

Algorithmic Game Theory and Applications

(Approximate) Mechanism Design on Restricted Domains

The Gibbard-Satterthwaite Theorem

Theorem (Gibbard 73 - Satterthwaite 75): In the unrestricted domain, when there are $m \geq 3$ candidates, a voting rule is **truthful** and **onto** if and only if it is **dictatorial**.

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Unrestricted Domain

A *social choice function*, or *voting rule*, or *mechanism* is a function $f: (\succ)^n \rightarrow A$ mapping preference profiles to candidates,

where \succ^n is the space of all possible preference profiles.

The unrestricted domain: \succ^n can contain any preference profile.

i.e., for any voter $i \in N$, \succ_i is the set of *all permutations* of $\{1, \dots, m\}$.

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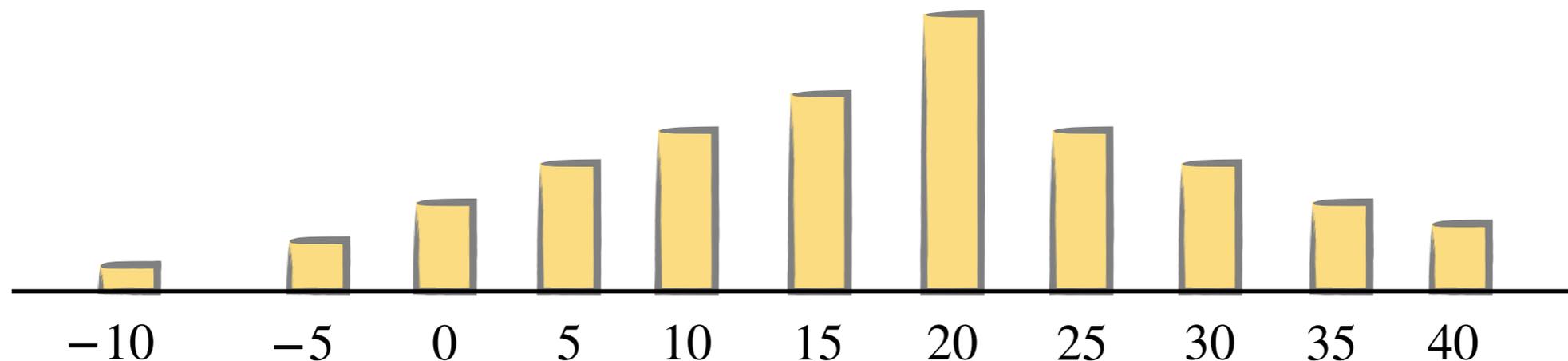
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Introduced by Black in 1948, as a domain for which Condorcet winners always exist.

Find the Condorcet winner

Recall: A Condorcet winner wins a pairwise majority election against any other candidate.

x_1

x_2

x_3

x_4

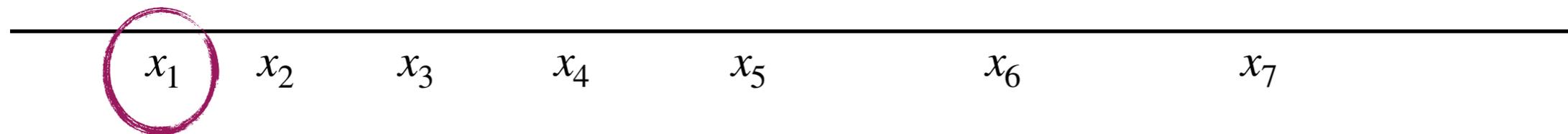
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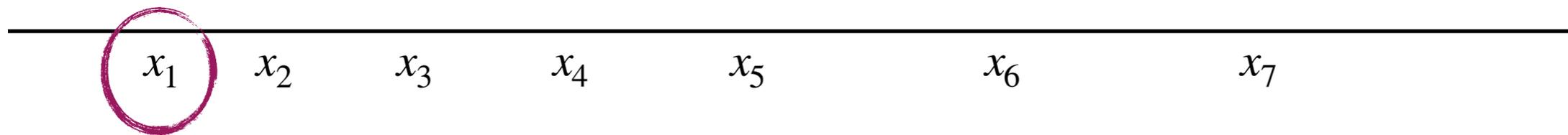
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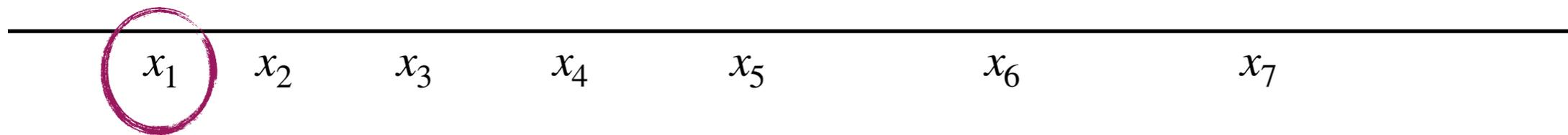
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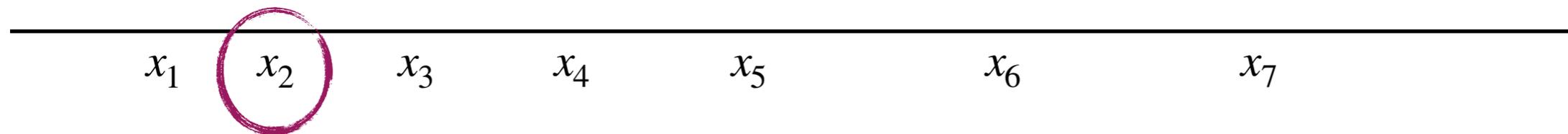
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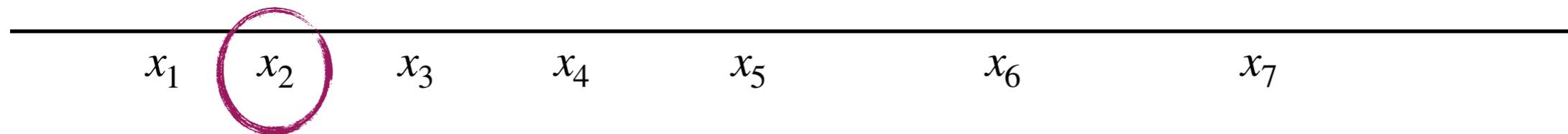


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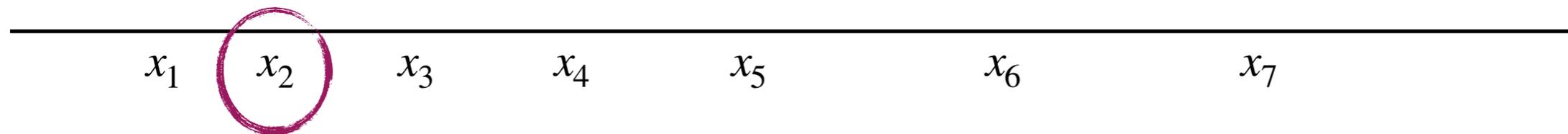


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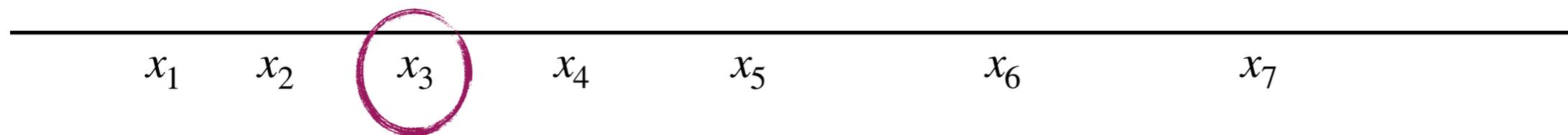


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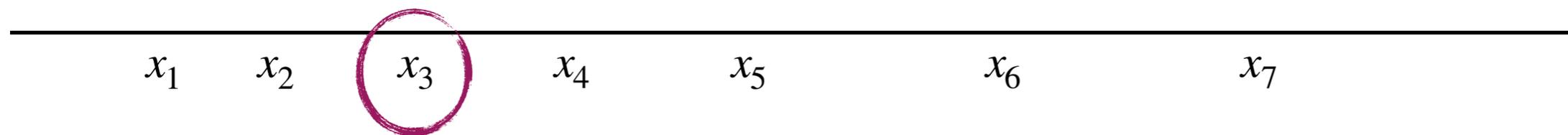
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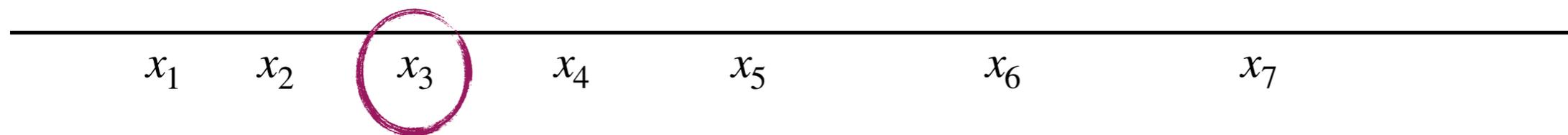
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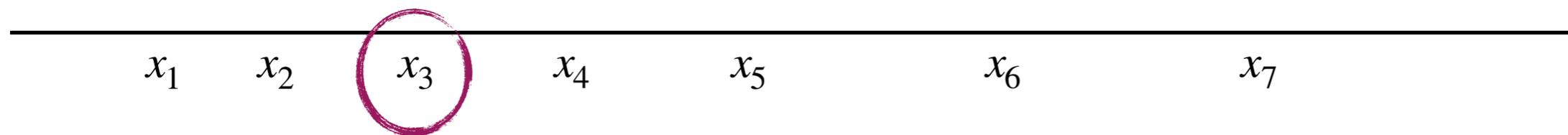
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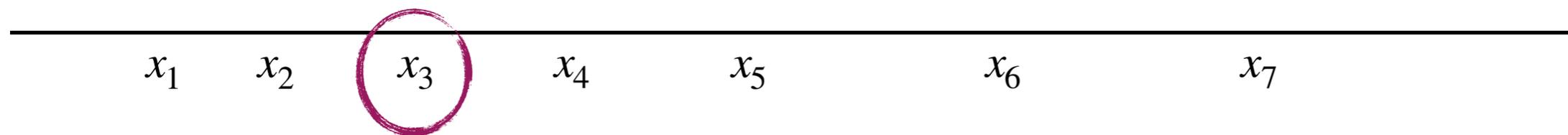
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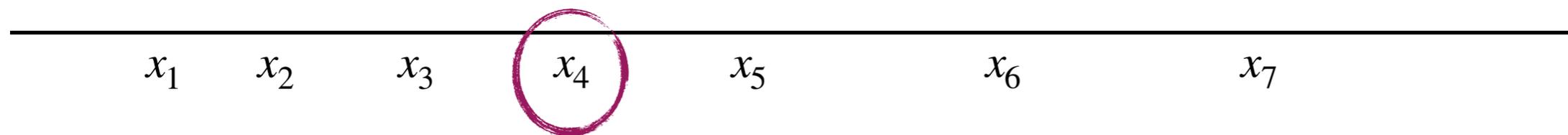
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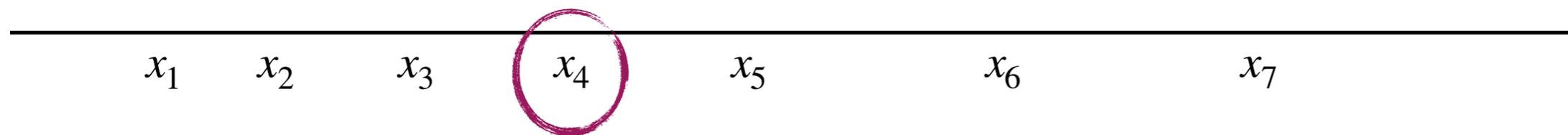
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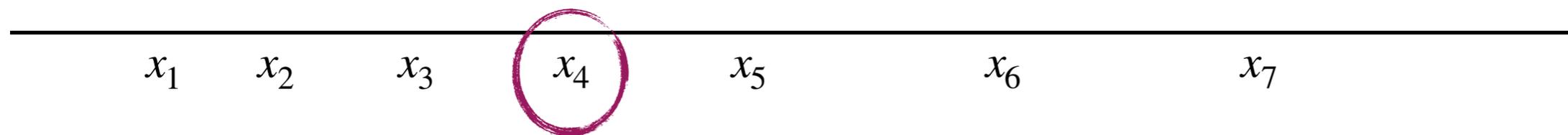
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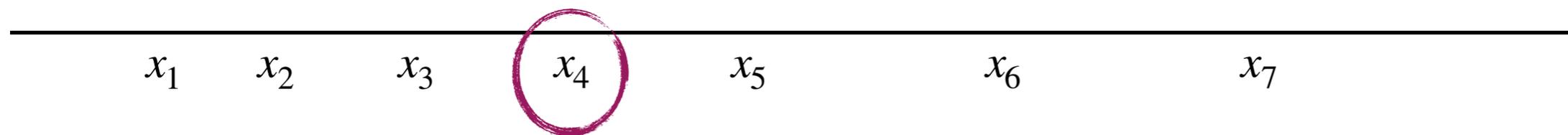
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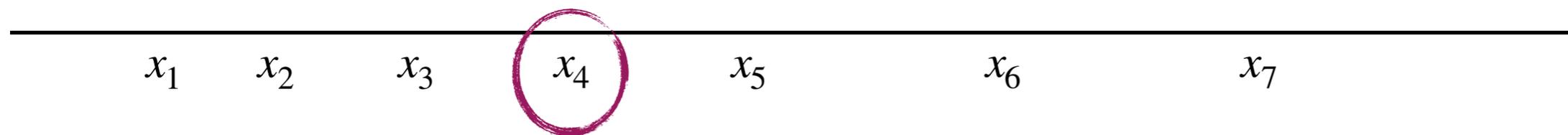
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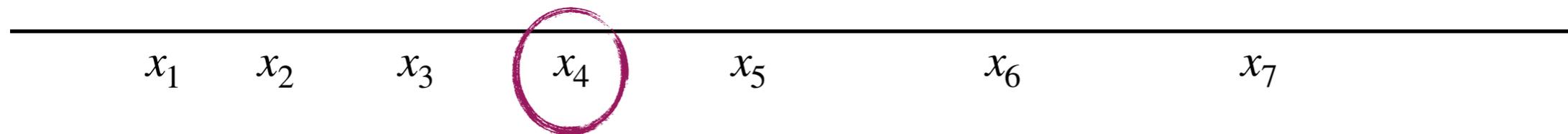
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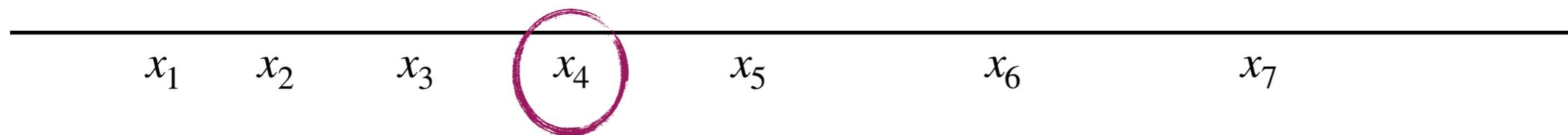
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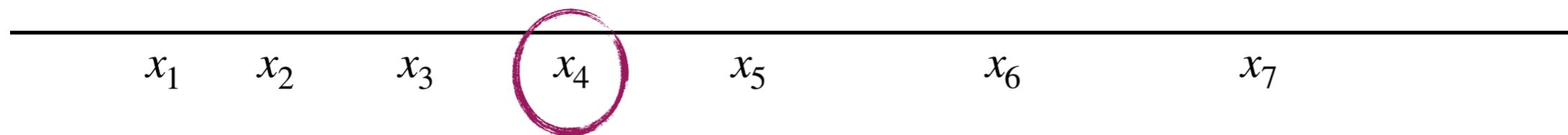
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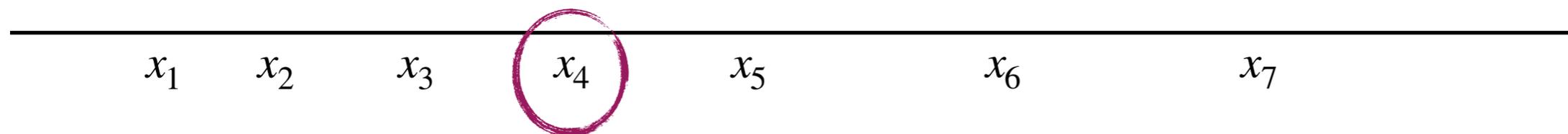
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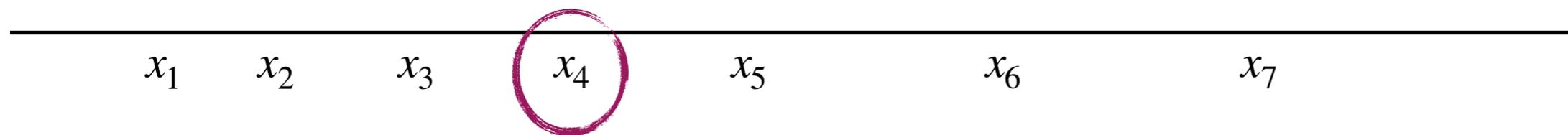
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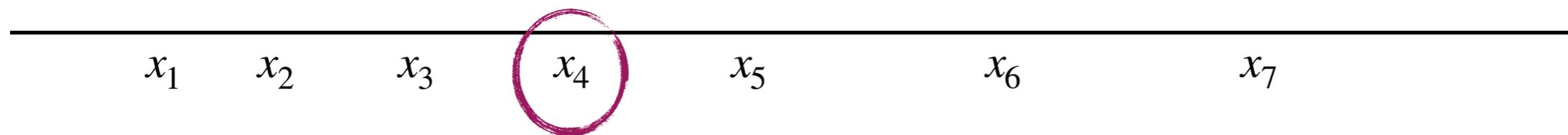
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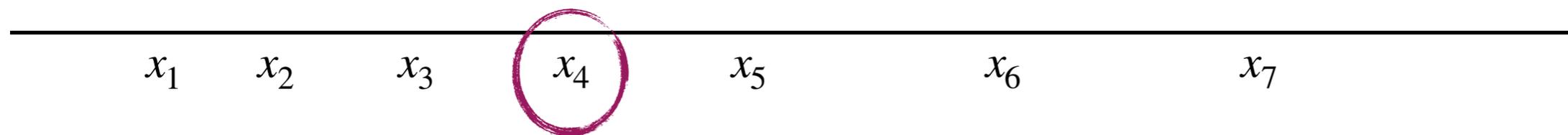
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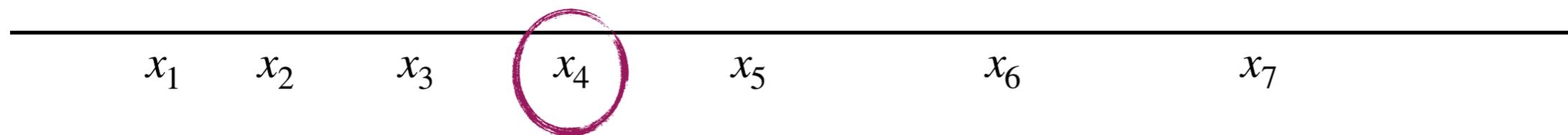


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What else is x_4 ?

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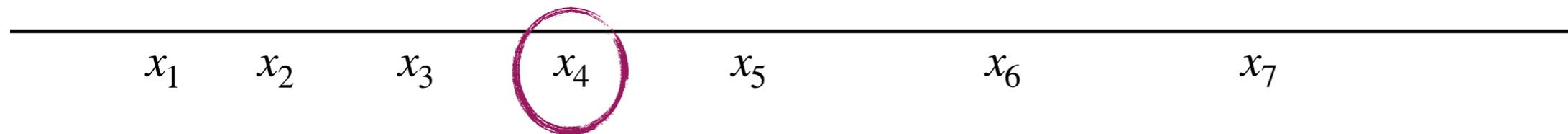
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Theorem: The median voter rule is **truthful**.

Proof: Easy, we've seen it before.

Median Voter Rule

The proof follows from the same monotonicity argument: To affect the median, a voter with $x_i \leq f(>)$ must report something to the right of the median. But then the median will move farther away from x_i .



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Proof: Virtually identical to before, check at home.

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e.g., “A voting rule is **truthful** and **onto** if and only if it is a ***k*-th order statistic voter rule**.”

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Here, x_i are the peaks, y_i are the other possible candidates, which are not peaks of any voter.

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Here in fact, the median candidate is the Condorcet winner among all candidates (not just the peaks).

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We can also have any k -th ordered statistic among all the candidates.

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If you are interested, check the AGT book Definition 10.3.

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Obviously we still want the voting rule to be robust to incentives, so we are still interested in **truthfulness**.

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General Question: Among all the truthful voting rules, which one is the best with respect to the global objective?

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Refined Question: Among all the truthful voting rules, or, in this context, *mechanisms*, what is the one with the smallest possible approximation ratio?

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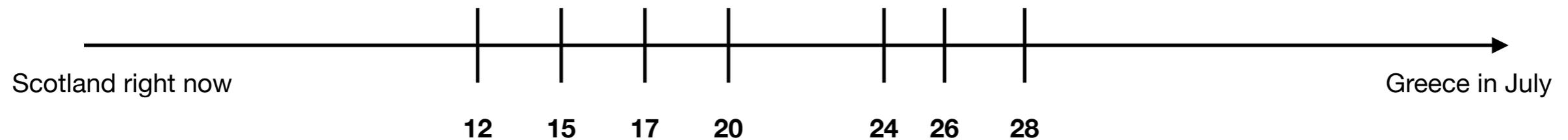
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We want to design a truthful mechanism (voting rule) for the problem that has the minimum possible approximation ratio for the **social cost objective**, i.e., the sum of agents' costs $\sum_{i \in N} |y - x_i|$

Example 2: Setting the temperature

The reports shown in the picture are the peaks, but any temperature is a possible outcome.



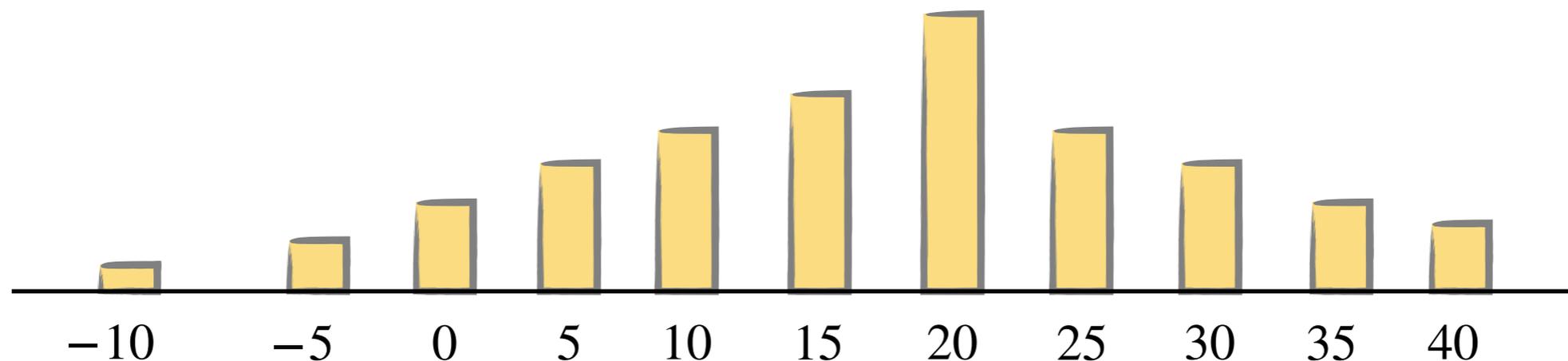
Single-Peaked Preferences

Assume that we have a set of possible temperatures for the thermostat, e.g., $\{-10, -5, 0, 5, 10, 15, 20, 25, 30, 35, 40\}$.

Let's say that your ideal temperature would be 20 degrees.

It is reasonable to assume that you would also prefer 25 degrees to 30 degrees, and likewise, 15 degrees to 10 degrees.

Generally, the “farther away” we move from your ideal temperature, the less happy you become.



Single-Peaked Preferences

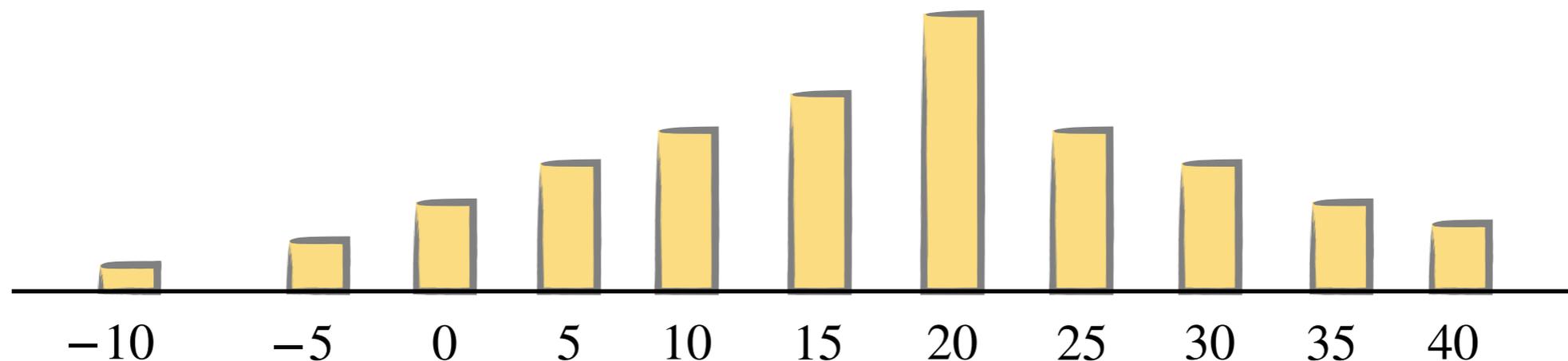
How is the facility location setting different from this?

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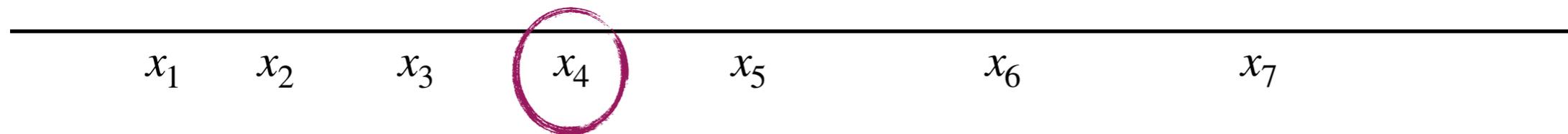
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Median Voter Mechanism

Let's use the median voter rule for the TFL problem.

The mechanism is truthful for the same reason as before.

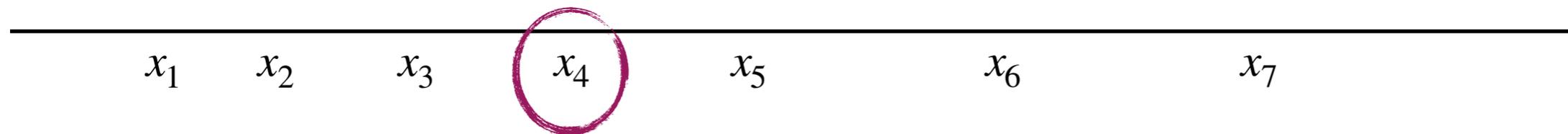


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Let's consider any other location $z \in \mathbb{R}$

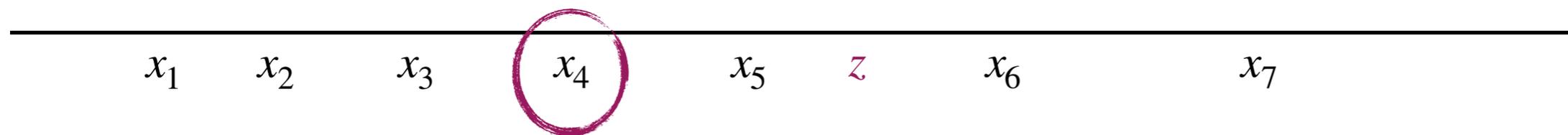


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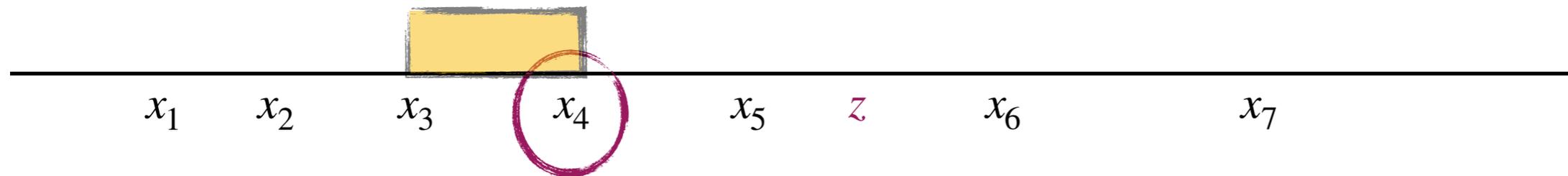


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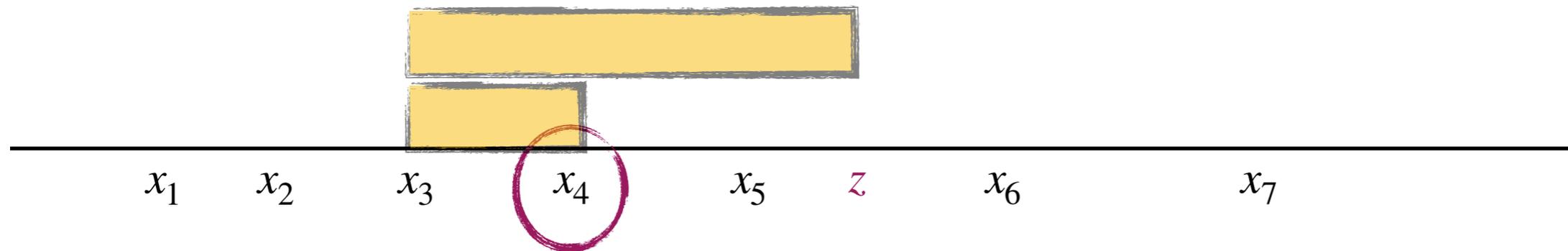


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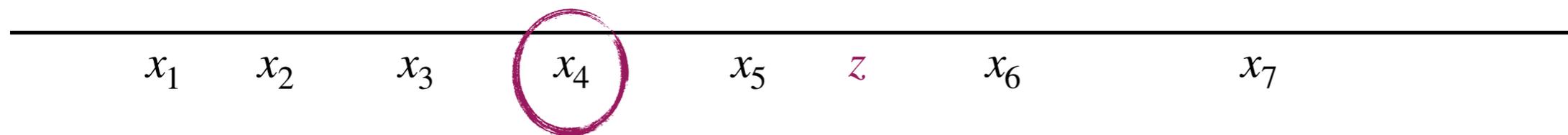


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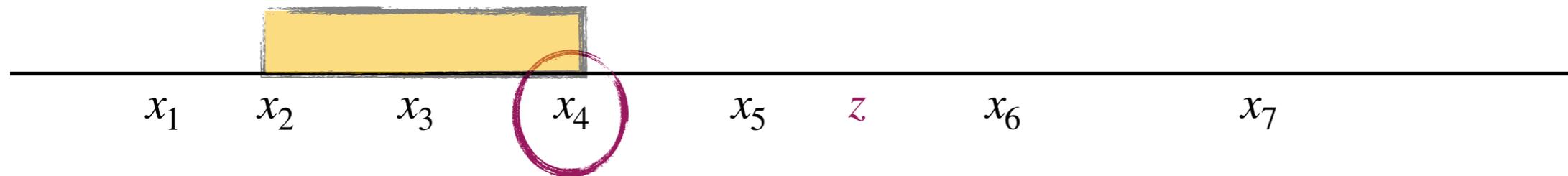


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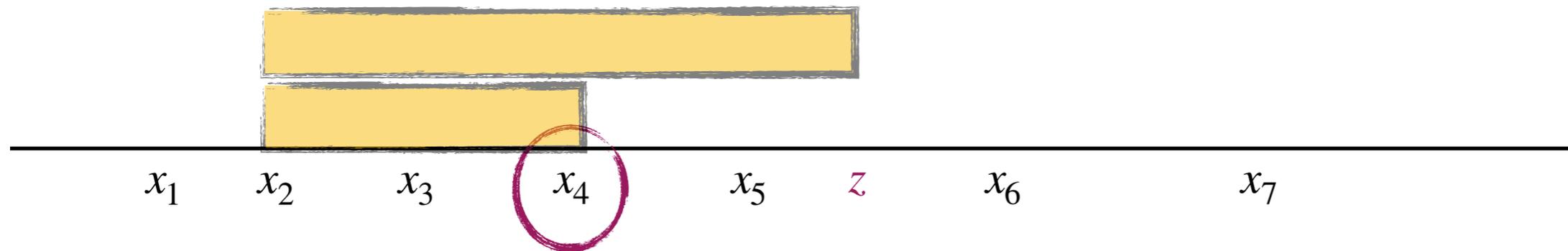


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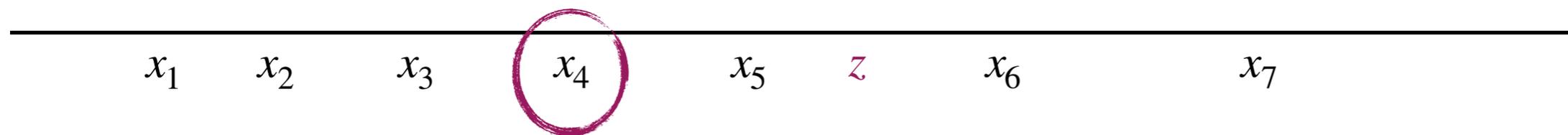


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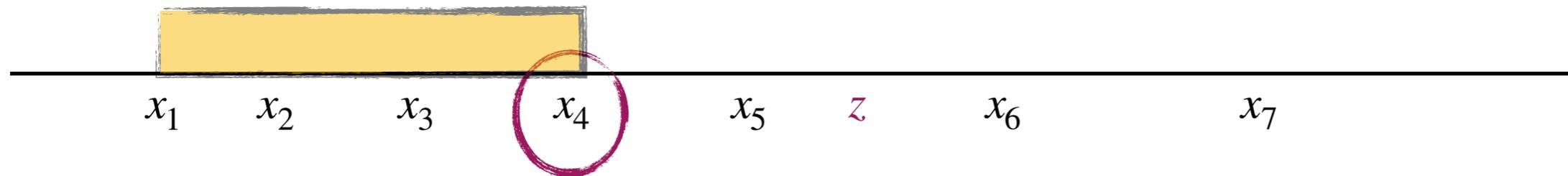


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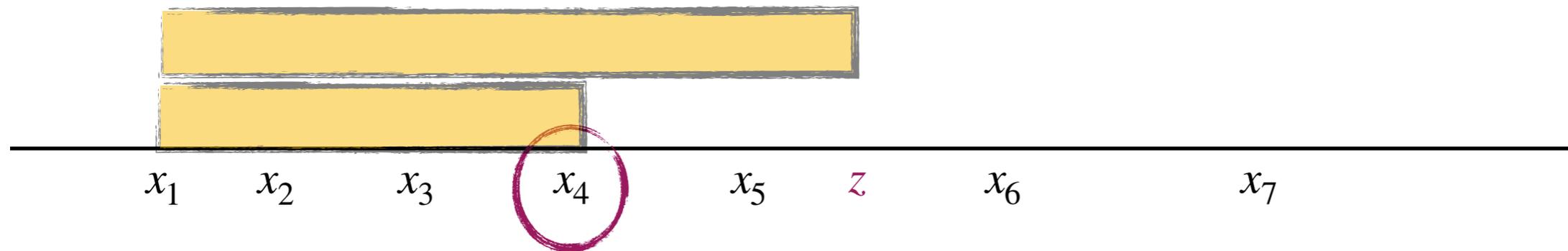


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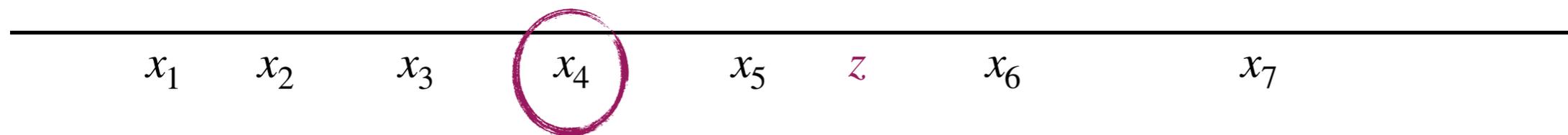
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At least half of the agents have a cost surplus

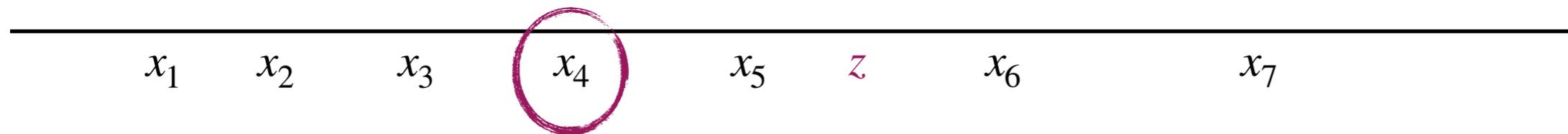


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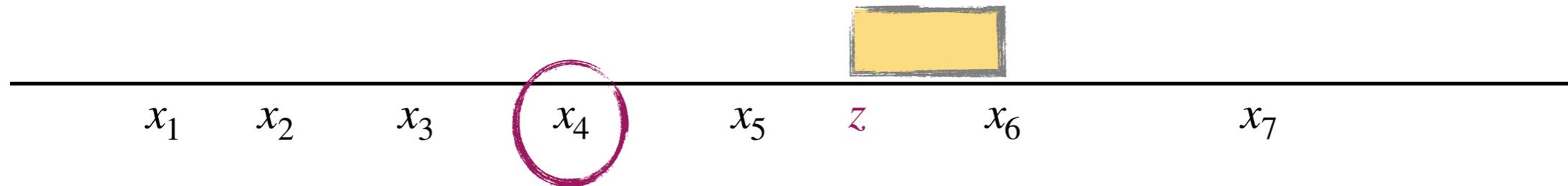


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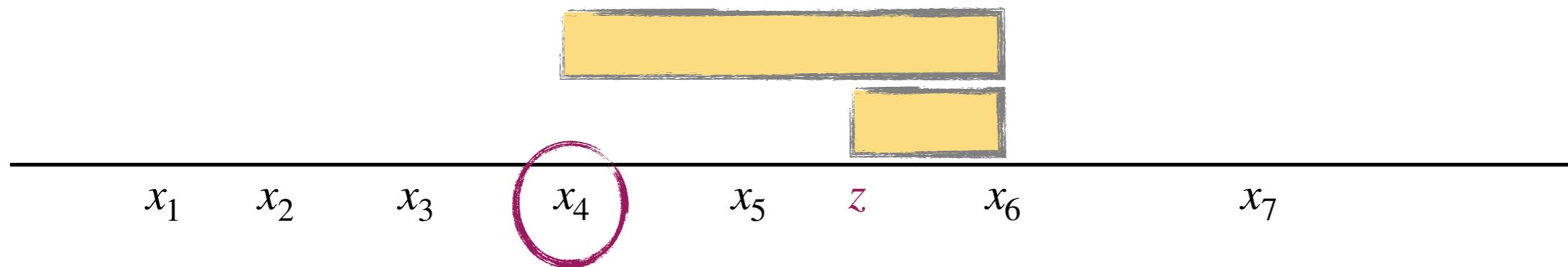


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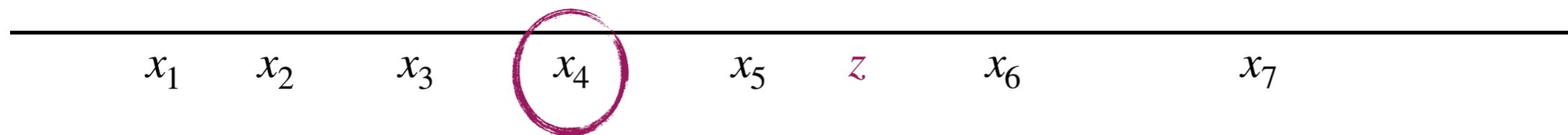


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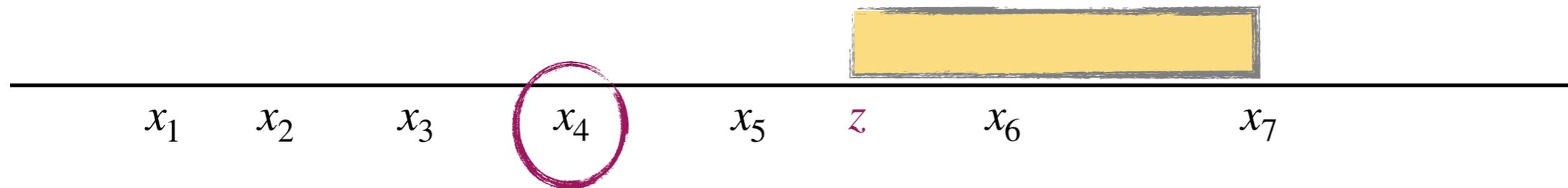


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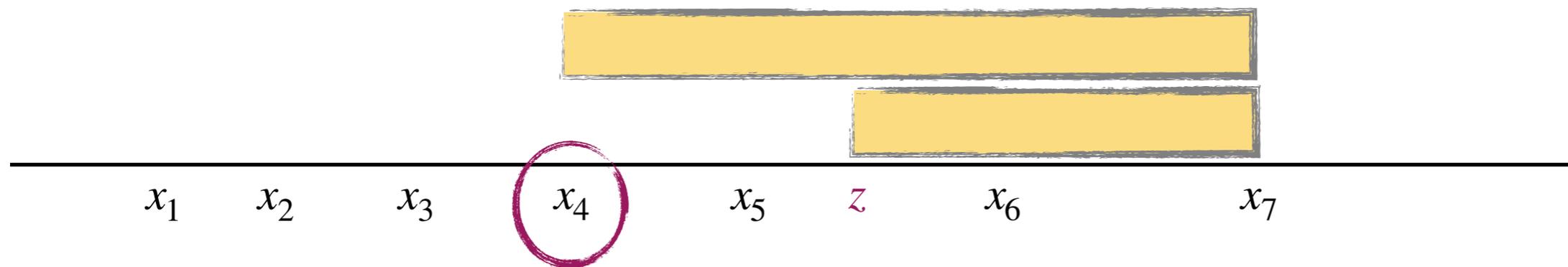


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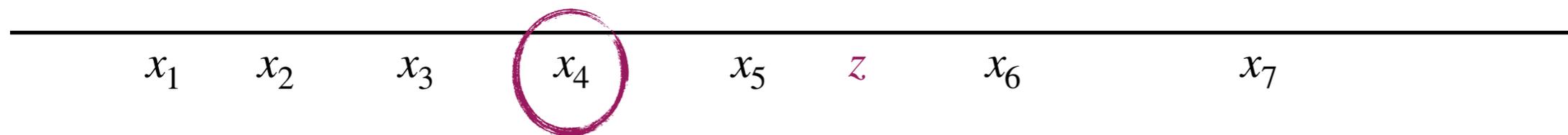


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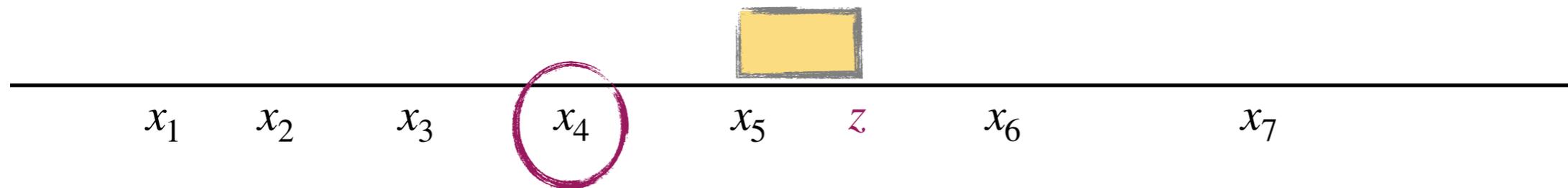


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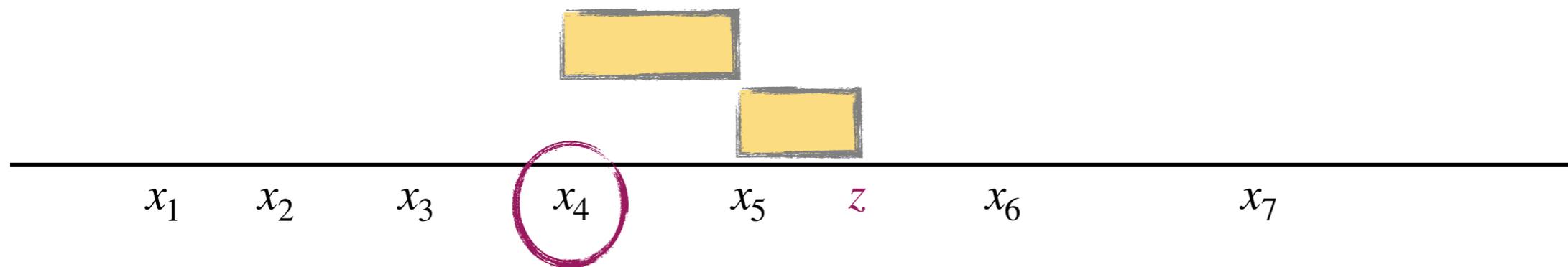


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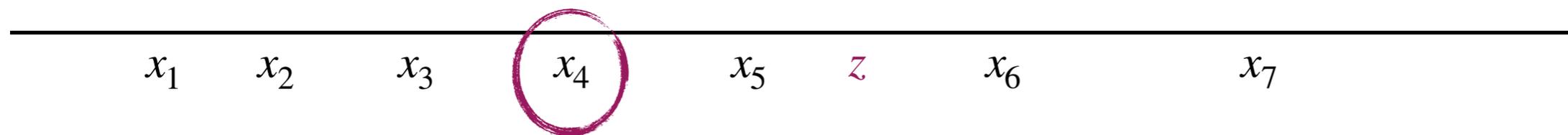
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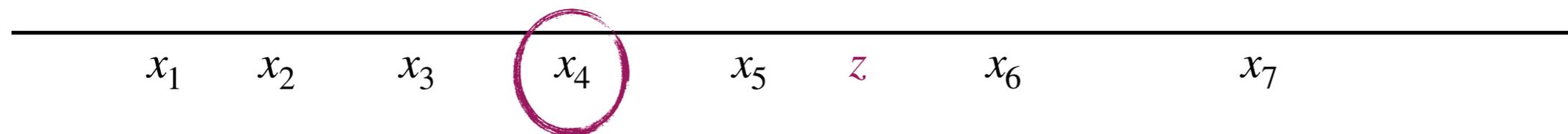
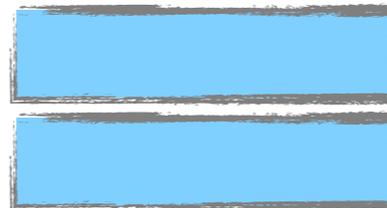
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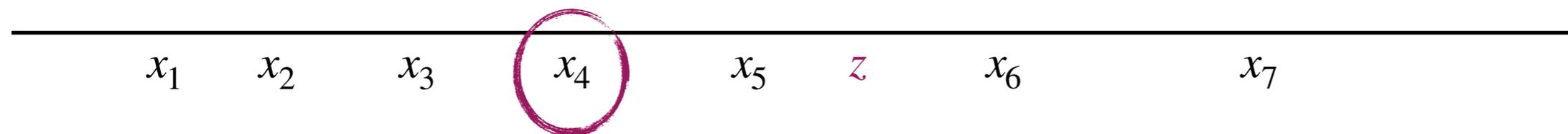
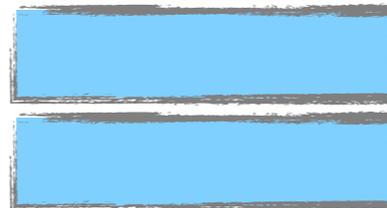
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What is the approximation ratio of the median voter mechanism?

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There is a set $N = \{1, \dots, n\}$ agents (voters), each of which has an ideal location (the “peak”) x_i on the real line \mathbb{R} .

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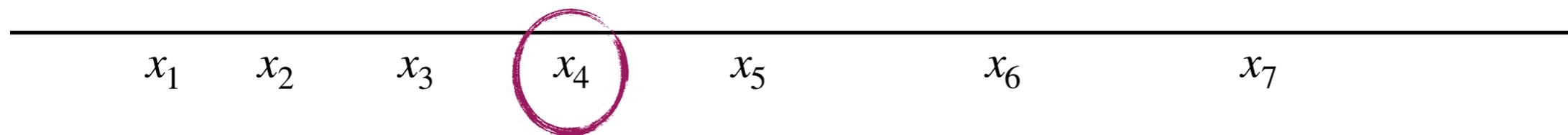
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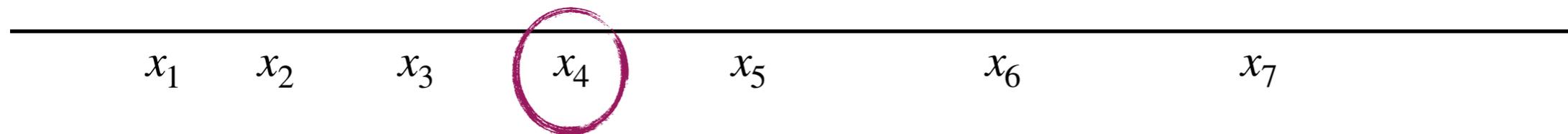


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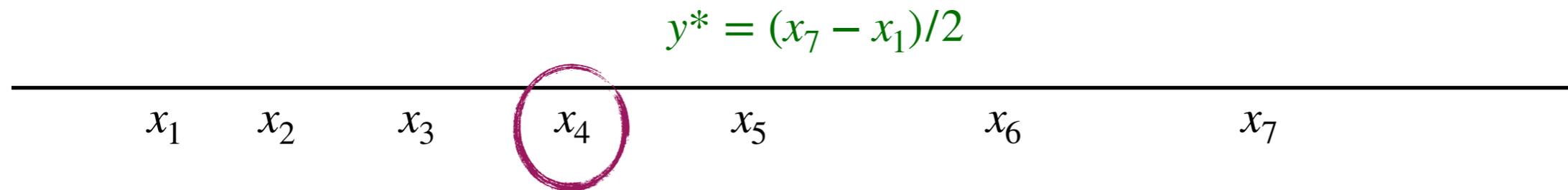


Median Voter Mechanism

Let's use the median voter rule for the TFL problem.

The mechanism is truthful for the same reason as before.

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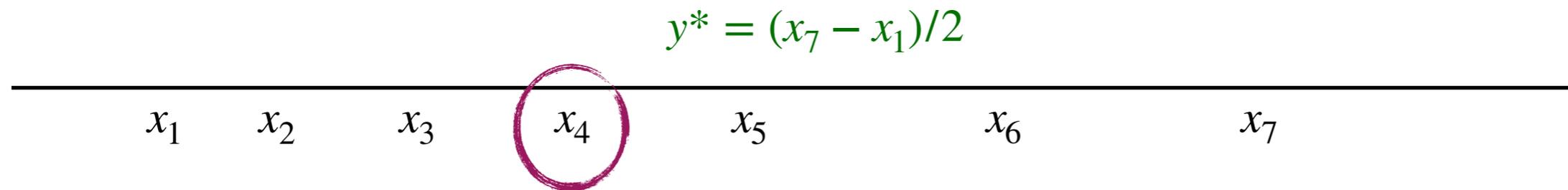
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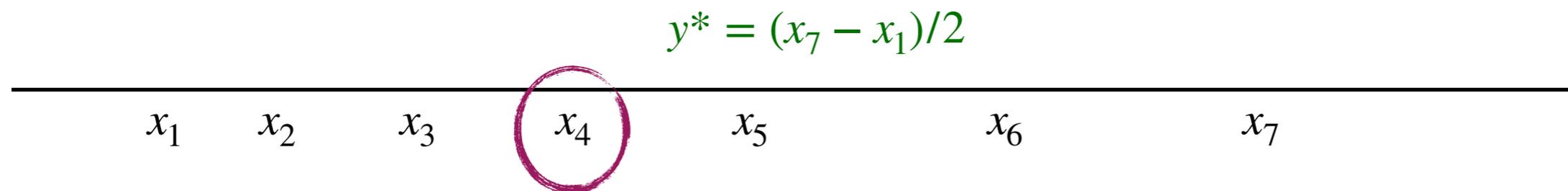
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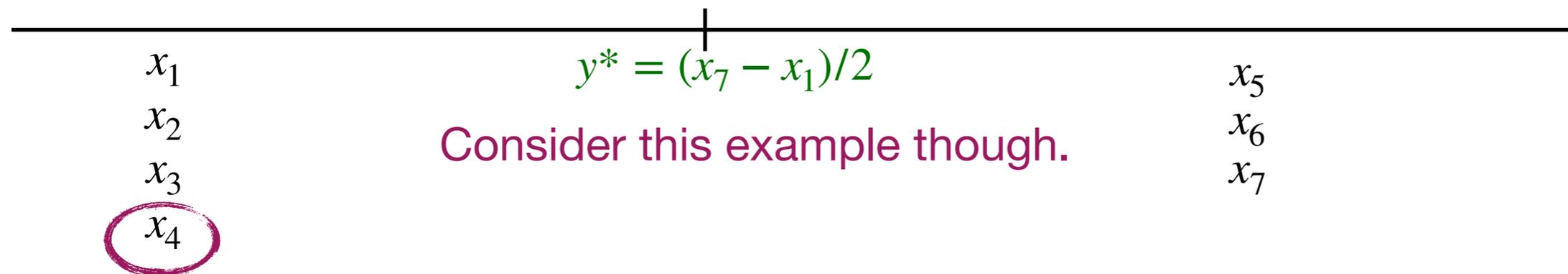
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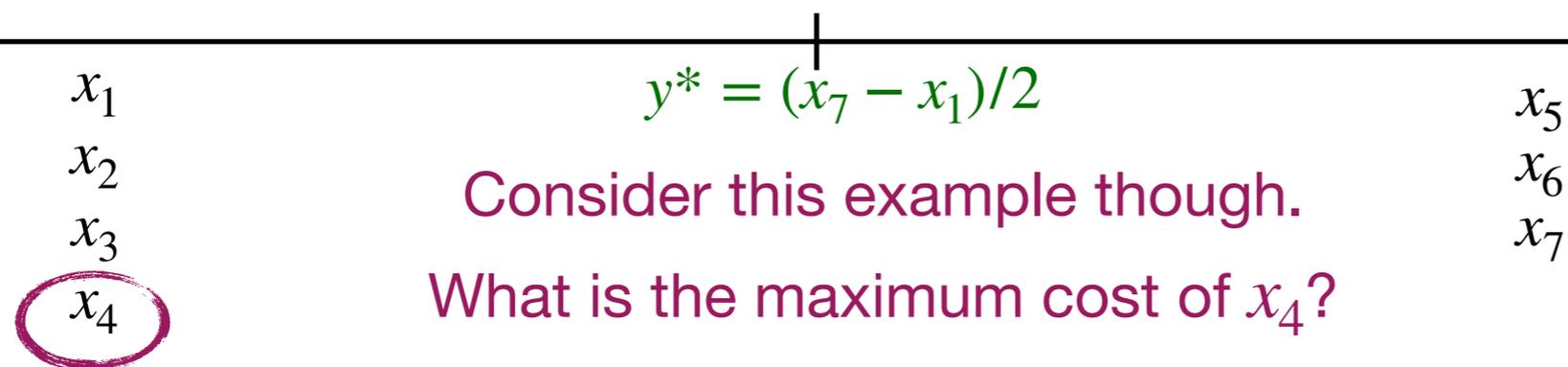
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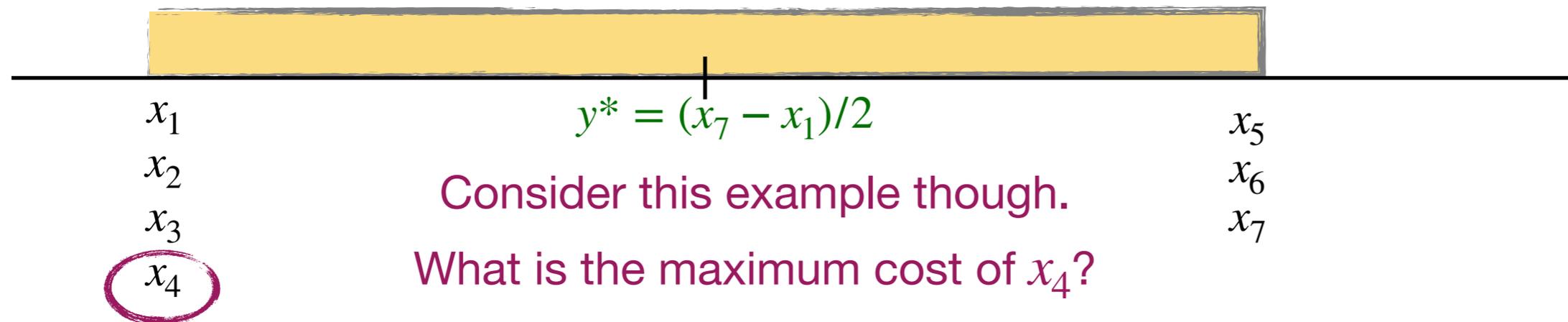
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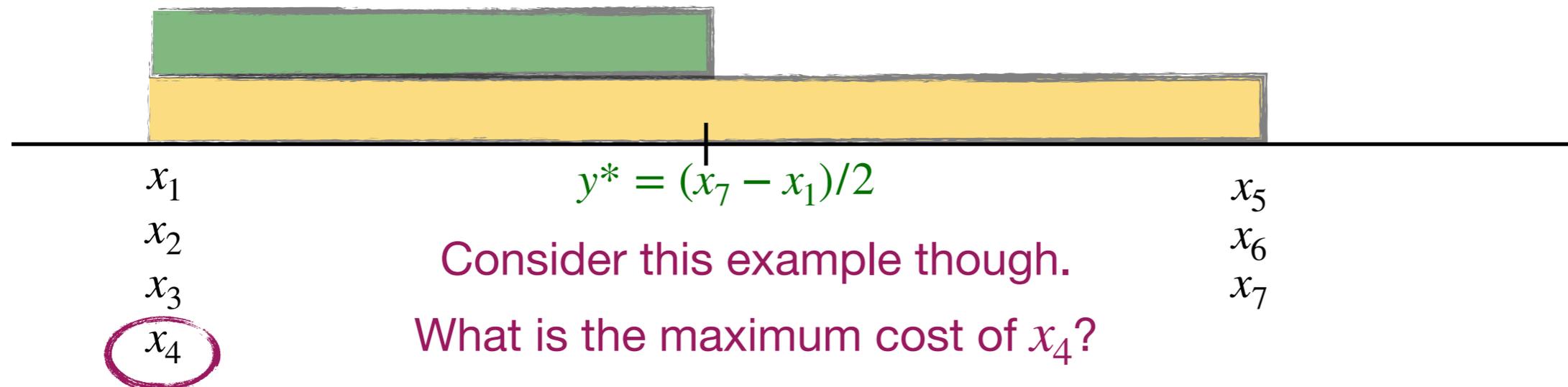
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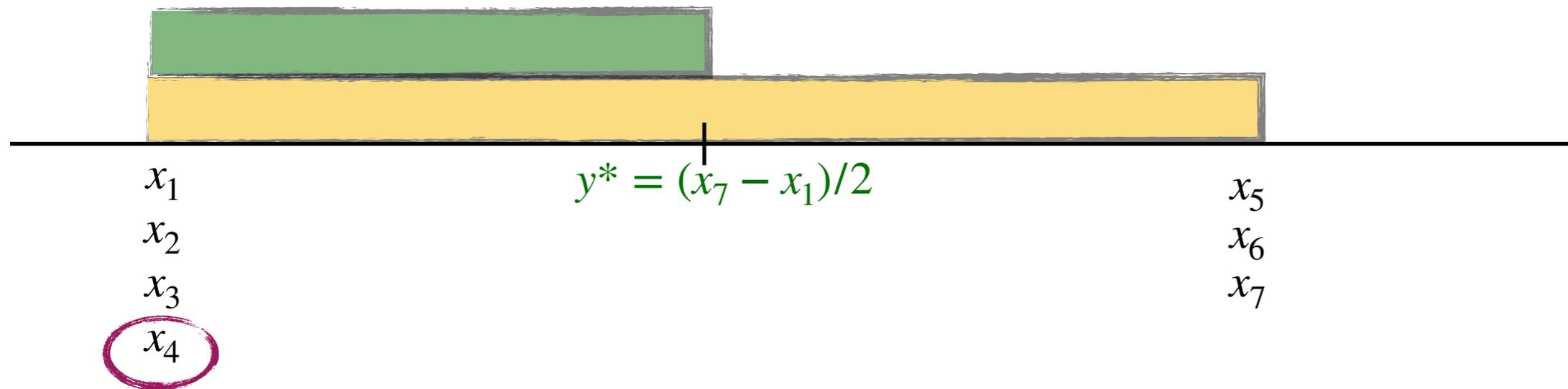
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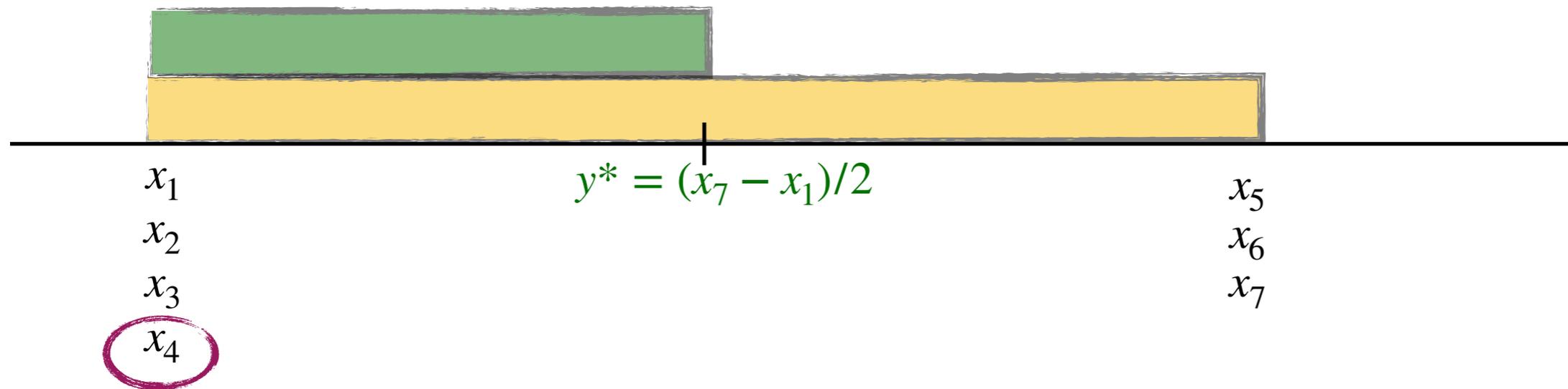


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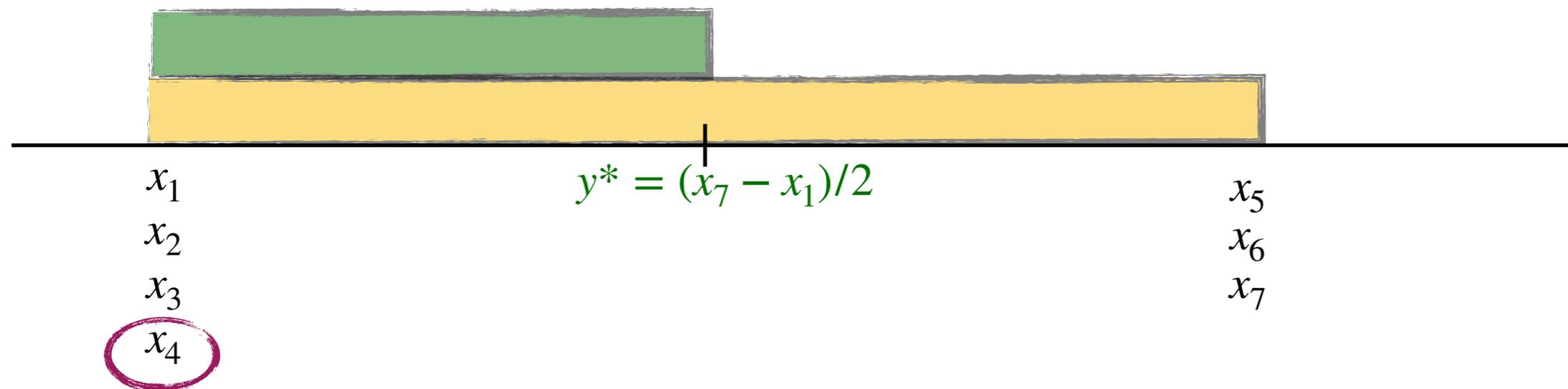
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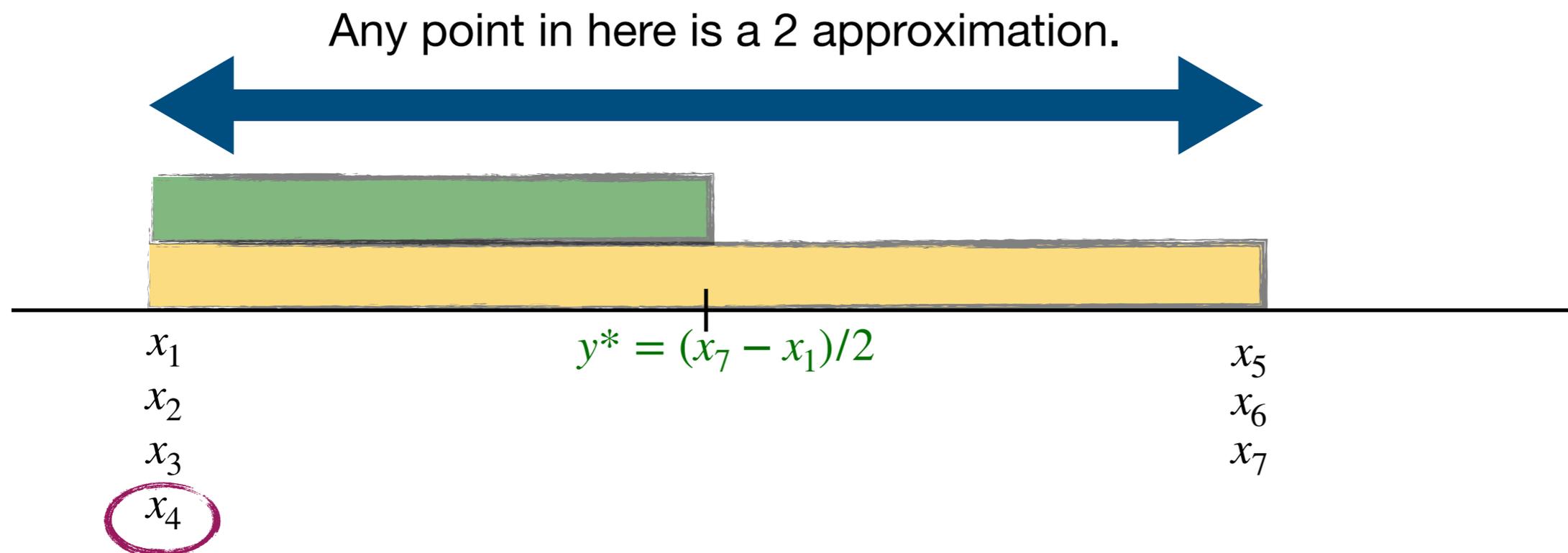
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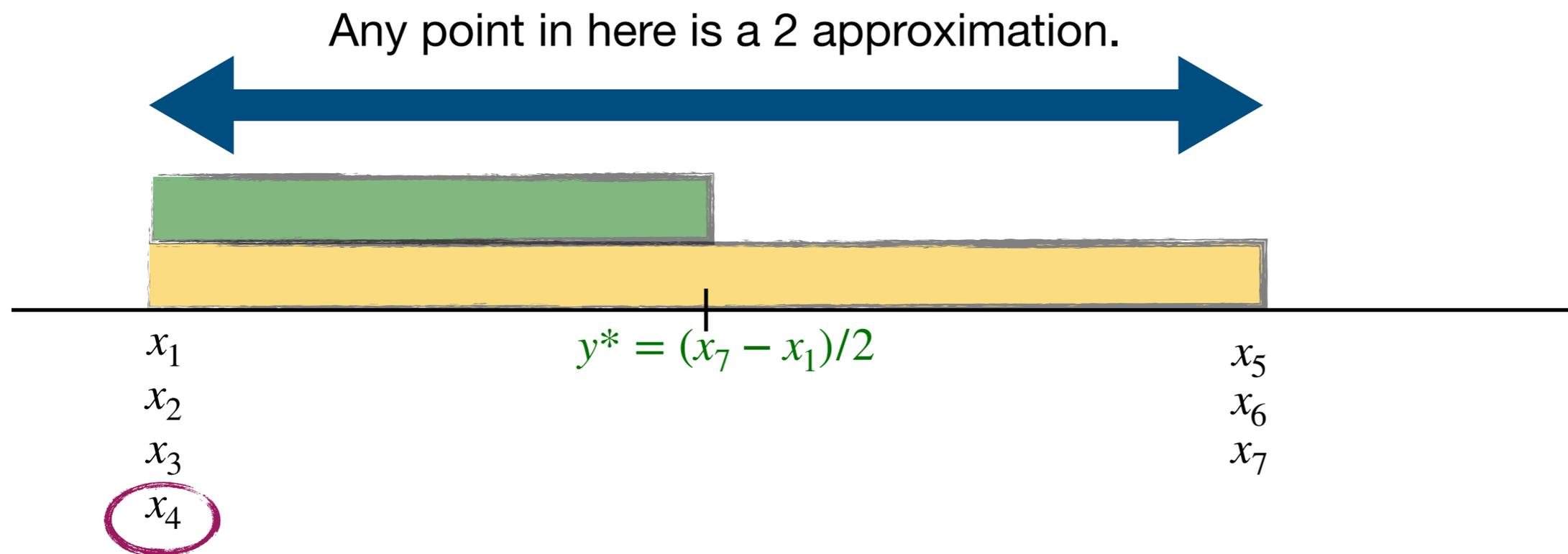


Median Voter Mechanism

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The approximation ratio of any k -th order statistic is exactly 2.



Truthful Facility Location (Procaccia and Tennenholtz 2010)

There is a set $N = \{1, \dots, n\}$ agents (voters), each of which has an ideal location (the “peak”) x_i on the real line \mathbb{R} .

We want to place a facility at some location $y \in \mathbb{R}$. Any location $y \in \mathbb{R}$ is a possible candidate.

Given a location $y \in \mathbb{R}$, the cost of agent i is defined as $|y - x_i|$, i.e., the distance between its peak and the location y .

A mechanism asks the agents to report their peaks x_i , and outputs a location $y = f(x_1, \dots, x_n)$.

Each agent aims to minimise its cost and reports its peak as \hat{x}_i accordingly.

We want to design a truthful mechanism (voting rule) for the problem that has the minimum possible approximation ratio for the **maximum cost objective**, i.e., the maximum of agents' costs $\max_{i \in N} |y - x_i|$

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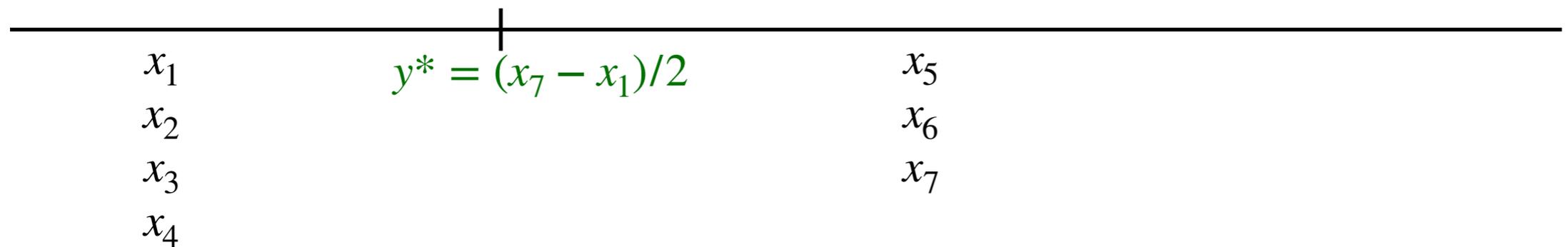
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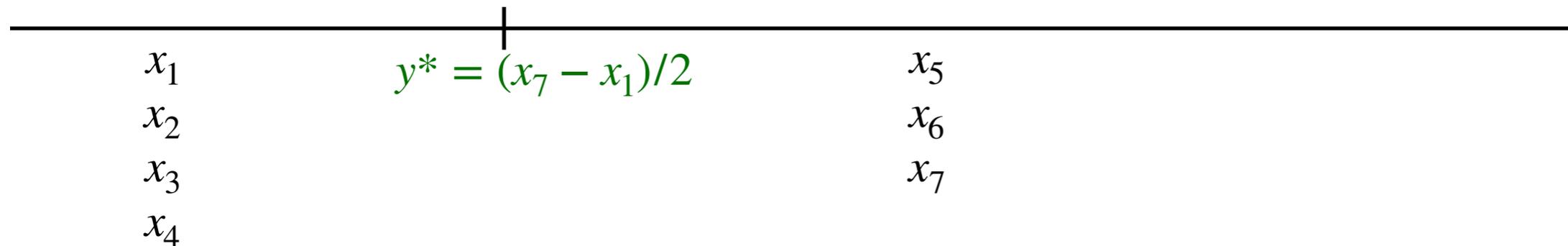
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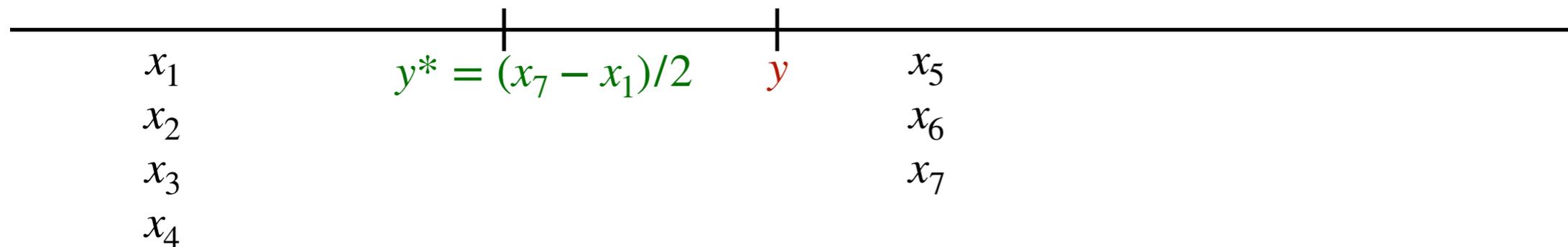


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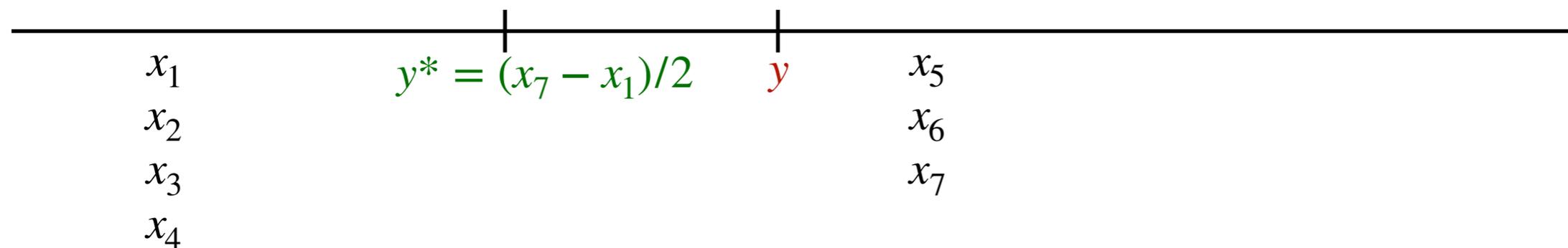
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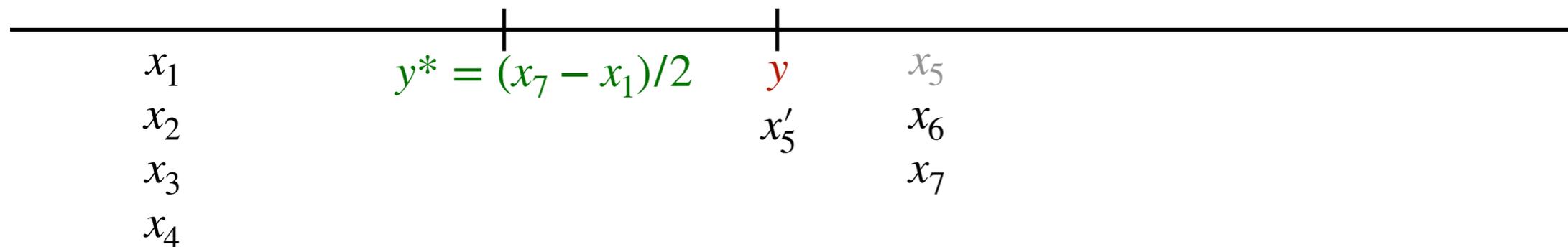
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Assume now that agent with peak x_5 reports $x'_5 = y$



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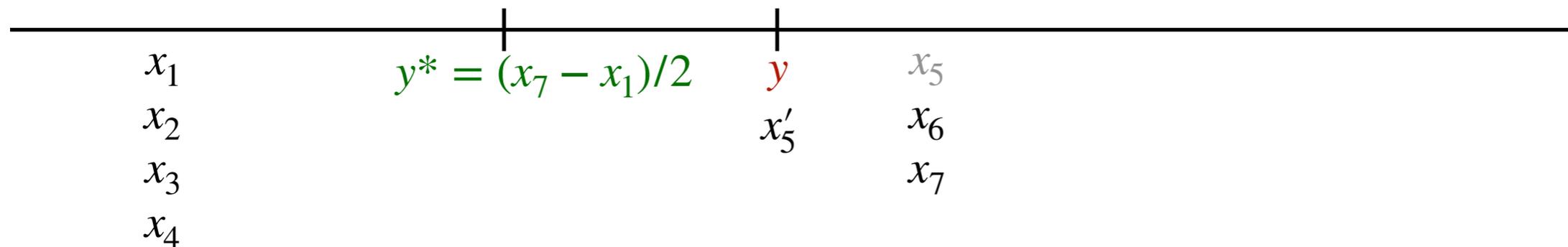
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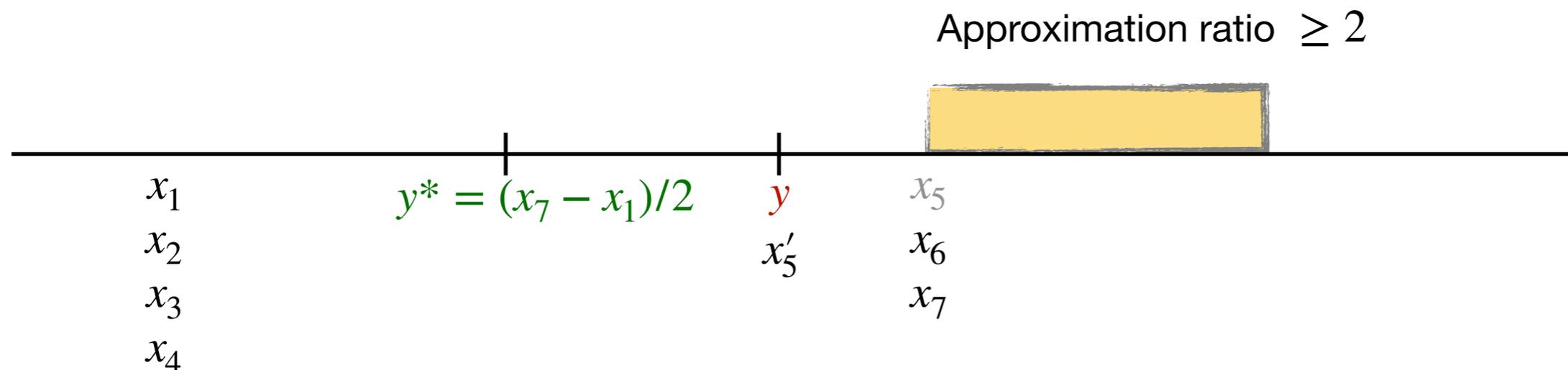
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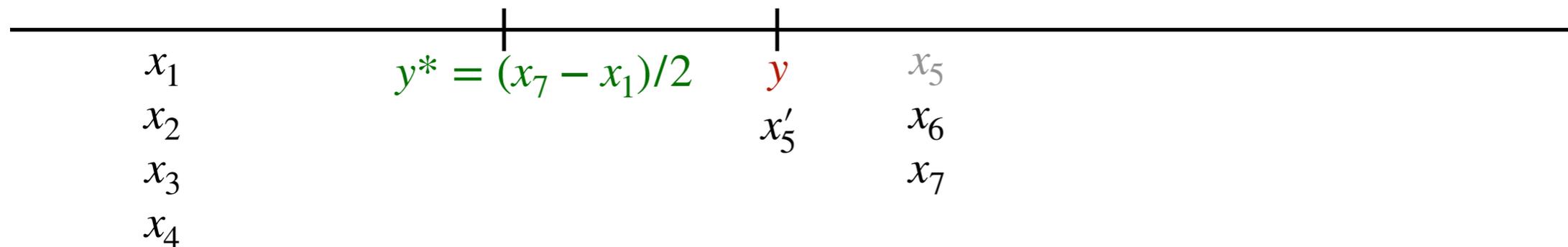
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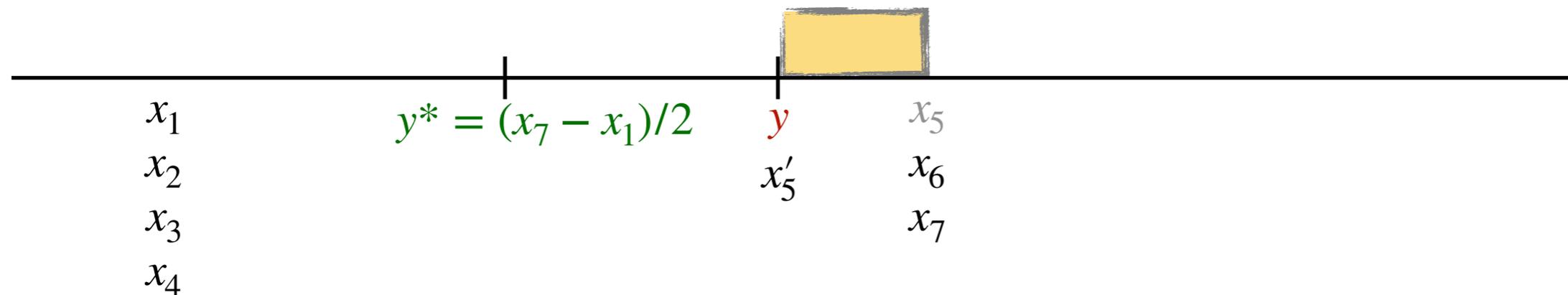


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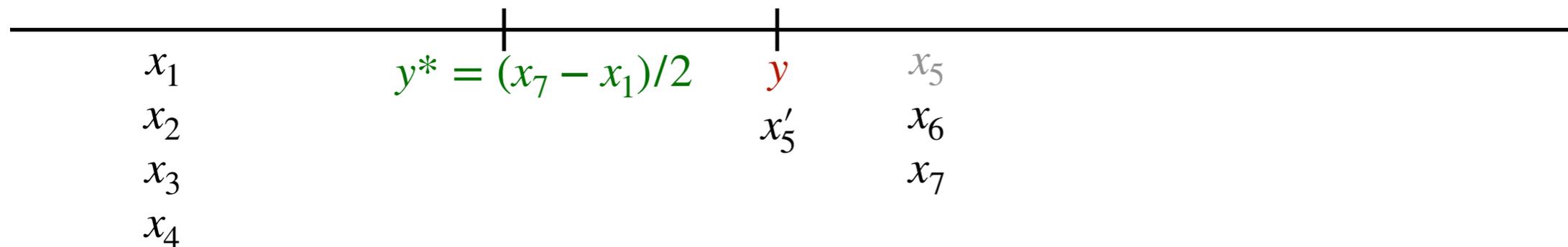
x_5 brings the facility closer to its true peak.



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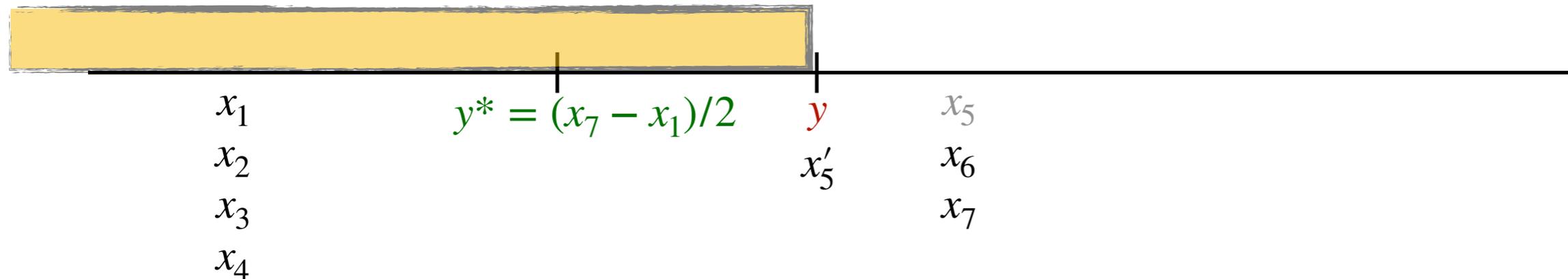


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It could be the case that x'_5 is the true peak and x_5 is the misreport.

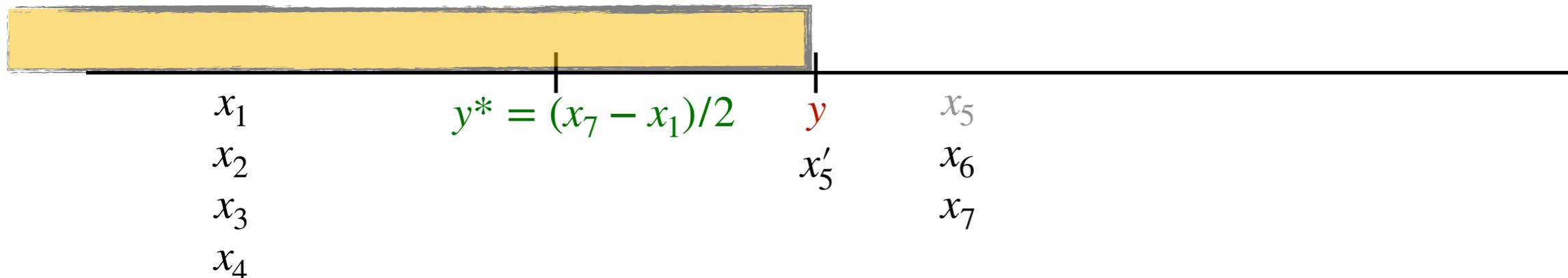


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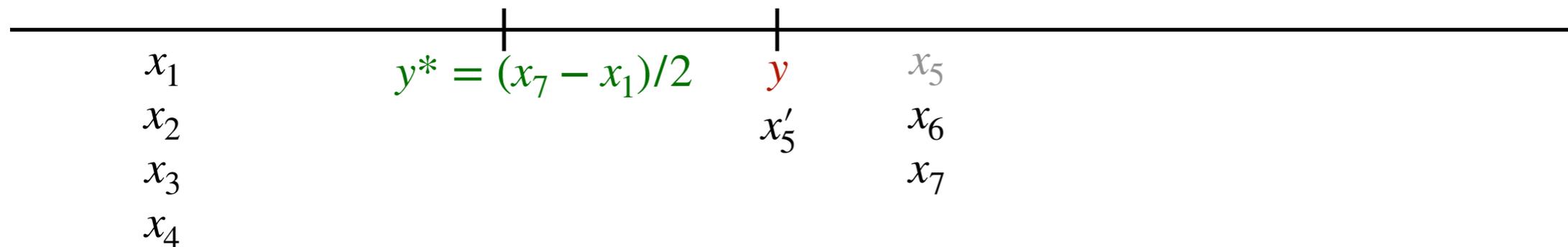
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In that case the misreport would bring the facility exactly on the true peak.



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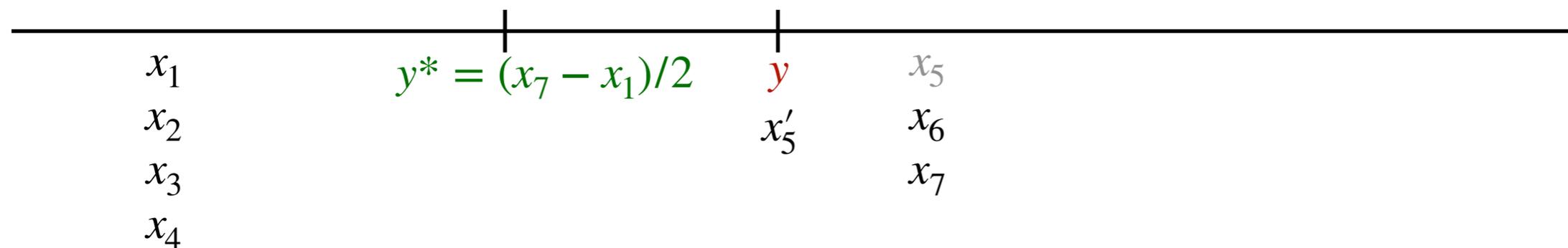


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We can use the same argument for x_6 and x_7 .

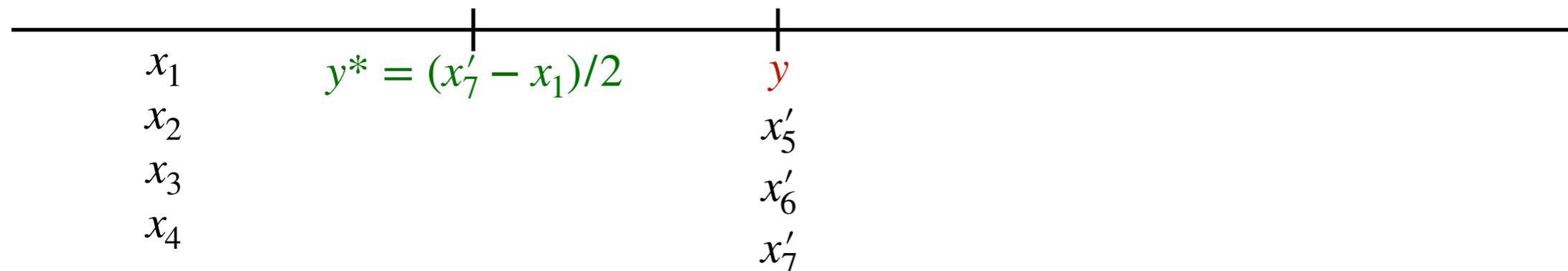


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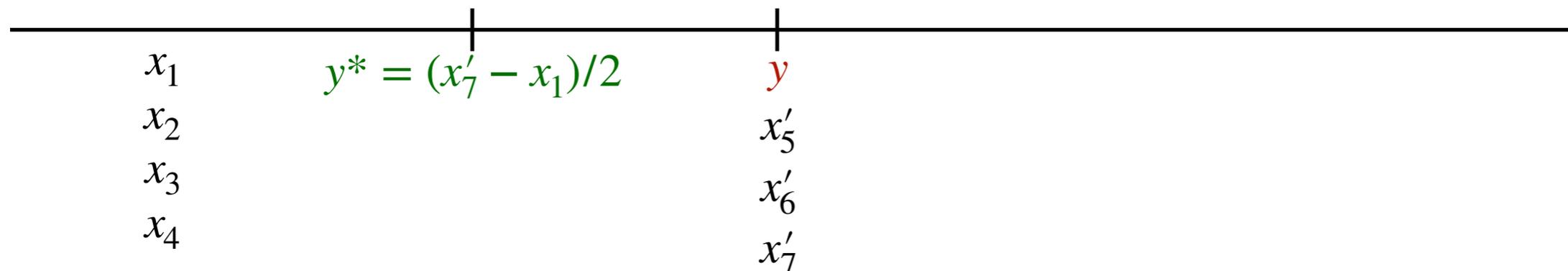
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What is the ratio on this instance?



Truthful Facility Location, max cost objective

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[Theorem \(Procaccia and Tennenholtz 2010\)](#): The best possible approximation ratio achieved by any truthful mechanism for the maximum cost objective is 2. This is achieved by any k -th ordered statistic mechanism.