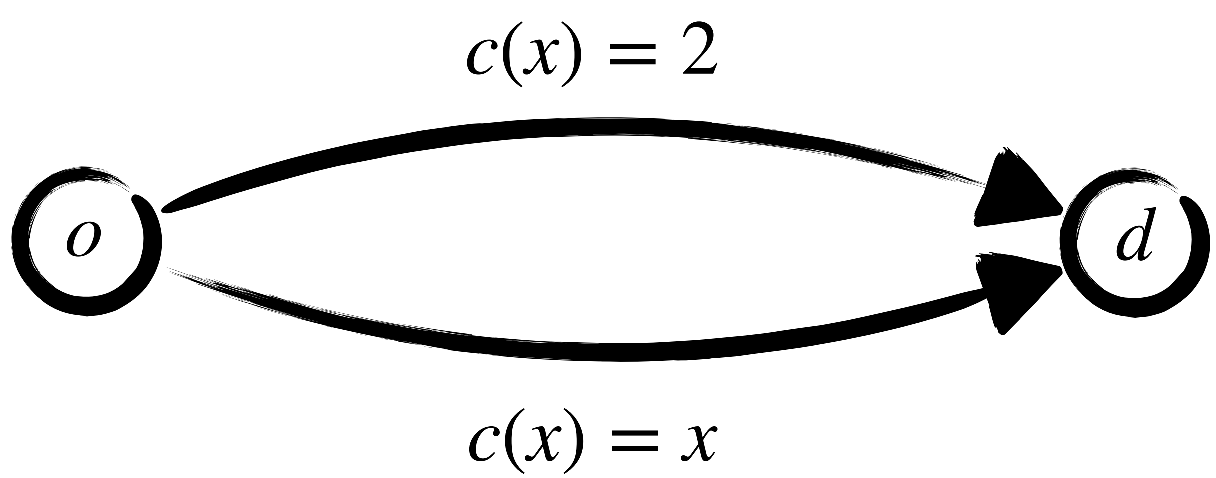
Back to congestion games



Suppose there are two players.

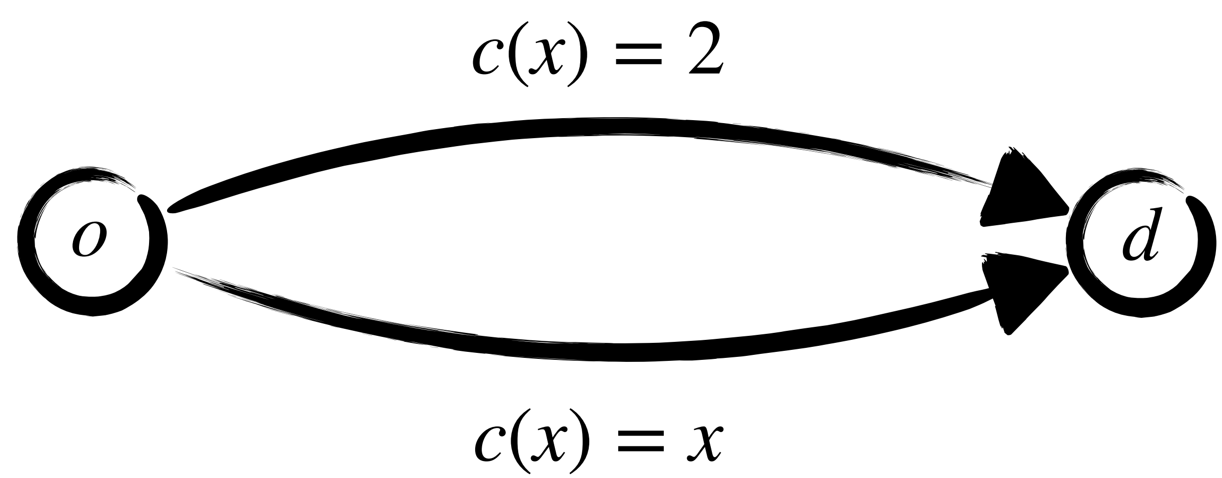
What is the optimal outcome here, i.e., the one the minimises the social cost?

s_1 = top, s_2 = bottom, SC$(s_1, s_2) = 3$

What is an equilibrium of this game?

The optimal solution is an equilibrium!

Back to congestion games



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s_1 = top, s_2 = bottom, $\text{SC}(s_1, s_2) = 3$

What is an equilibrium of this game?

The optimal solution is an equilibrium!

Any other equilibria?

s_1 = bottom, s_2 = bottom, $\text{SC}(s_1, s_2) = 4$

What is the Price of Anarchy of the game?

What is the Price of Stability of the game?

Atomic Network Congestion Games

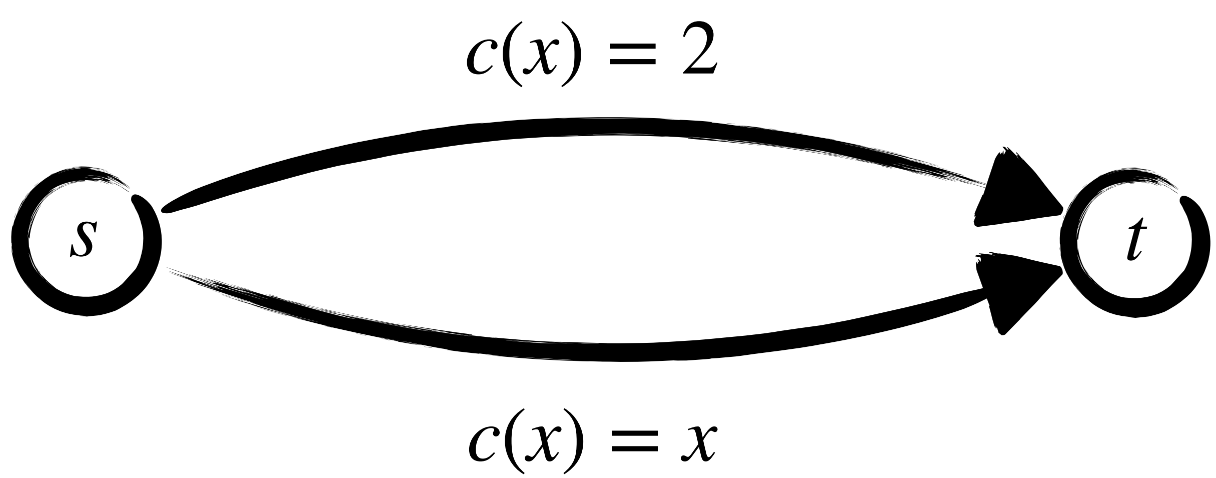
Definition: An (atomic) network congestion game is a congestion game in which the resources are edges in a directed graph, and each player must choose a set of edges that forms a (simple) path from a given source s_i to a given sink t_i.

On every edge there e is a cost function $c_e(x)$ which is a function of the number of players that have e in their chosen paths.

For example: $c_e(x)$ could be a linear function

$$c_e(x) = \alpha_e x + \beta_e$$

Back to congestion games



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What can we say about the PoA / PoS of network congestion games?

Definitions Lookup

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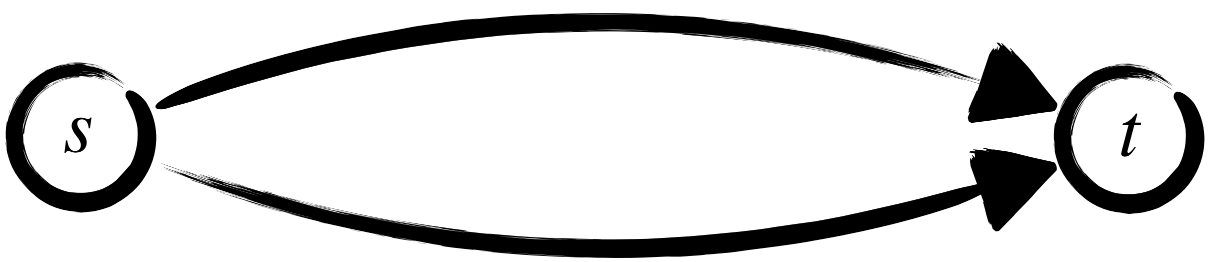
Back to congestion games



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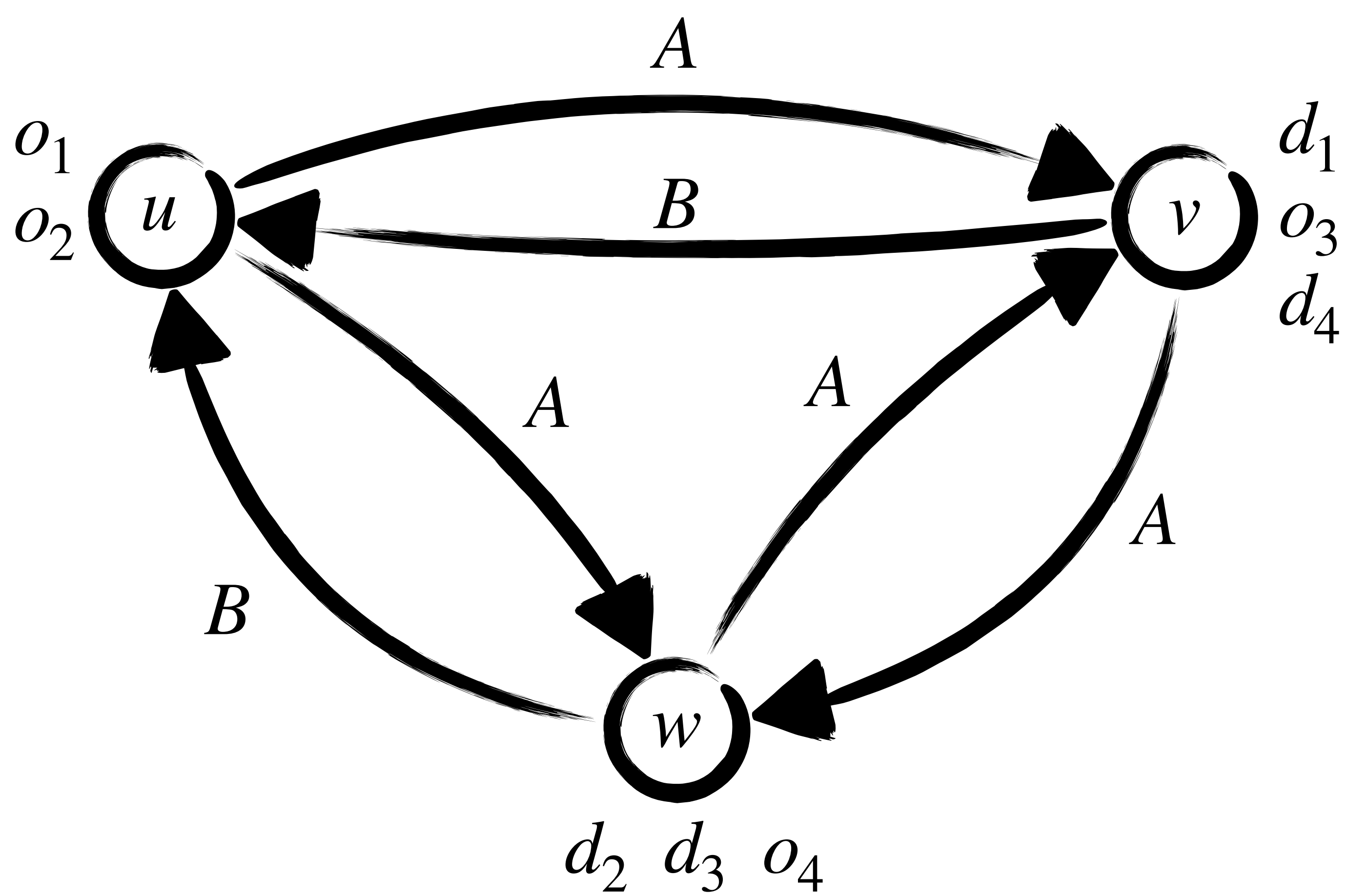
s_1 = bottom, s_2 = bottom, $\mathrm{SC}(s_1, s_2) = 4$

What can we say about the PoA / PoS of network congestion games?

PoA $(\mathscr{G}_{\mathrm{NC}}) \geq 4/3$

PoS $(\mathscr{G}_{\mathrm{NC}}) \geq 1$

Another network congestion game

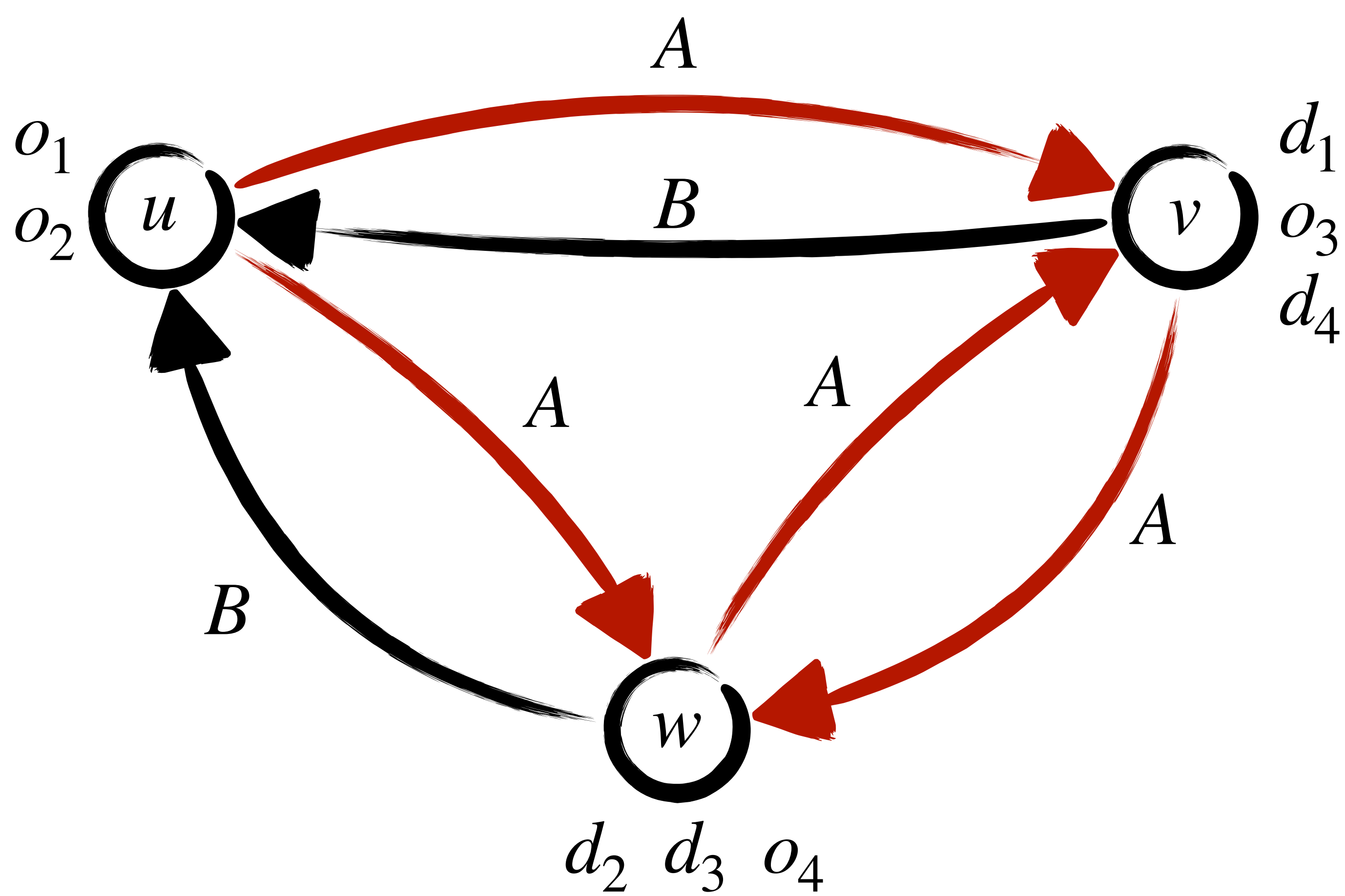


A . $c(x) = x$

B . $c(x) = 0$

What is the optimal outcome? Every player takes the one-hop path.

Another network congestion game



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SC(one-hop, one-hop, one-hop, one-hop) $= 1 + 1 + 1 + 1 = 4$

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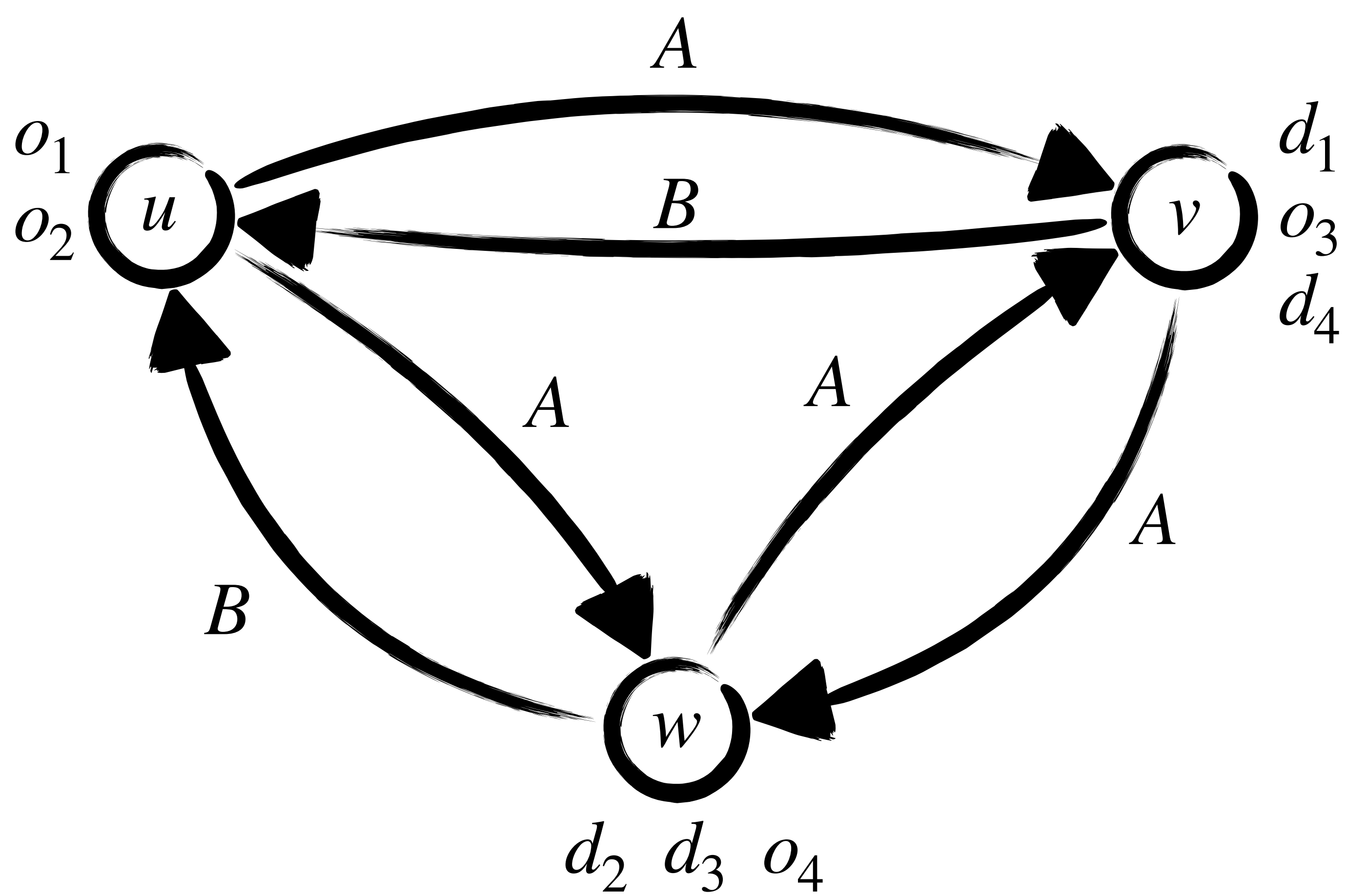

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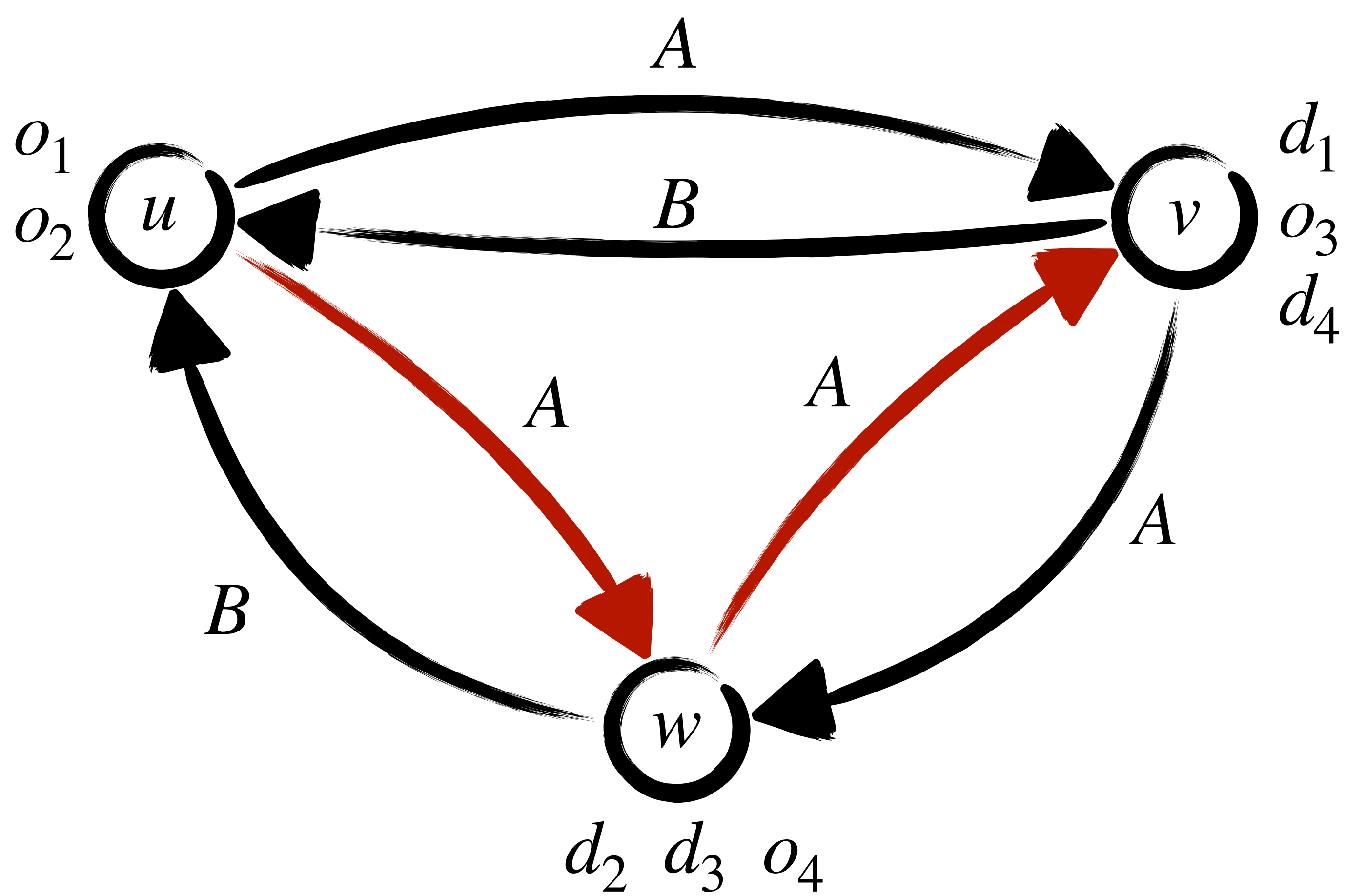
Let s be such that every player takes the two-hop path.

Another network congestion game



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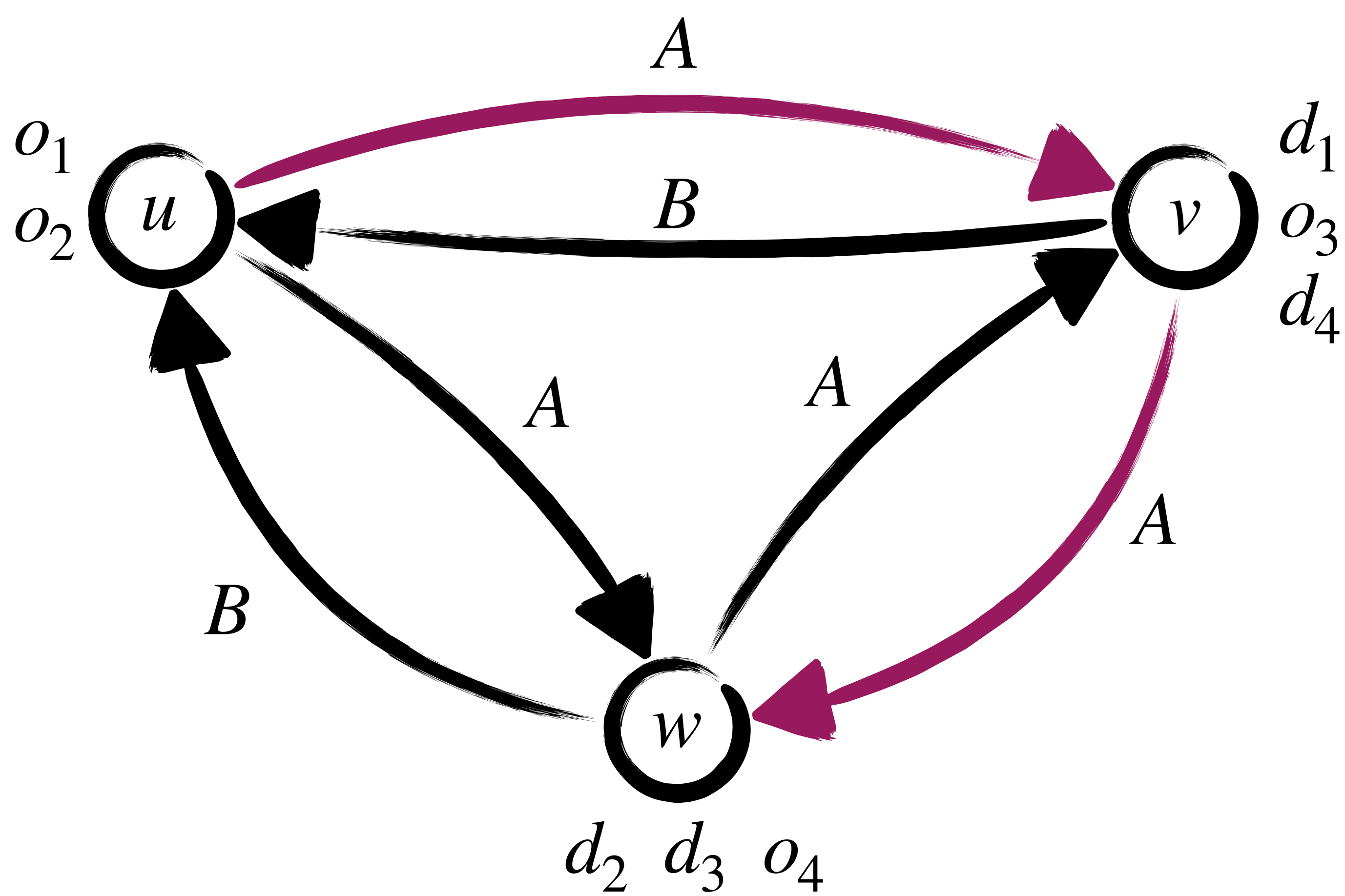

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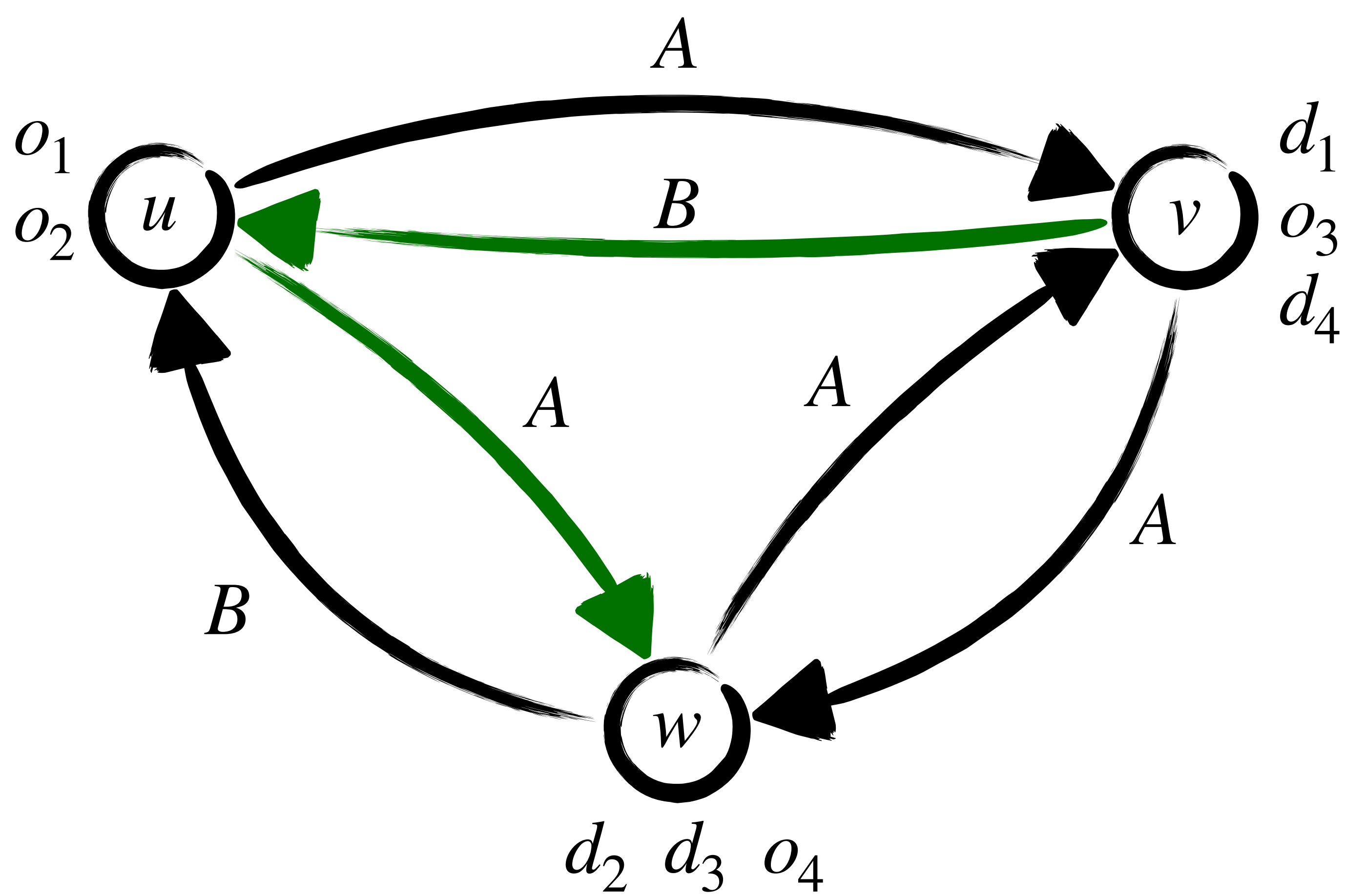

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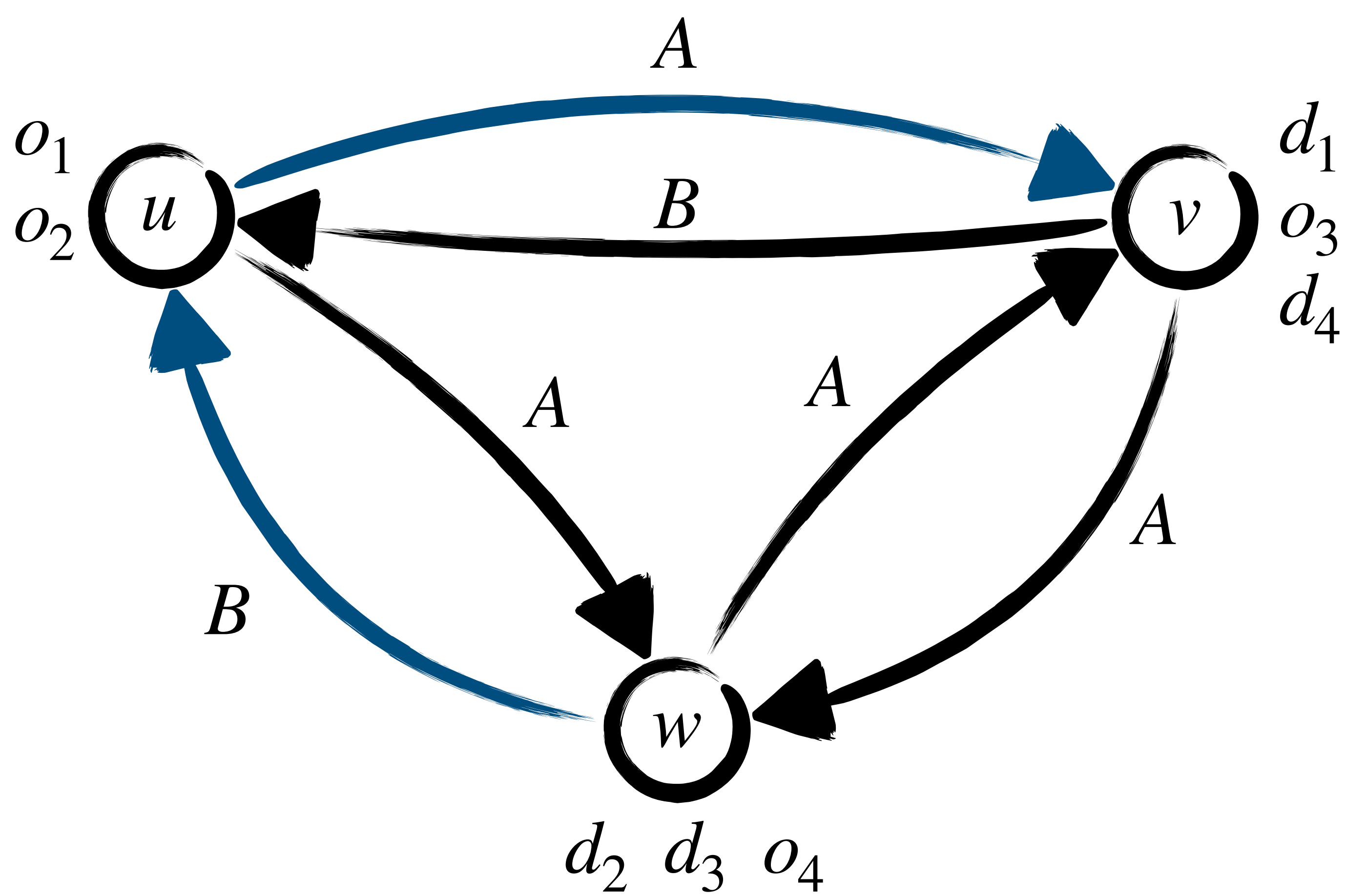

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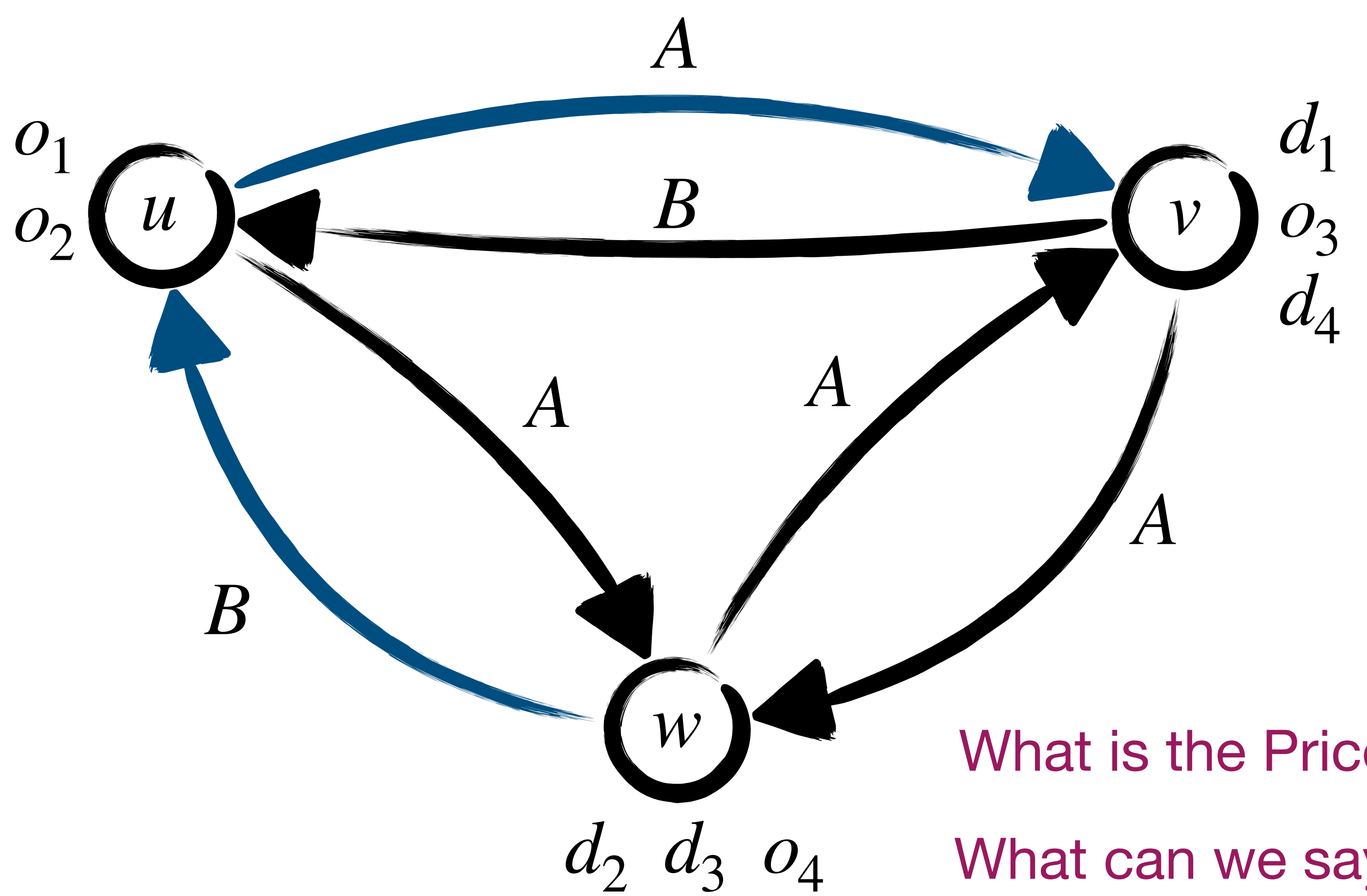

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What is the Price of Anarchy of the game?

What can we say about the PoA of network congestion games?

What is the optimal outcome? Every player takes the one-hop path.

SC(one-hop, one-hop, one-hop, one-hop) $= 1 + 1 + 1 + 1 = 4$

PoA ($\mathcal{G}_{\mathrm{NC}}$) $\geq 5/2$

Let s be such that every player takes the two-hop path.

This is an equilibrium! (Verify at home)

SC(two-hop, two-hop, two-hop. two-hop) $= 3 + 3 + 2 + 2 = 10$

A Price of Anarchy guarantee

Theorem (Christodoulou and Koutsoupias 2005): The Price of Anarchy of any congestion game with linear cost functions is at most $5/2$.

Proof:

Let s be any PNE and let s^* be an optimal profile (which minimises the social cost). Recall the definition of $\#(r, s)$ and $\#(r, s^*)$. For ease of notation, let us refer to those as n_r and n_r^* respectively.

By definition of the social cost, we have that

$$\text{SC}(s) = \sum_{r \in R} n_r \cdot c_r(n_r) = \sum_{r \in R} n_r(a_r n_r + b_r) = \sum_{r \in R} \left(a_r n_r^2 + b_r n_r\right)$$

linear cost function

$$\text{SC}(s^*) = \sum_{r \in R} \left(a_r (n_r^*)^2 + b_r n_r^*\right)$$

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Proof:

Furthermore, since s is a PNE, we have for every player $i \in N$:

PNE condition

$$\sum_{r \in s_i} c_r(n_r) \leq \sum_{r \in s_i^* \cap s_i} c_r(n_r) + \sum_{r \in s_i^* \setminus s_i} c_r(n_r + 1) \leq \sum_{r \in s_i^*} c_r(n_r + 1)$$

why? why?

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Theorem (Christodoulou and Koutsoupias 2005): The Price of Anarchy of any congestion game with linear cost functions is at most $5/2$.

Proof:

We have established for every player $i \in N$:

$$\sum_{r \in s_i} c_r(n_r) \leq \sum_{r \in s_i^*} c_r(n_r + 1)$$

Summing over all players:

$$\text{SC}(s) = \sum_{i \in N} \sum_{r \in s_i} c_r(n_r) \leq \sum_{i \in N} \sum_{r \in s_i^*} c_r(n_r + 1)$$

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$$= \sum_{r \in R} n_r^*(a_r(n_r + 1) + b_r)$$

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$$= \sum_{r \in R} n_r^*(a_r(n_r + 1) + b_r) \; = \sum_{r \in R} \left[a_r n_r^*(n_r + 1) + b_r n_r^*\right]$$

Useful Lemma

Lemma: For all $y, z \in \mathbb{N}$, $y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$

Plugging in, we get:

$$\text{SC}(s) \leq \sum_{r \in R} \left[a_r n_r^*(n_r + 1) + b_r n_r^*\right]$$

$$\leq \sum_{r \in R} \left[a_r \left(\frac{5}{3}(n_r^*)^2 + \frac{1}{3}n_r^2\right) + b_r n_r^*\right]$$

$$= \sum_{r \in R} \left[\frac{5}{3}a_r(n_r^*)^2 + b_r n_r^*\right] + \sum_{r \in R} \frac{1}{3}a_r n_r^2$$

$$\leq \frac{5}{3}\sum_{r \in R} \left[a_r(n_r^*)^2 + b_r n_r^*\right] + \frac{1}{3}\sum_{r \in R} a_r n_r^2 + b_r n_r$$

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$$= \frac{5}{3}\mathsf{SC}(s^*) + \frac{1}{3}\mathsf{SC}(s)$$

Proving Price of Anarchy bounds

To prove that the PoA of a class of games $\mathscr{G}$ is α, we need to prove that it is at most α and at least α in the worst case. In particular, we need to:

- Construct *some* game $G \in \mathscr{G}$ and argue that the social cost of *some* equilibrium of this game is *at least* a factor α away from the optimal.
- Construct an argument that for *any* game $G \in \mathscr{G}$ and argue that the social cost of *any* equilibrium of this game is *at most* a factor α away from the optimal.

A Price of Stability guarantee

Theorem: The Price of Stability of any congestion game with linear cost functions is at most 2.

Proof sketch:

The proof will use the following lemma:

Lemma (Potential Method): Assume that Φ is a potential function for the game G. If there exist constants $c, d > 0$ such that, for any strategy profile s, it holds that $c \cdot \text{SC}(s) \leq \Phi(s) \leq d \cdot \text{SC}(s)$, then $\text{PoS}\ (G) \leq \dfrac{d}{c}$.

Proof of Potential Method lemma

Proof:

Suffices to show that there exists some PNE $\tilde{s}$ such that

$\mathrm{SC}(\tilde{s}) \leq \frac{d}{c} \cdot \mathrm{SC}(s^*)$.

$\tilde{s}$ is a potential minimiser

Let $\tilde{s} \in \arg\min \Phi(s)$ be a minimiser of the potential function. As we argued in the last lecture, $\tilde{s}$ is indeed a PNE.

From the statement of the lemma, it holds that

$c \cdot \mathrm{SC}(\tilde{s}) \leq \Phi(\tilde{s}) \leq \Phi(s^*) \leq d \cdot \mathrm{SC}(s^*)$

LHS of statement for $\tilde{s}$ RHS of statement for s^*

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$$\text{PoS}\ (G) \leq \frac{d}{c}.$$

One can show that Rosenthal's potential function satisfies the condition with $c = 1/2$ and $d = 1$.

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