

Algorithmic Game Theory and Applications

Single-Parameter Domains

Domains

Unrestricted Domain



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Unrestricted Domain

Gibbard-Satterthwaite 73-75

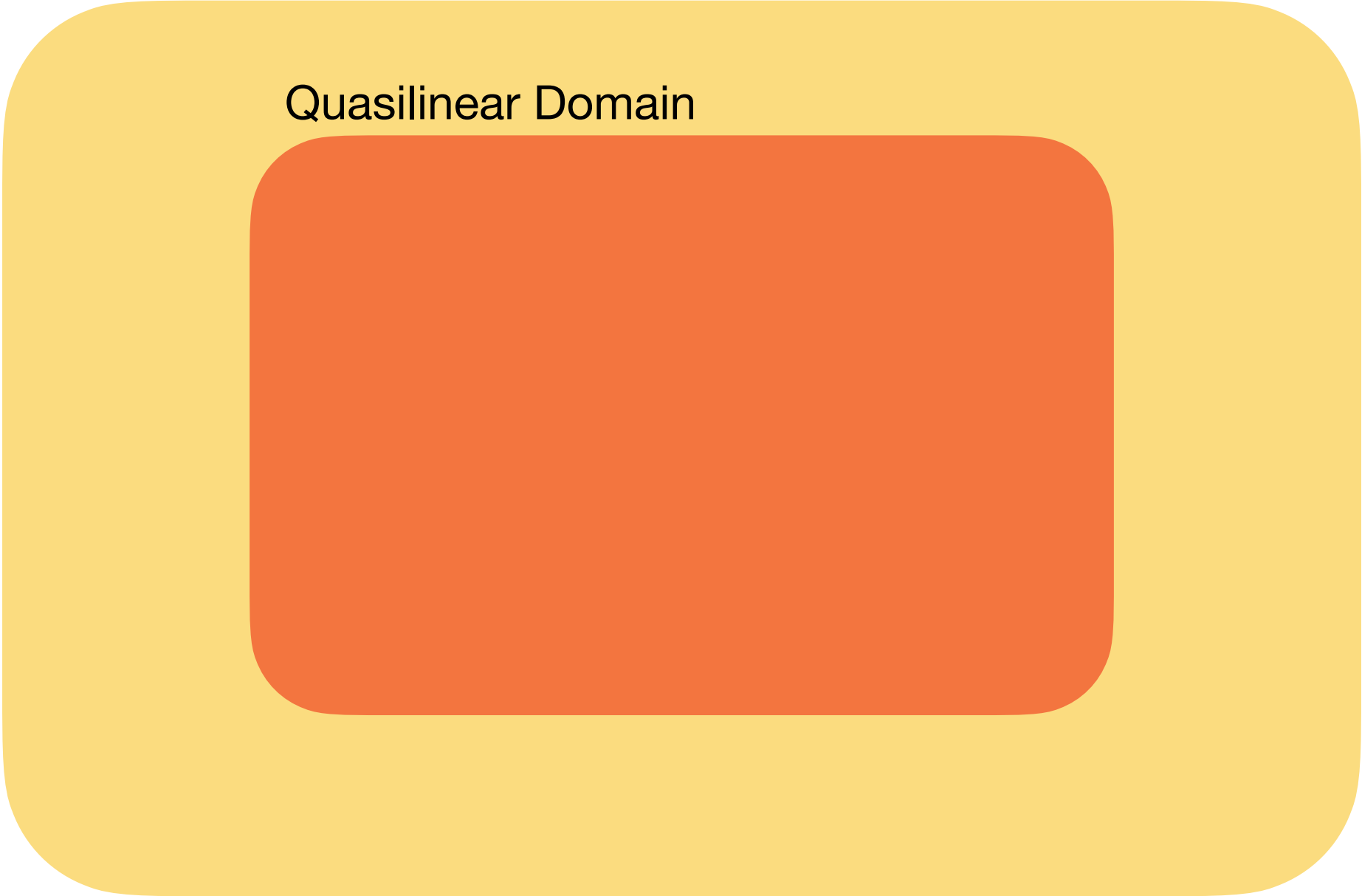


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Quasilinear Domain

A diagram illustrating nested domains. It consists of two rounded rectangles. The outer rectangle is yellow and represents the 'Unrestricted Domain'. The inner rectangle is orange and represents the 'Quasilinear Domain'. The orange rectangle is centered within the yellow rectangle, visually demonstrating that the quasilinear domain is a subset of the unrestricted domain.

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Roberts 79



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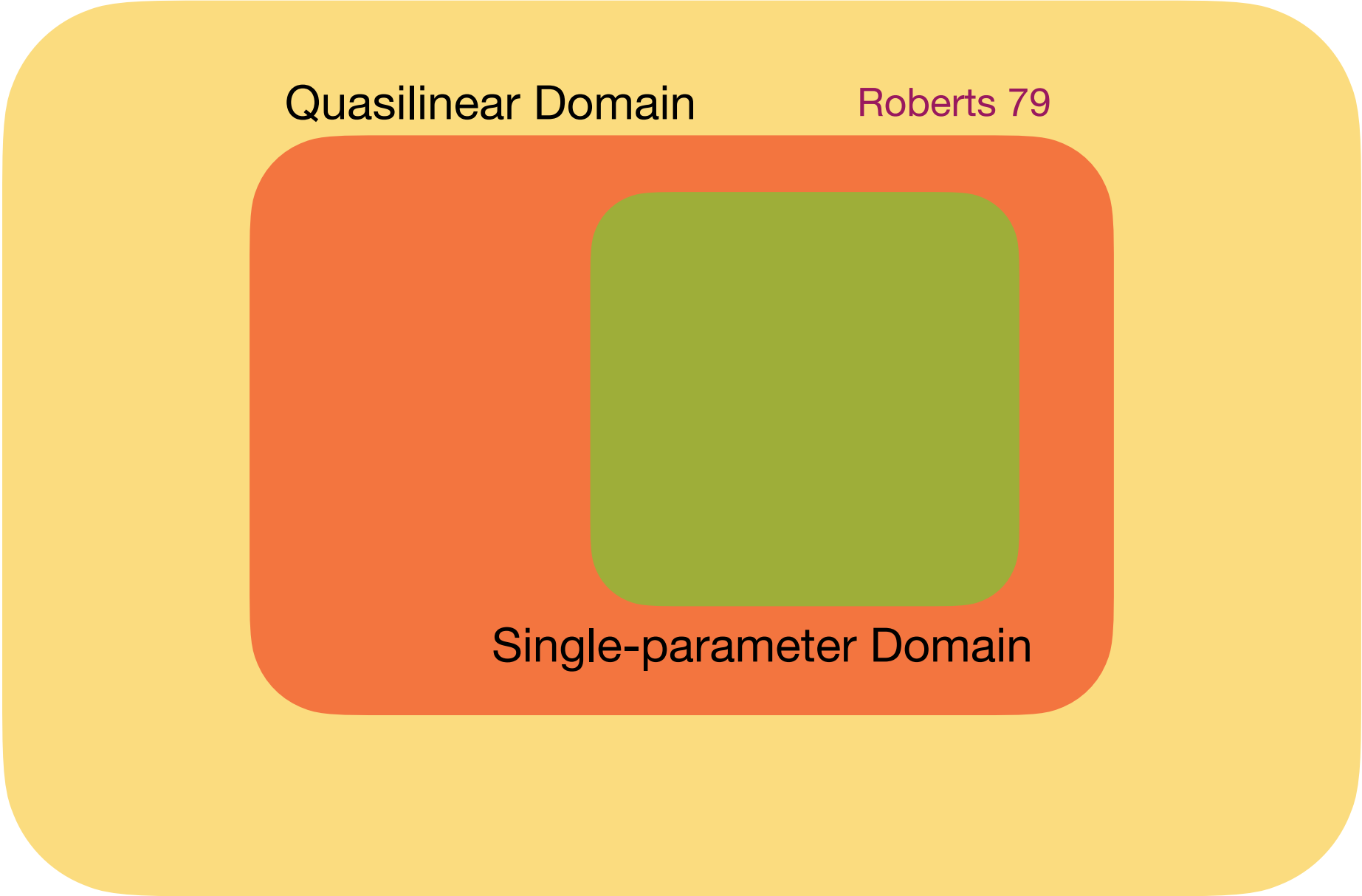
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Single-parameter Domain



Single-item Auctions

There are n bidders from a set $N = \{1, \dots, n\}$.

There is one item for sale.

Every bidder has a value v_i for the item - this is the bidder's willingness to buy it.

Each bidder chooses a bid $b_i = \beta(v_i)$ according to some function β .

The allocation function $f : B^n \rightarrow \{0, 1\}^n$ decides who wins given the bids.

The payment function $p : B^n \rightarrow \mathbb{R}^n$ decides how much each bidder will pay.

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Public project: A public project with cost C is to be implemented, which is valued by each citizen at v_i . The government wants to implement the project if $\sum_i v_i > C$.

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It may make sense to think of single-item auctions, keeping in mind that the results that we will present next are much more general.

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Monotonicity

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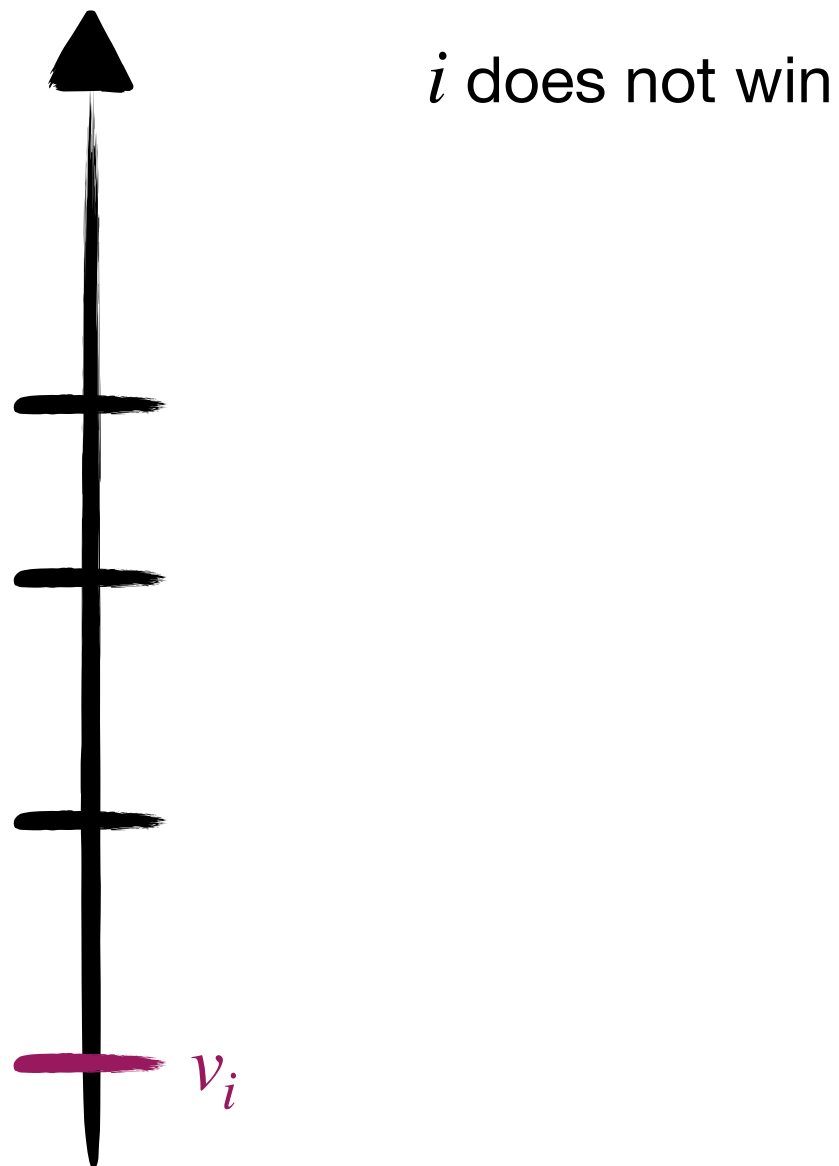
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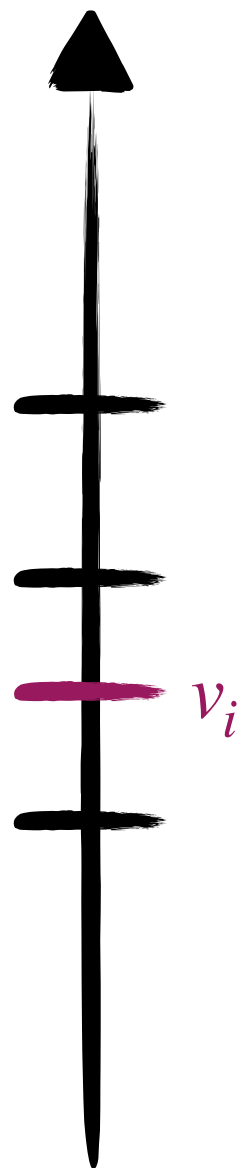
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i.e., if the value of agent i increases, then, if i was winning before, i is still winning.

Pictorially, in a single-item auction

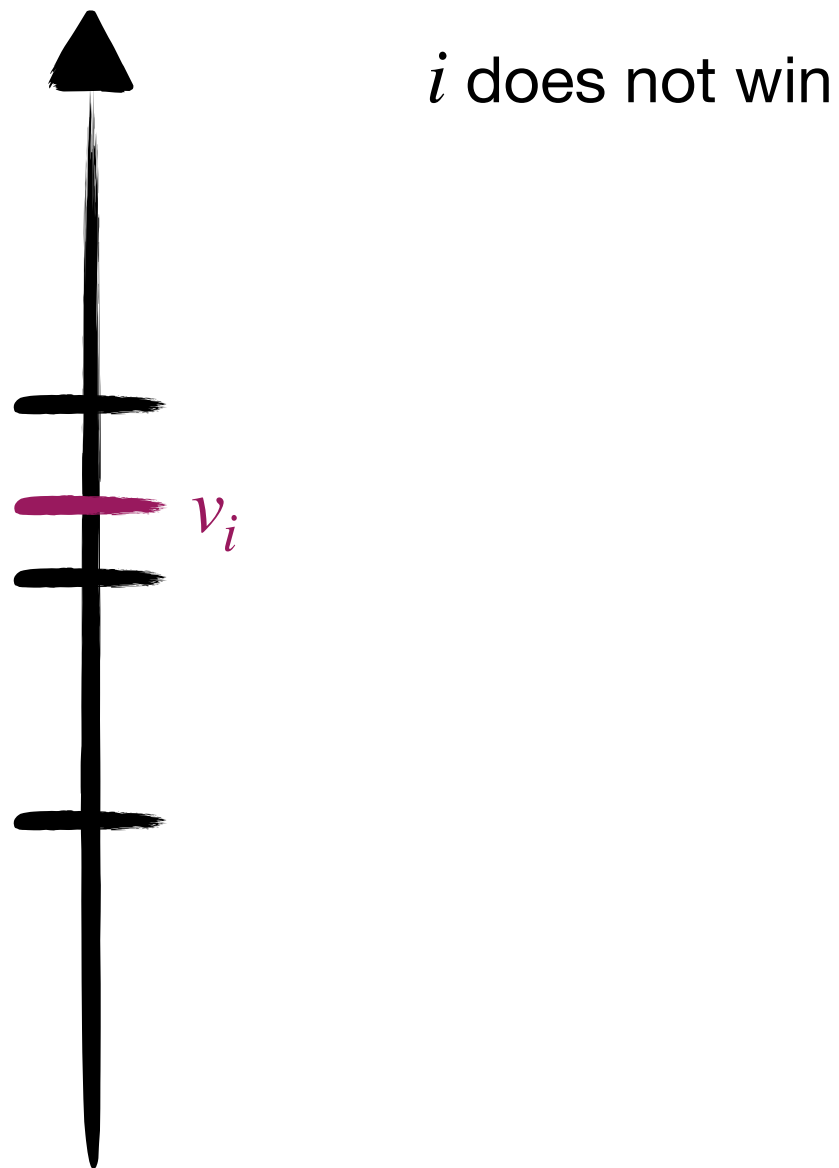


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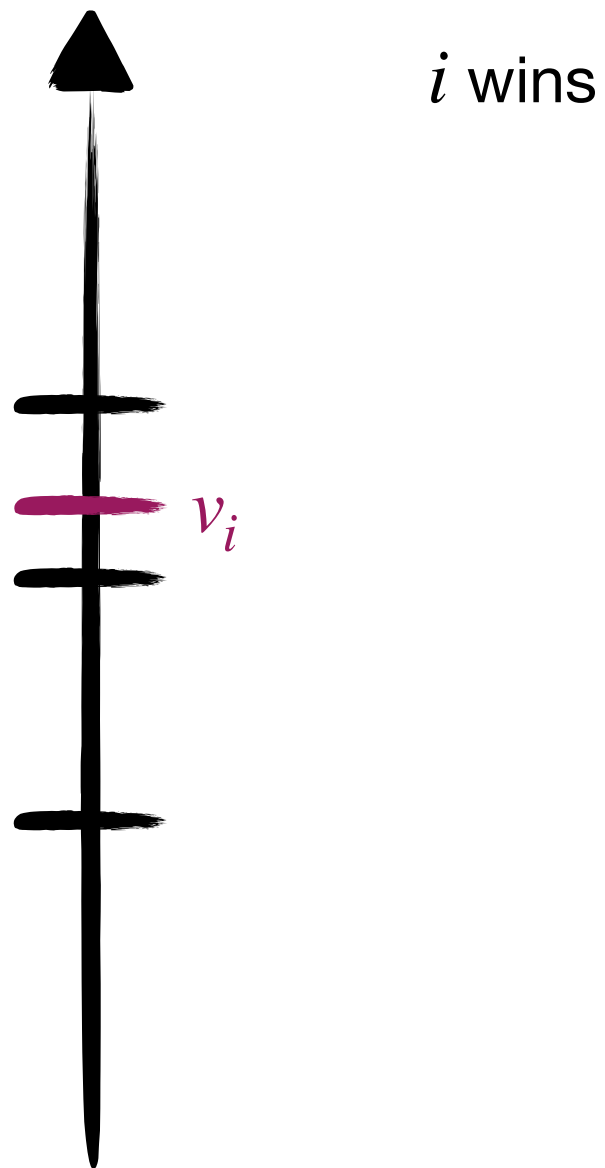


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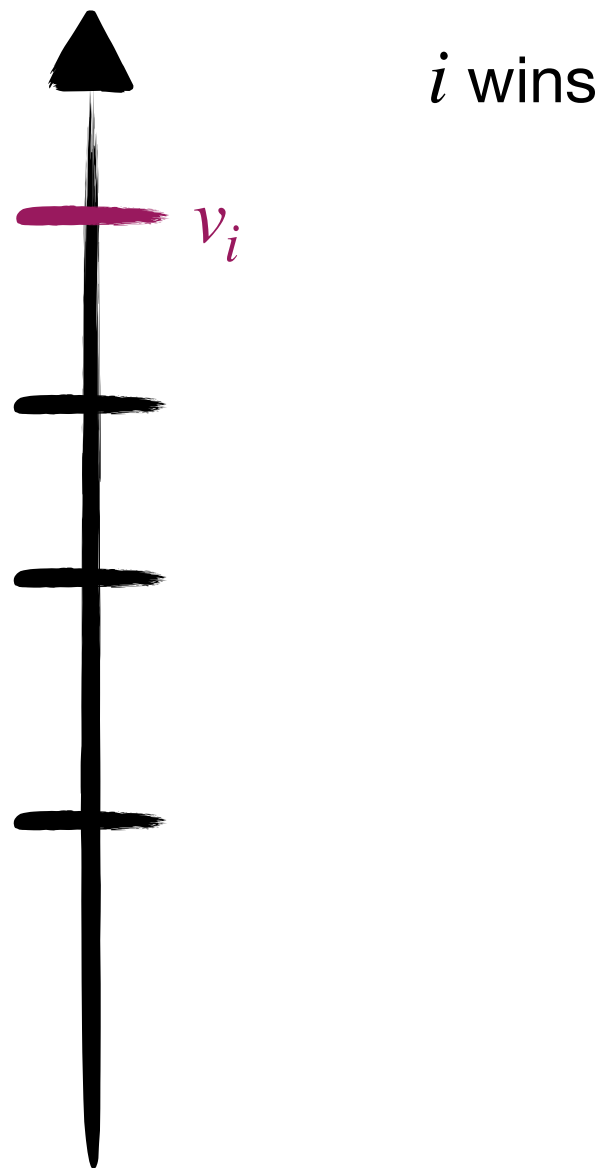
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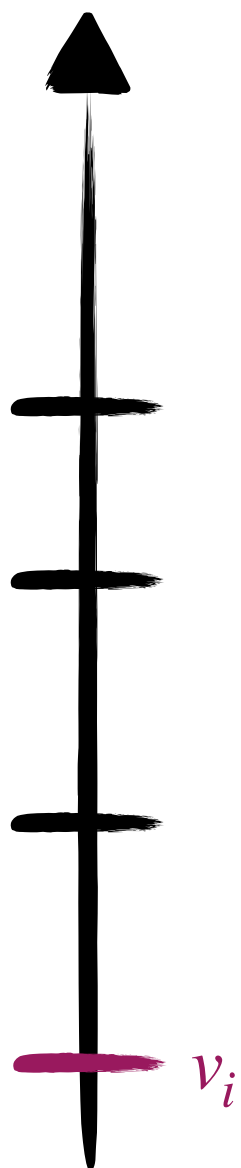
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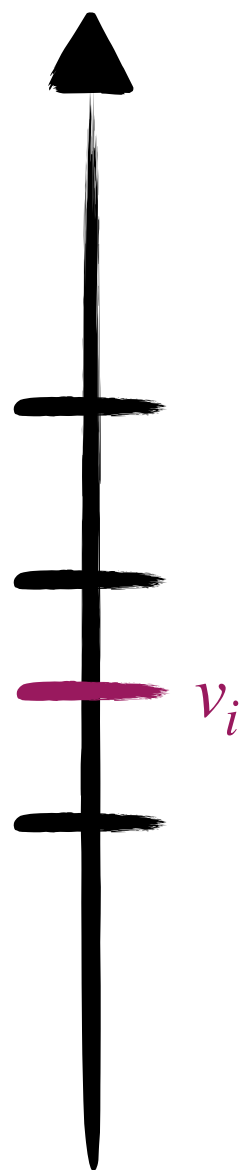


Critical value



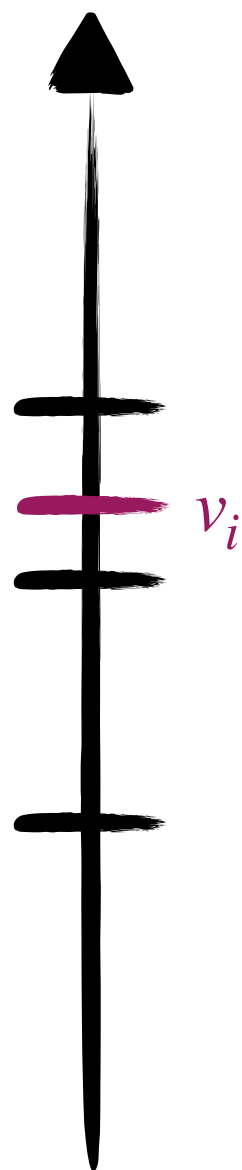
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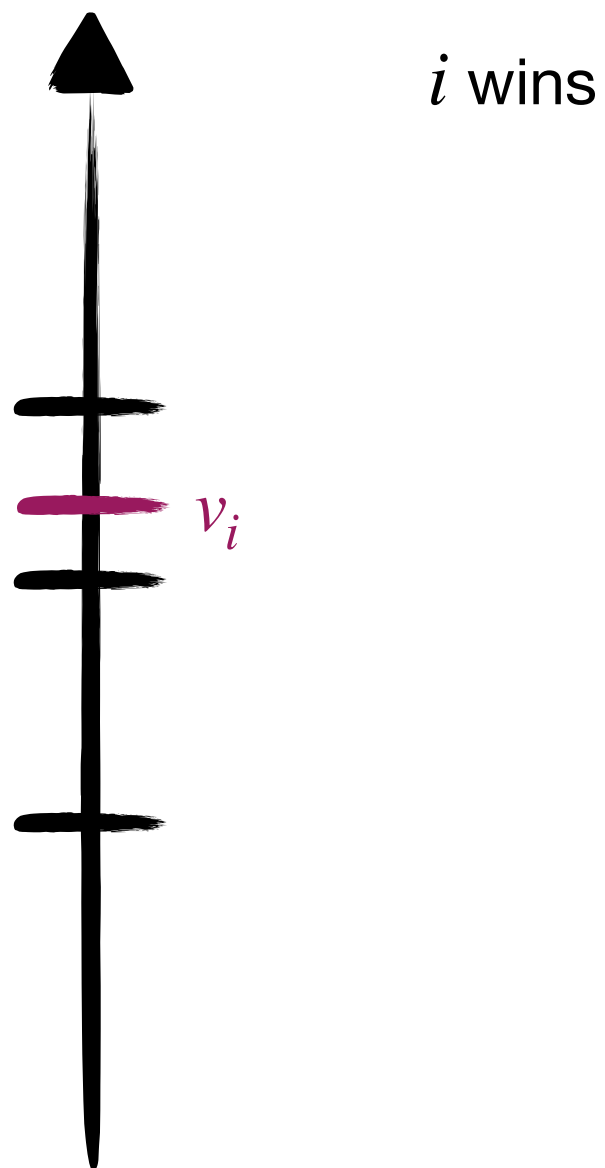
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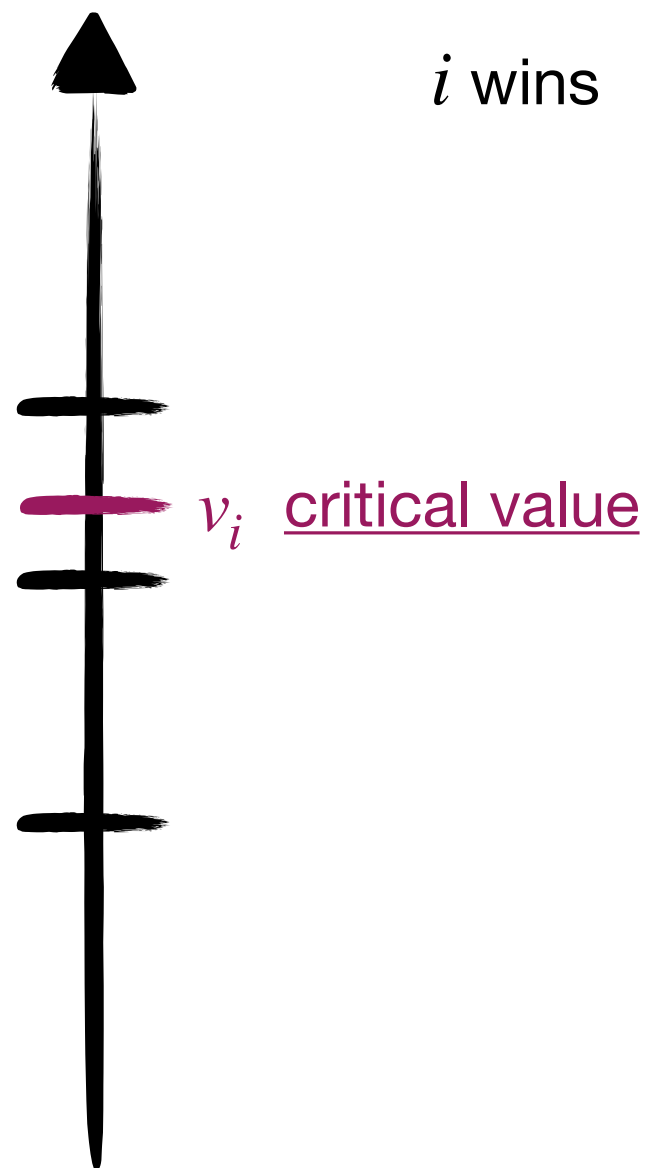


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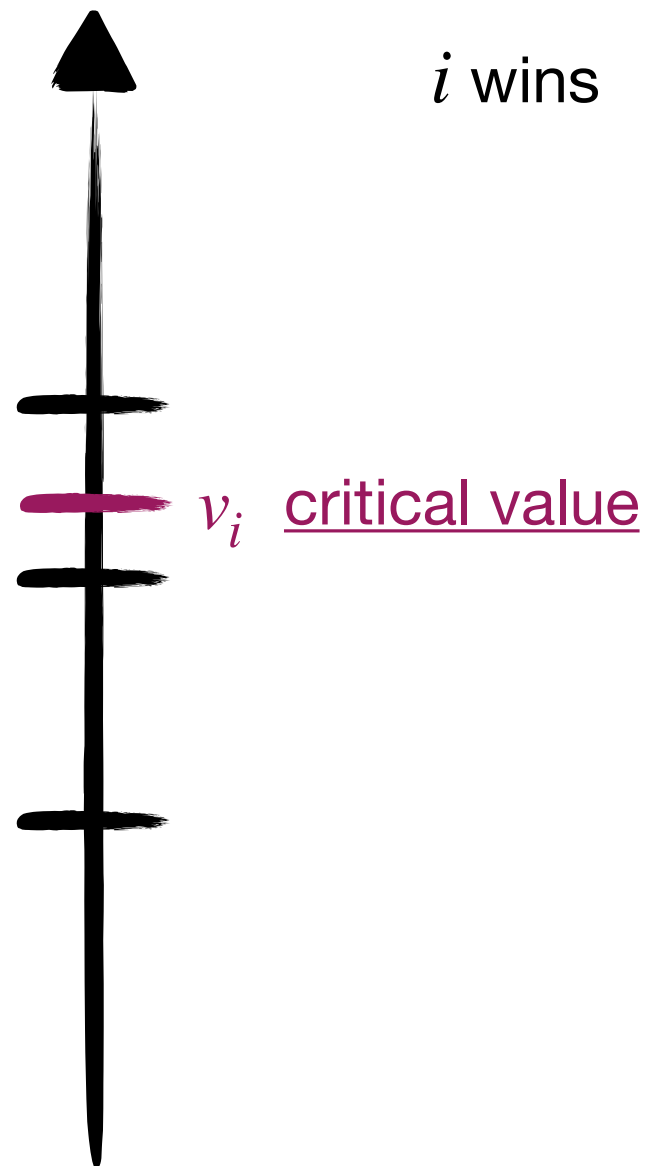
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Definition (critical value): The critical value of a social choice function f in the single-parameter domain is

$$c_i(v_{-i}) = \sup_{v_i: f(v_i, v_{-i}) \neq W_i} v_i$$

Second-price auctions (SPA)

These are also *sealed-bid auctions*.

Each bidder submits their bid independently, without seeing the bids of the other bidders.

The winner is the bidder with the *highest bid*.

If there are multiple such bidders, one is chosen *at random*.

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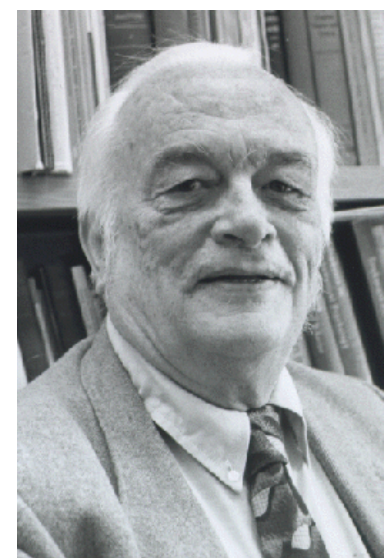
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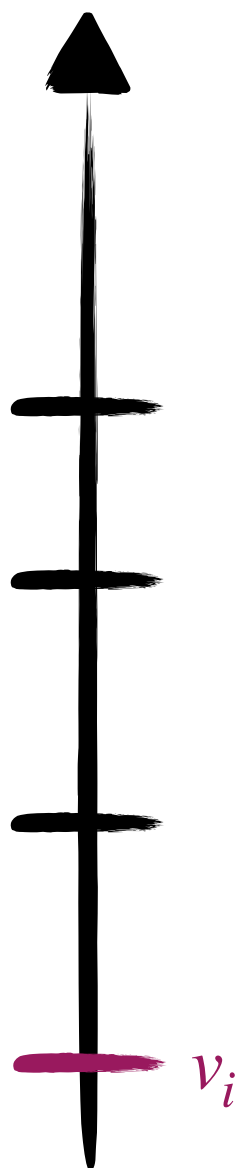
What is the critical value in the SPA with reserve? What is the payment?

Why is the FPA not truthful then?

Possible reason: The SCF (allocation) is not monotone. Is it?

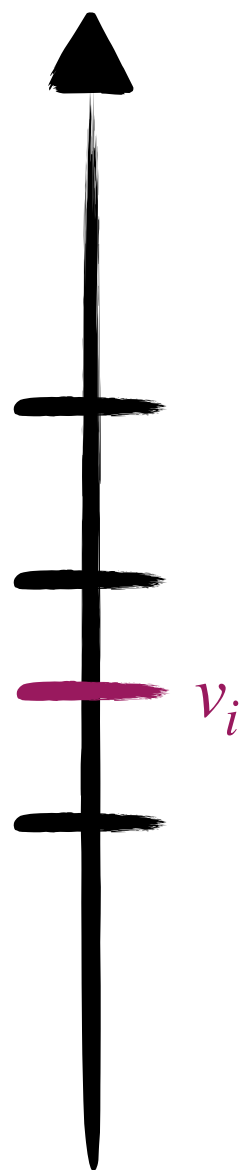
Possible reason: The payment is not the critical value.

Critical value



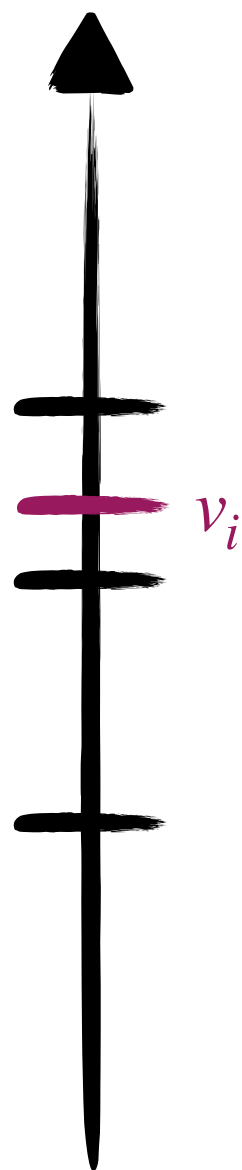
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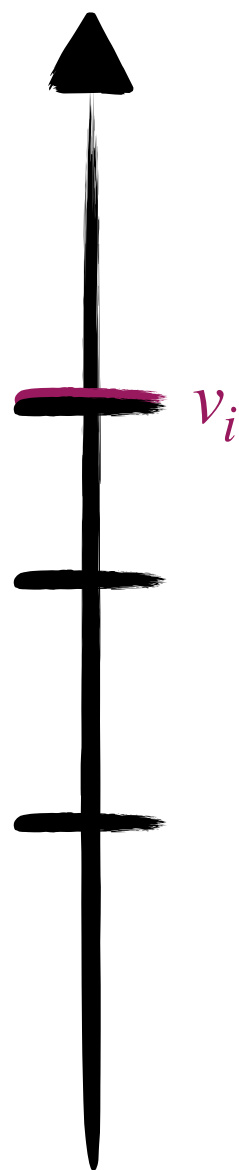
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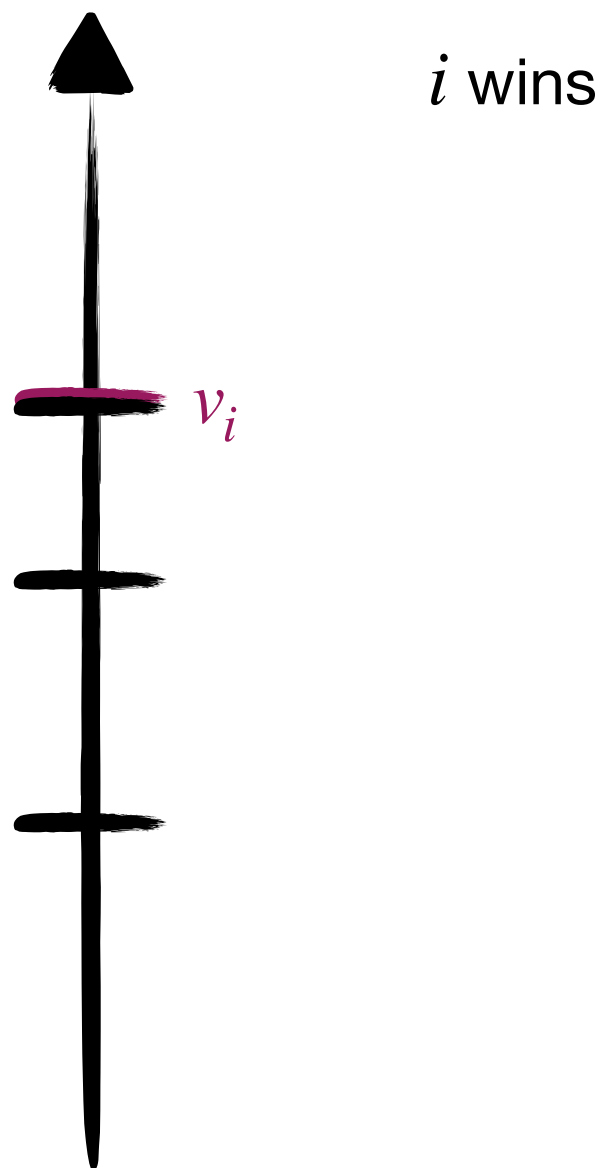
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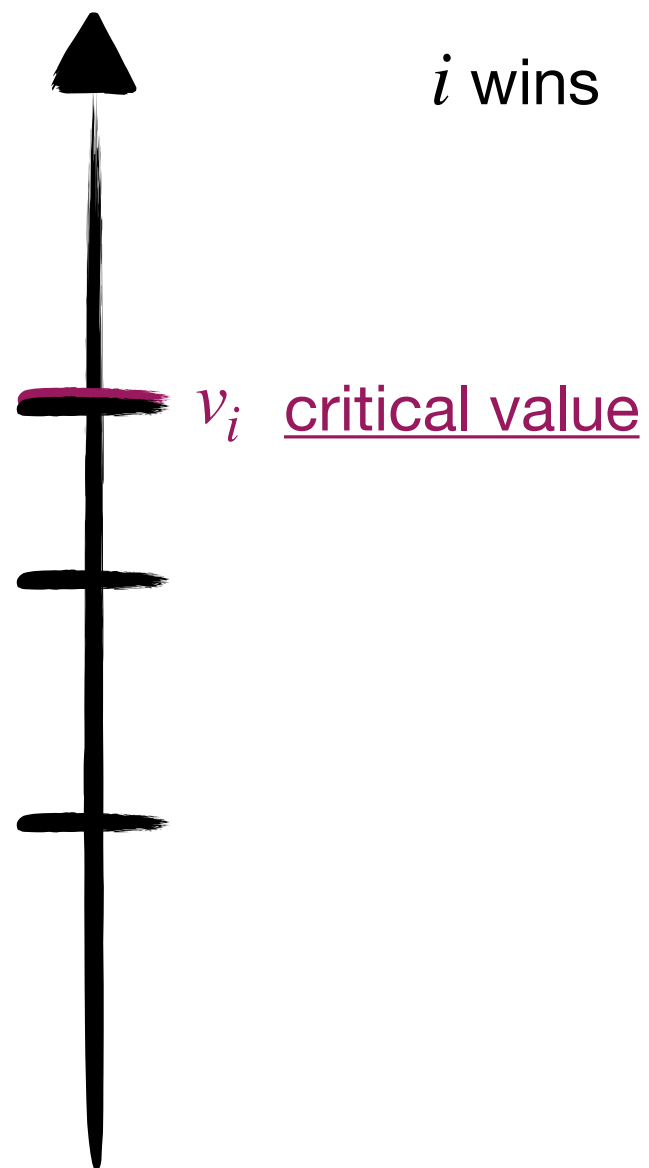


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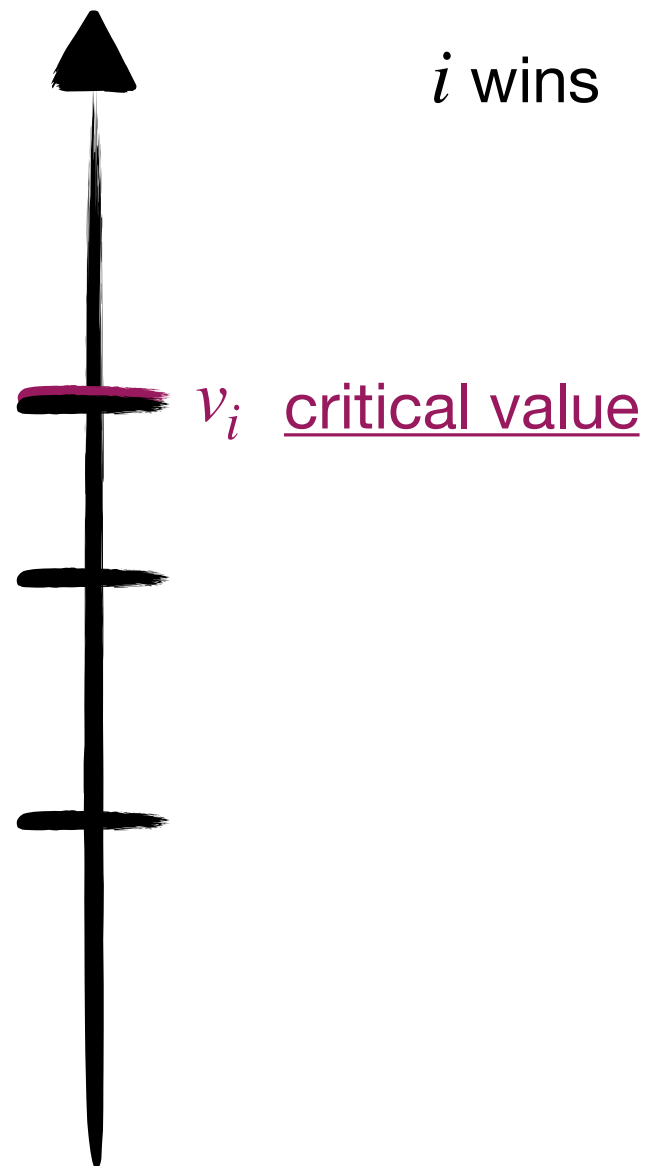
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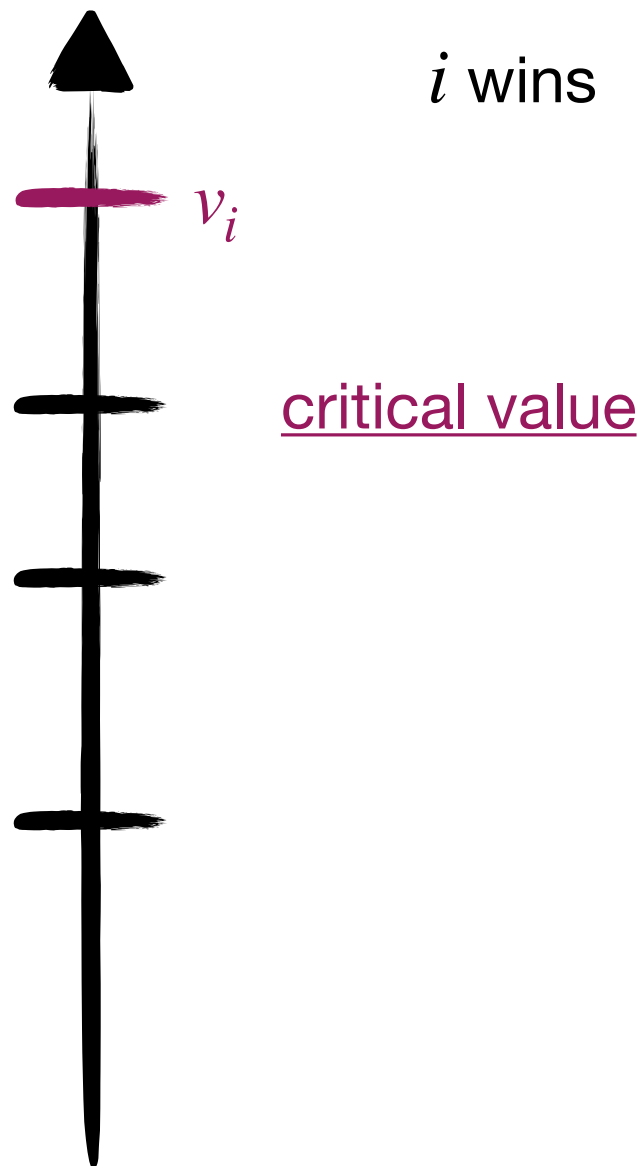


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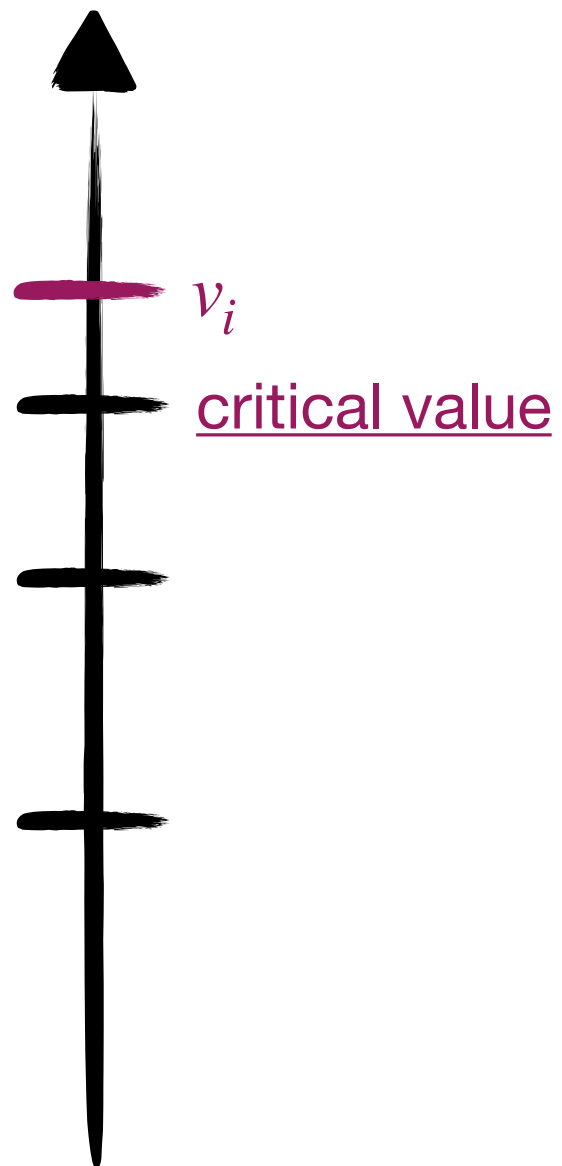
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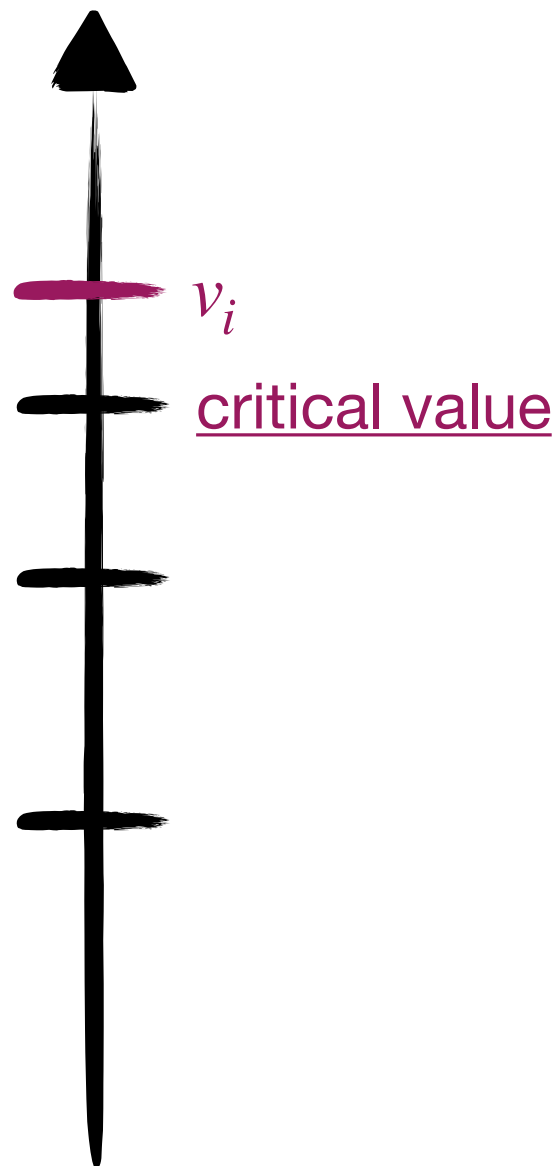
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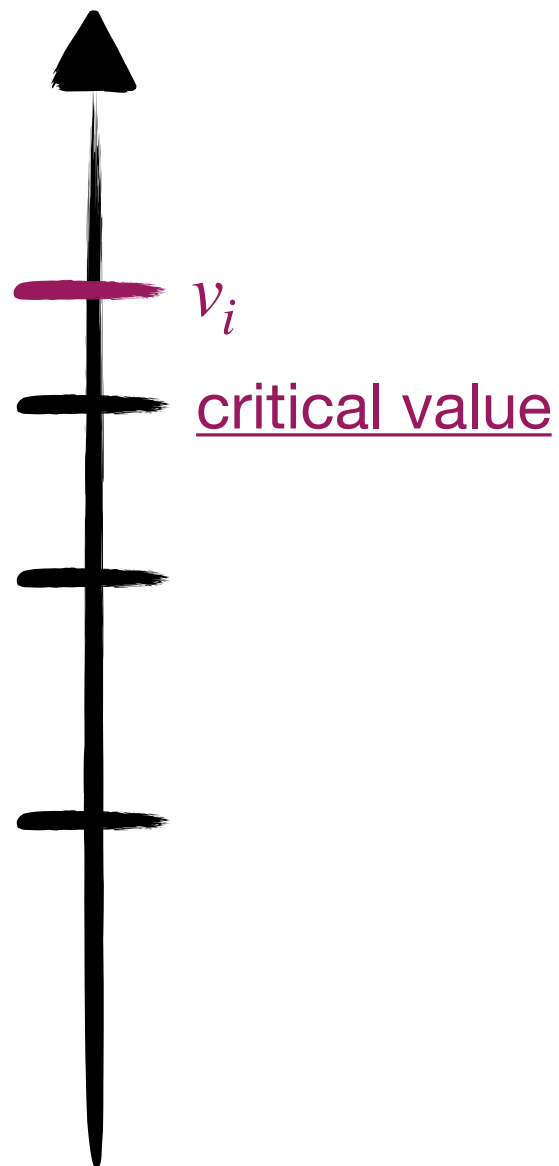


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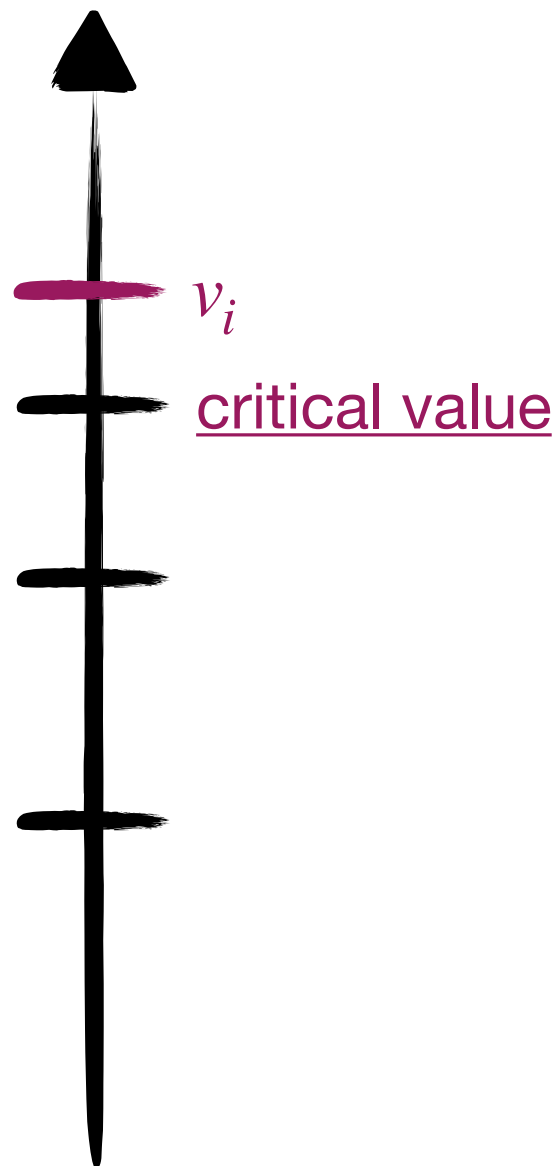
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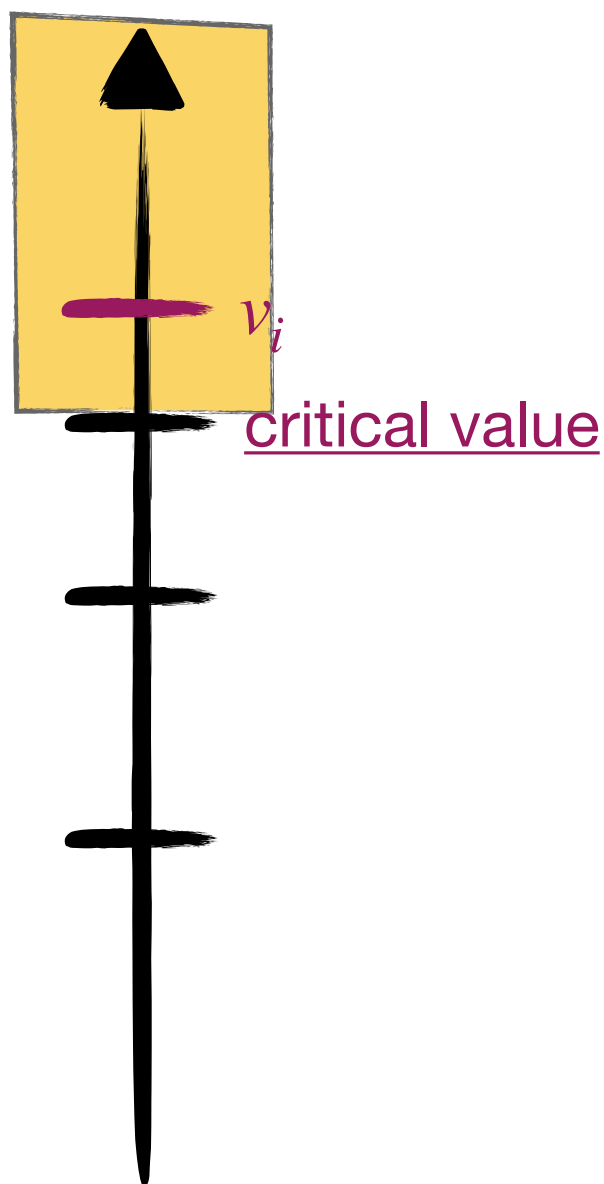


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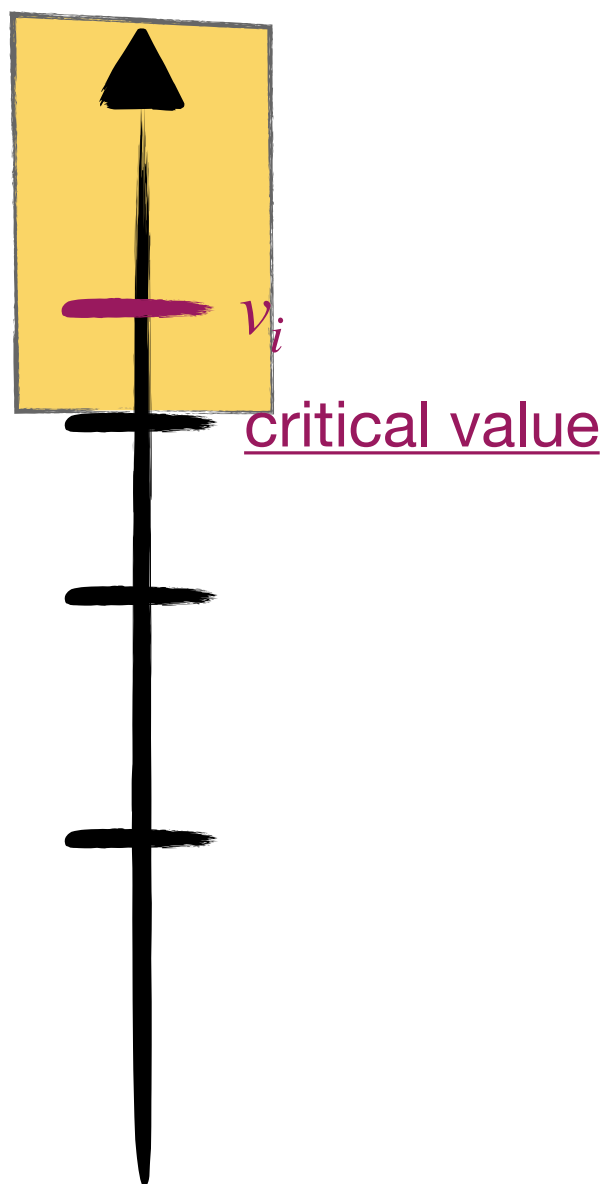


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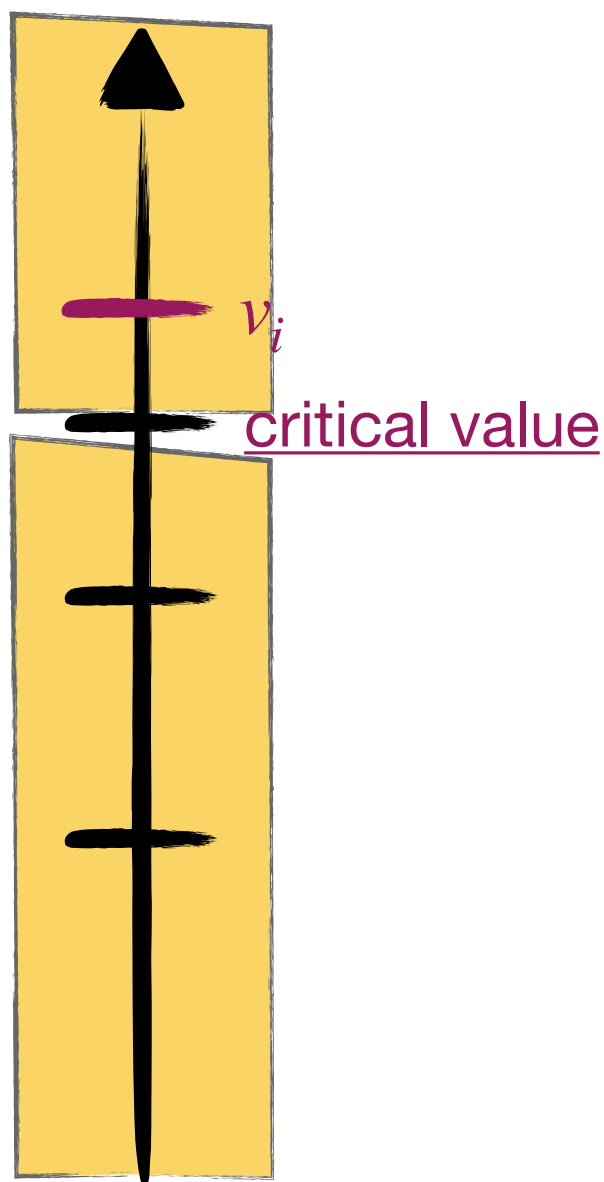
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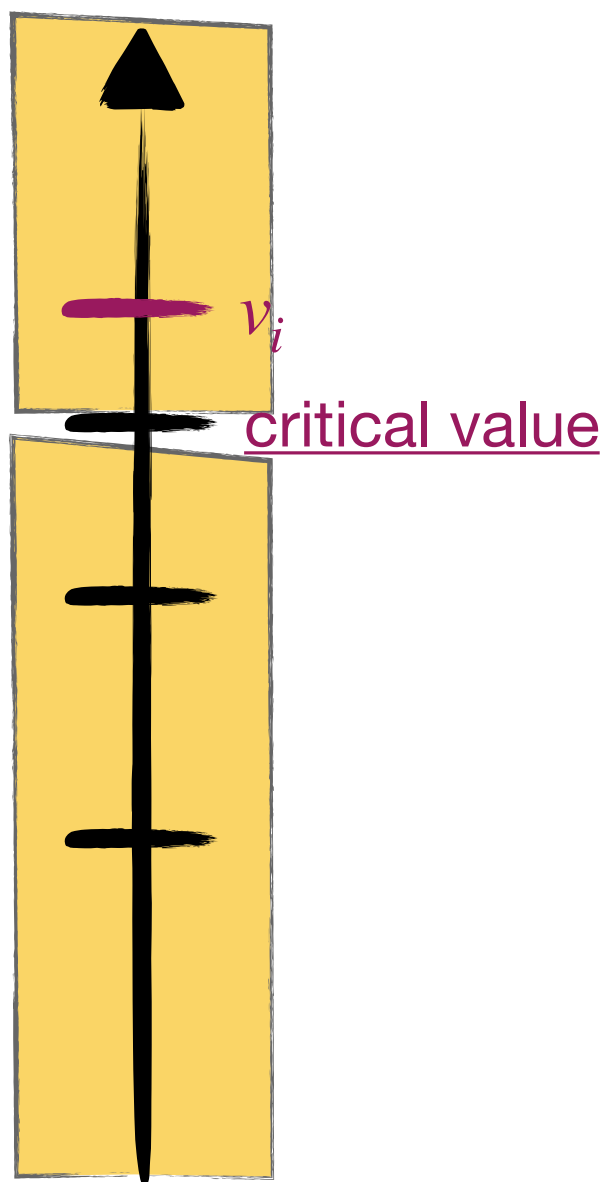
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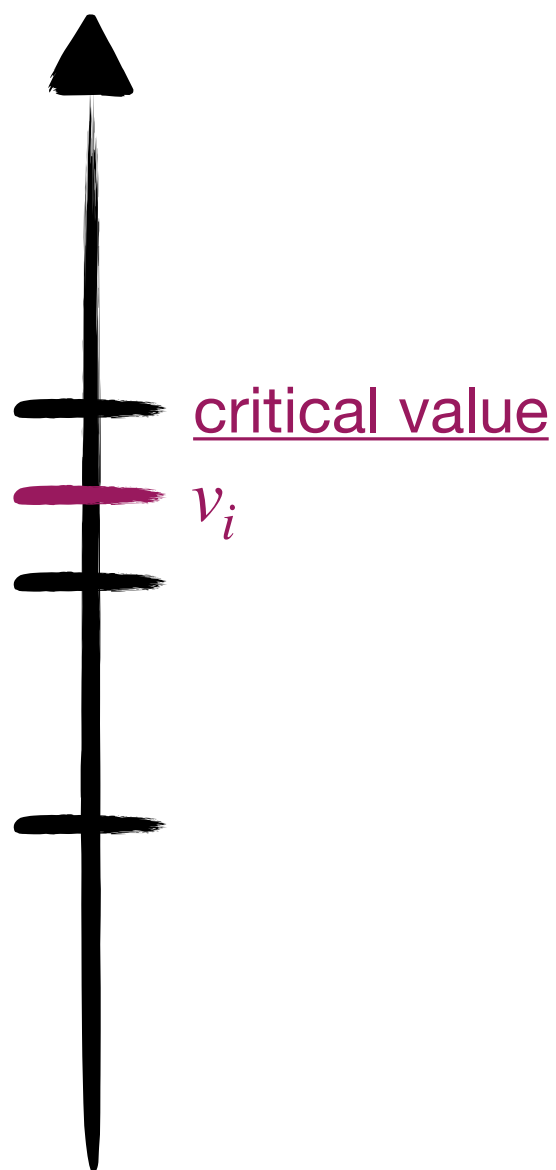
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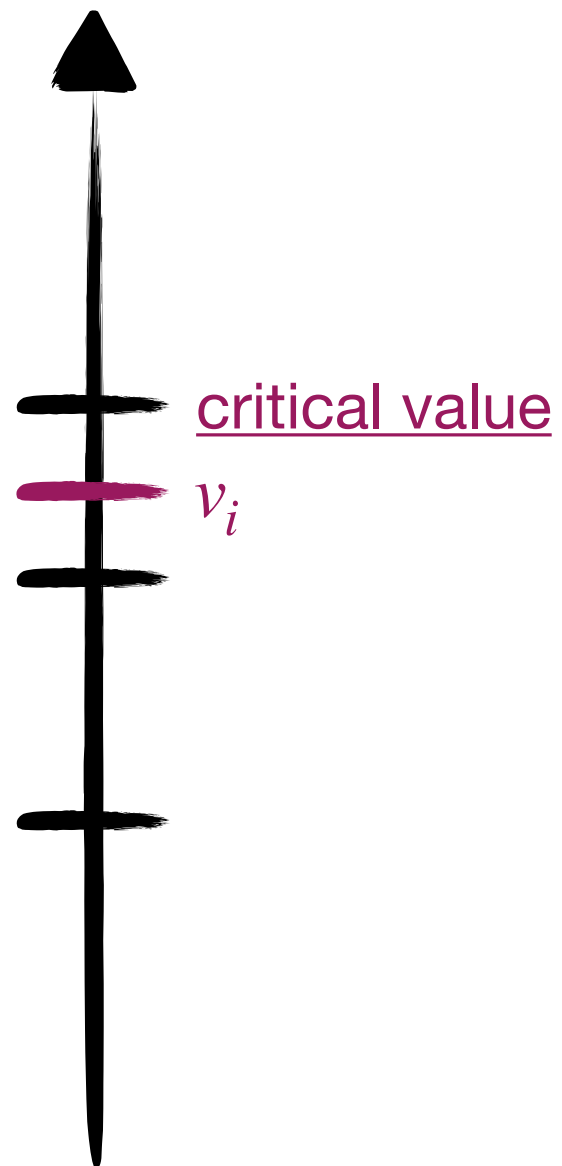
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May be losing, then the utility is 0.

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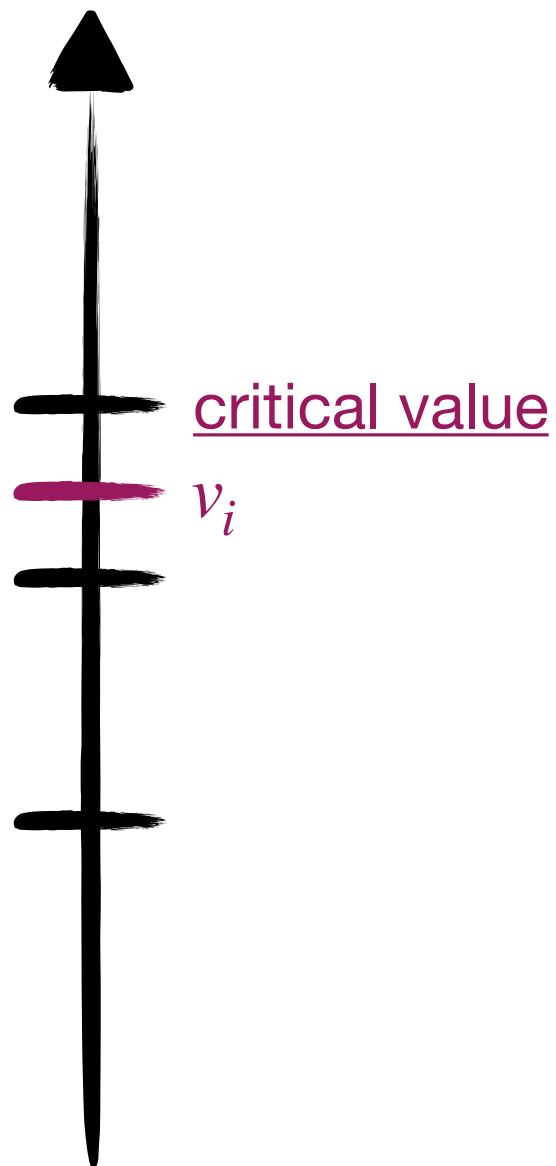


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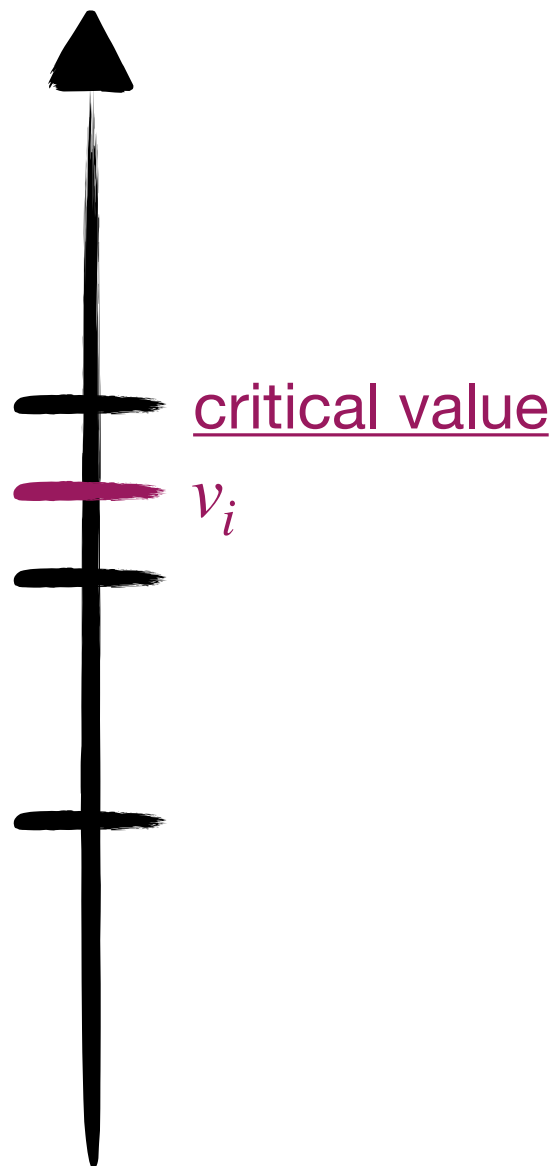
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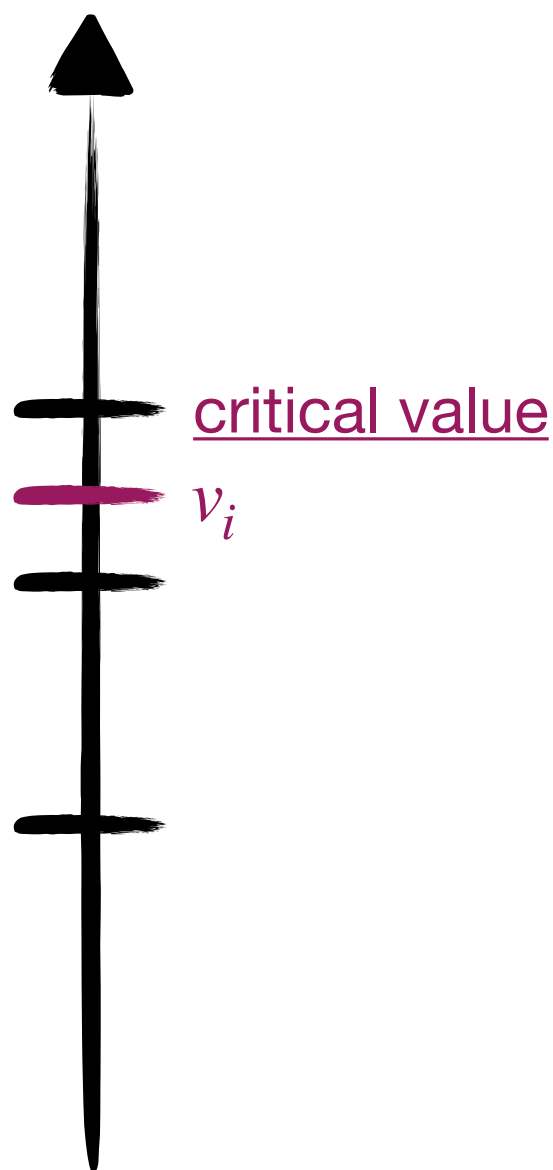


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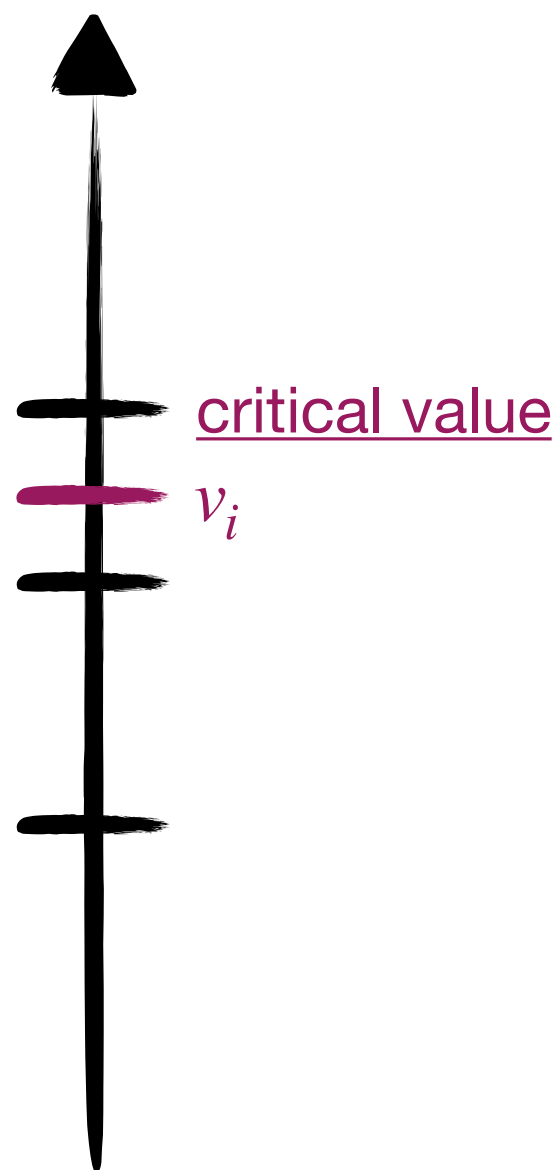
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Where was monotonicity used really?

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By truthfulness, $u_i(v'_i, p_i) = v'_i - p(v_i, v_{-i}) \leq 0$, as otherwise the agent with real value v'_i would have an incentive to misreport v_i , win, and get a positive utility.

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Claim: Fix any v_{-i} . Let v_i and v'_i be such that bidder i wins with both. Then $p_i(v_i, v_{-i}) = p_i(v'_i, v_{-i})$.

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By the Claim above, $p(v'_i, v_{-i}) = p(v_i, v_{-i})$.

An agent with true value v'_i now has negative utility, so it would prefer to bid 0 and lose, violating truthfulness.

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Now assume by contradiction that some winning agent pays

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An agent with true value v'_i has 0 utility, and it would prefer to bid v_i and win, gaining positive utility, and violating truthfulness.

Myerson's Characterisation

Theorem (Myerson's Characterisation or Myerson's Lemma, Myerson 1981): Let $(f, p_1, \dots, p_n)$ be a mechanism on a single-parameter domain, *for which losers pay 0*. Then, $(f, p_1, \dots, p_n)$ is truthful if and only if the following conditions hold:

- (1) Condition on the SCF (allocation): f is *monotone*.
- (2) Condition on the payments: The payment p_i of every winner is the *critical value*.

Formally, for every i , v_i , and v_{-i} such that $f(v_i, v_{-i}) \in W_i$, we have that $p_i = c_i(v_{-i})$.

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Myerson's characterisation can be generalised for these mechanisms as well!

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We will consider *normalised* mechanisms in which the lowest v_i has 0 probability of winning, i.e., $w_i(v_i^\ell, v_{-i}) = 0$ for $v_i^\ell = \min_i v_i$ and incurs 0 payment, i.e., $p_i(v_i^\ell, v_{-i}) = 0$.

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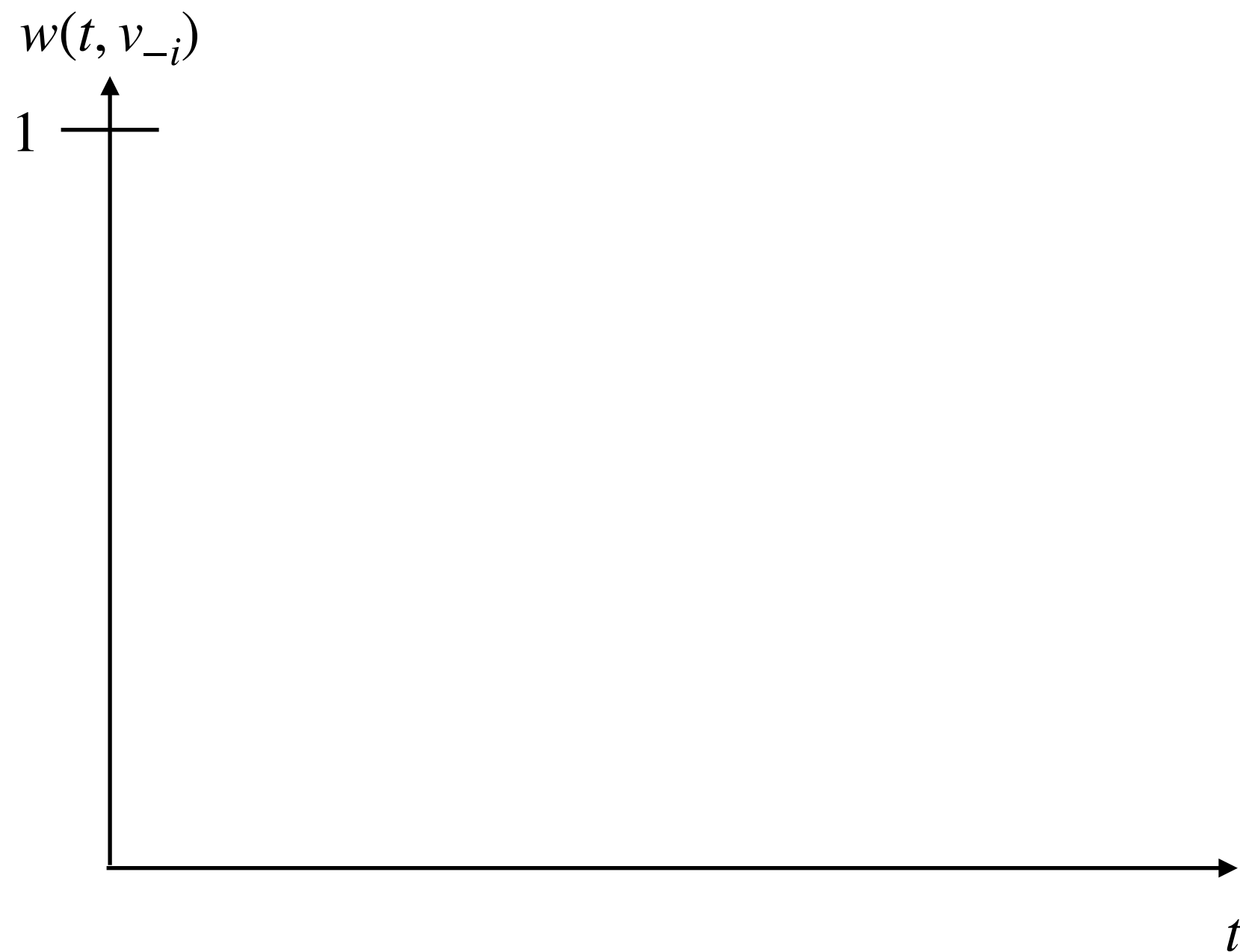
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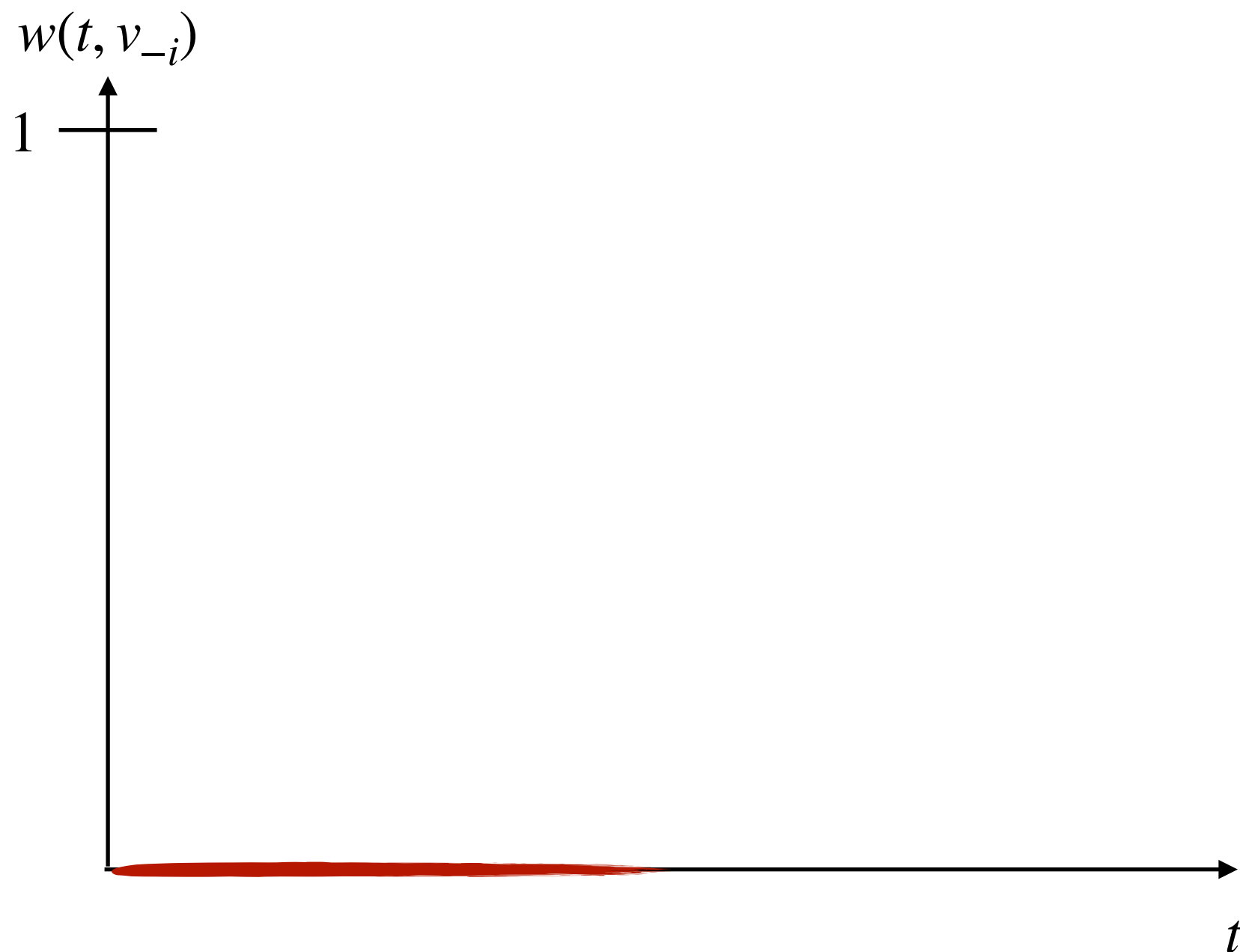
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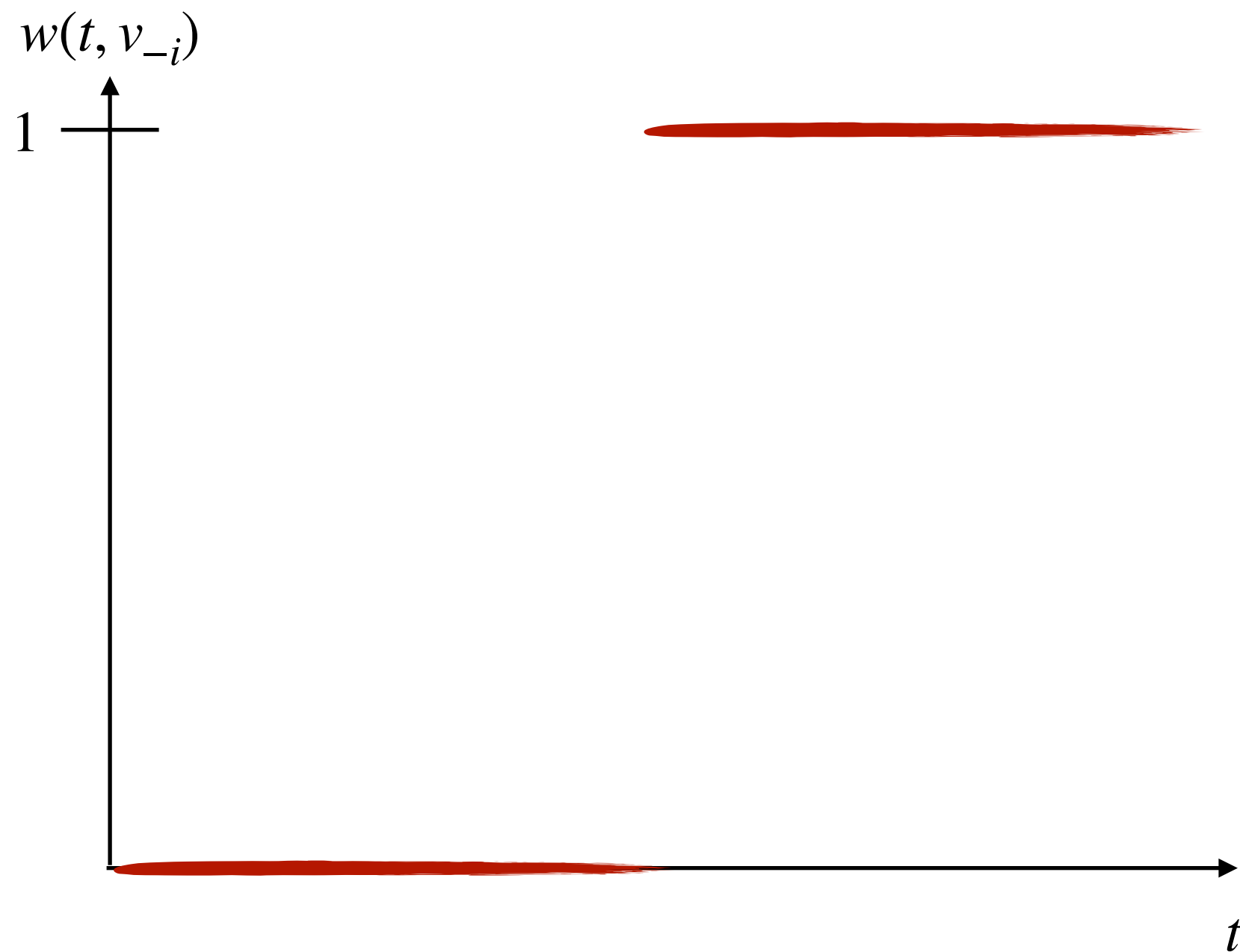
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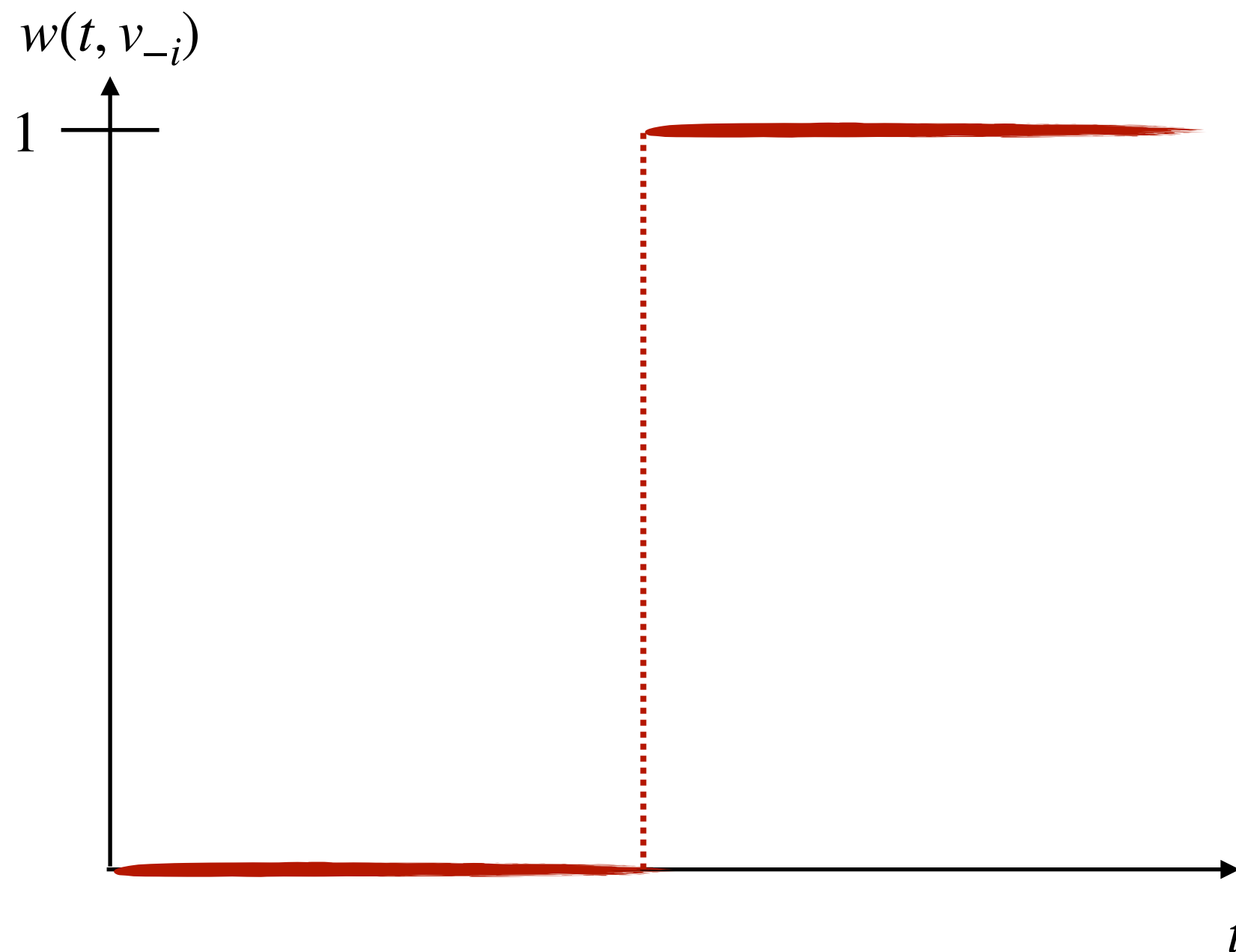
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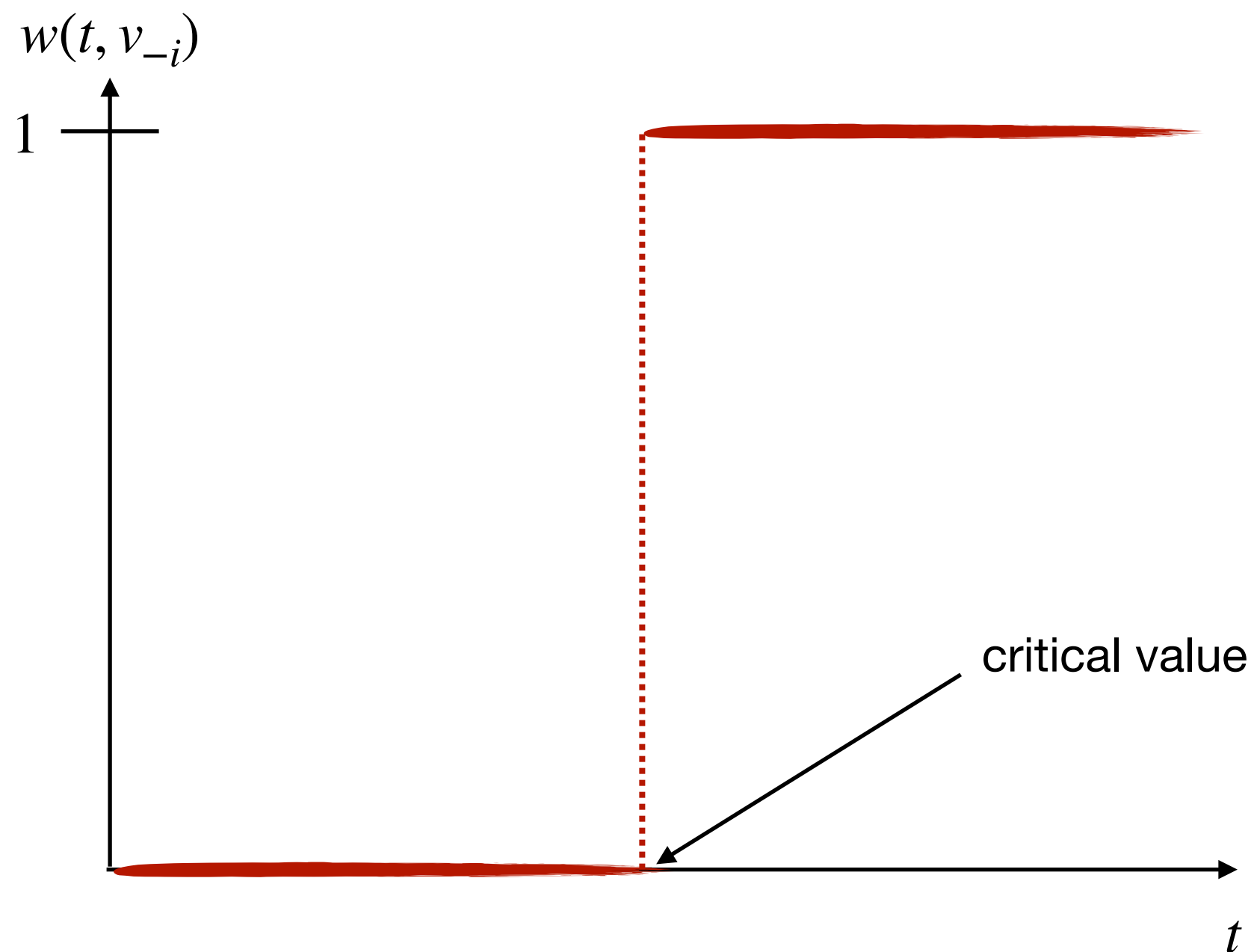
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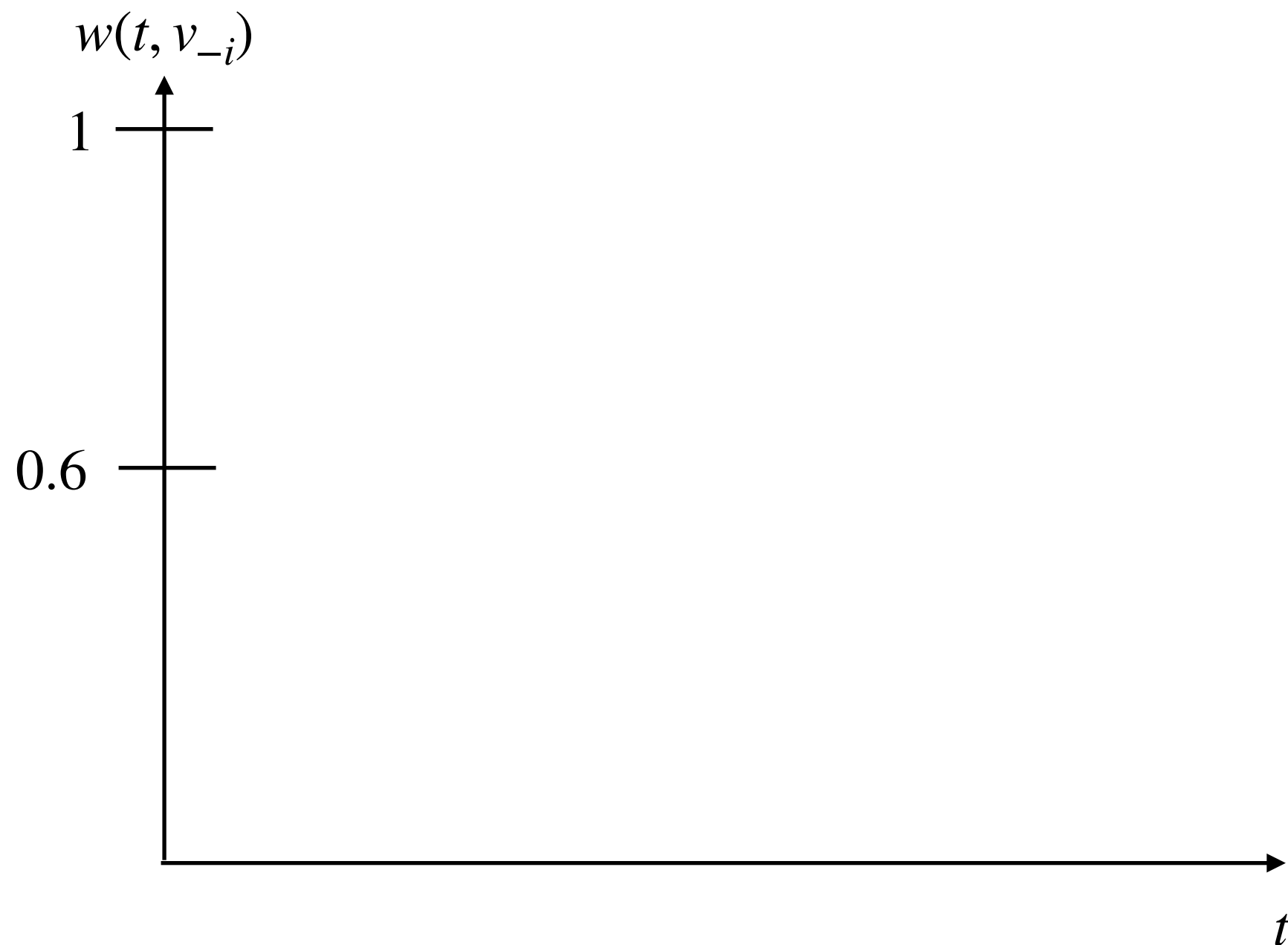
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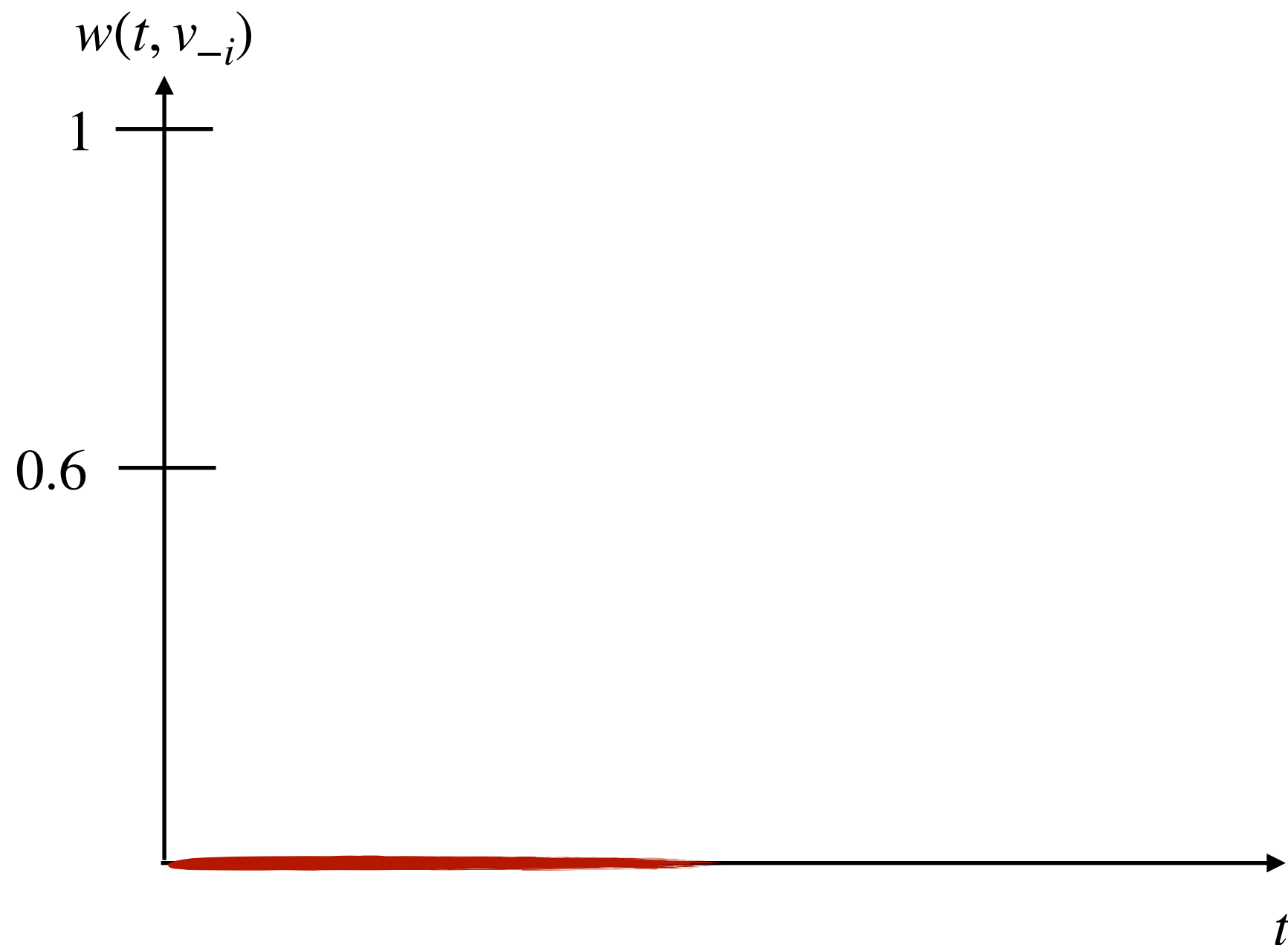
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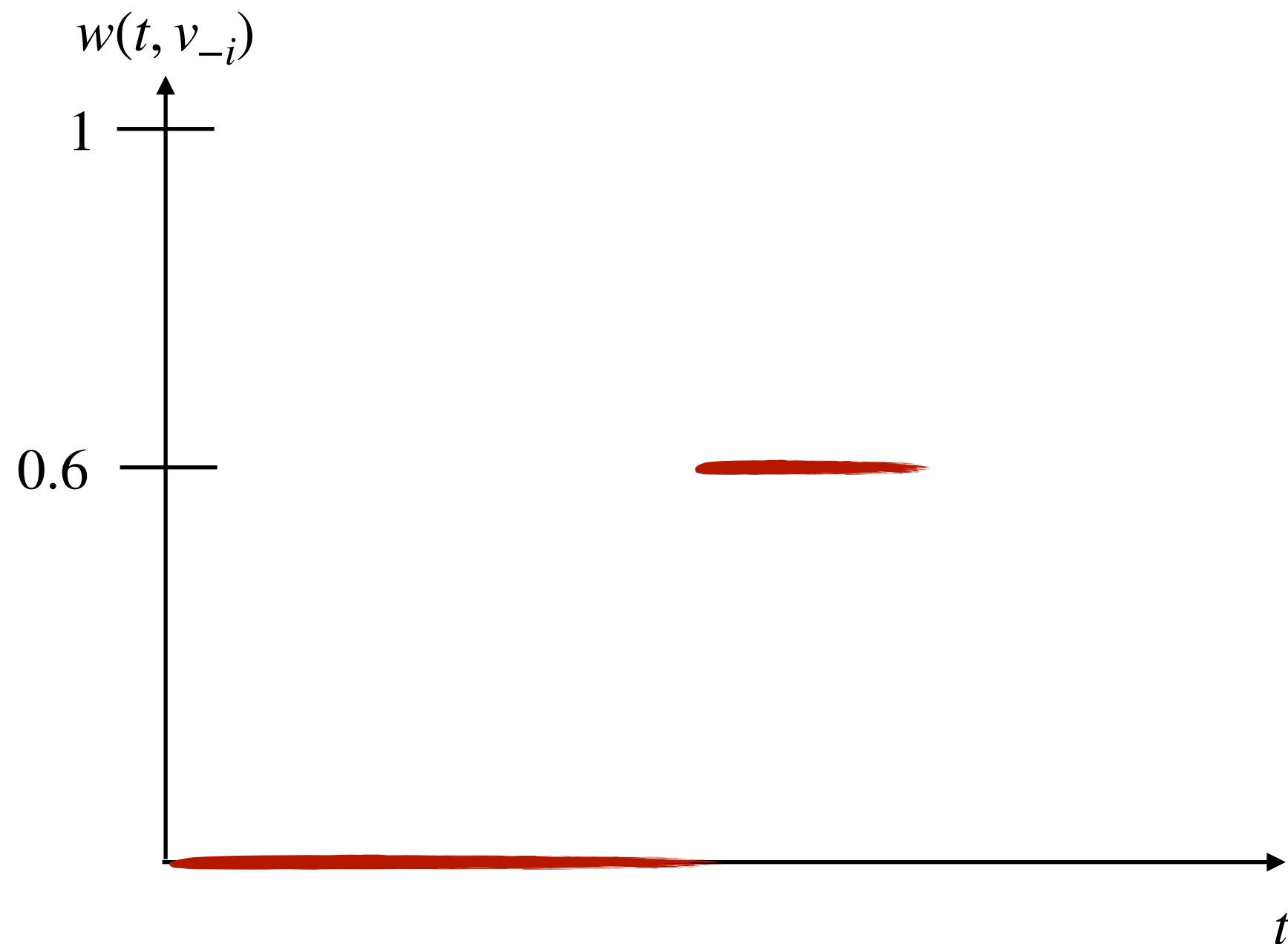
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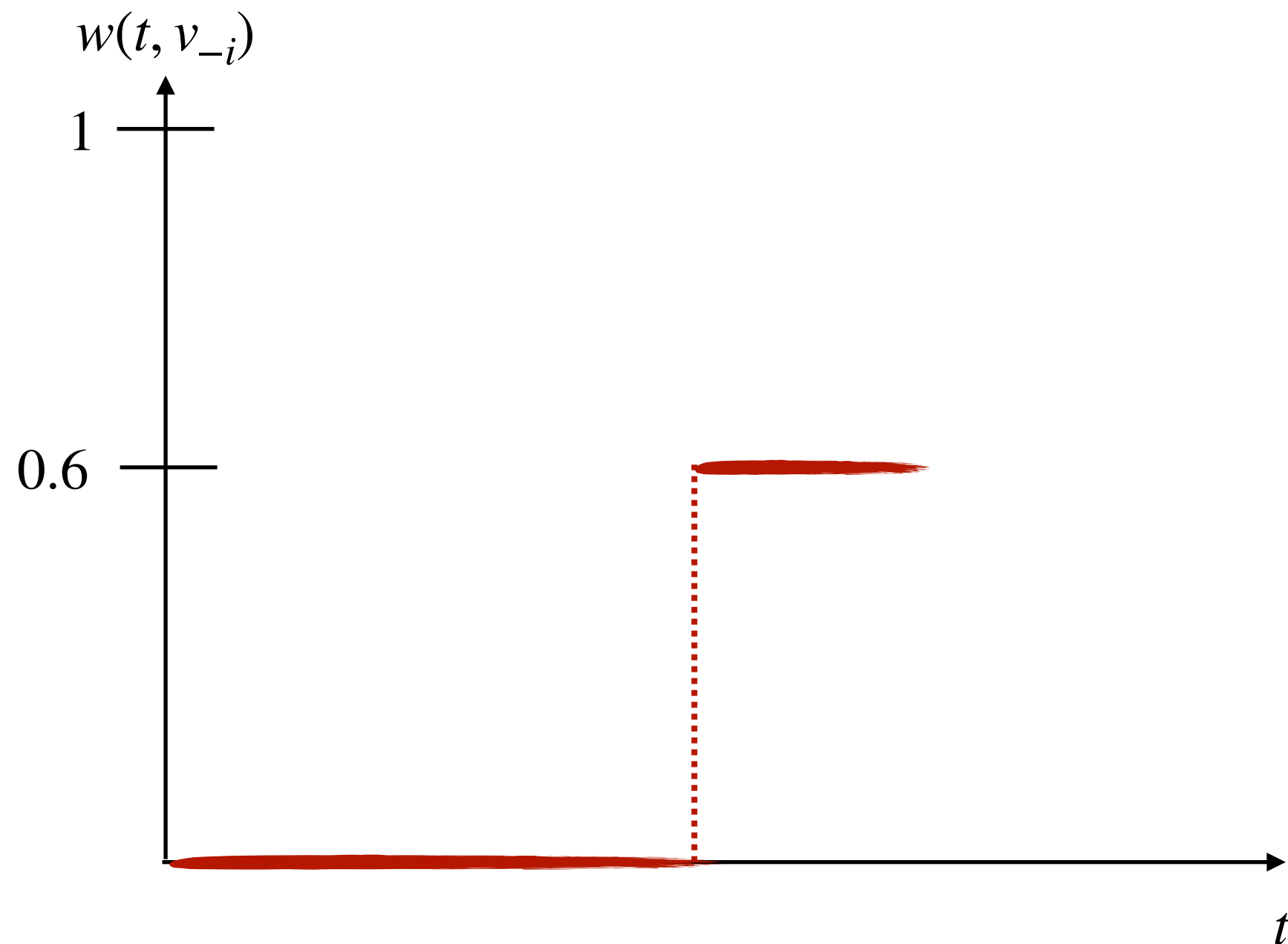
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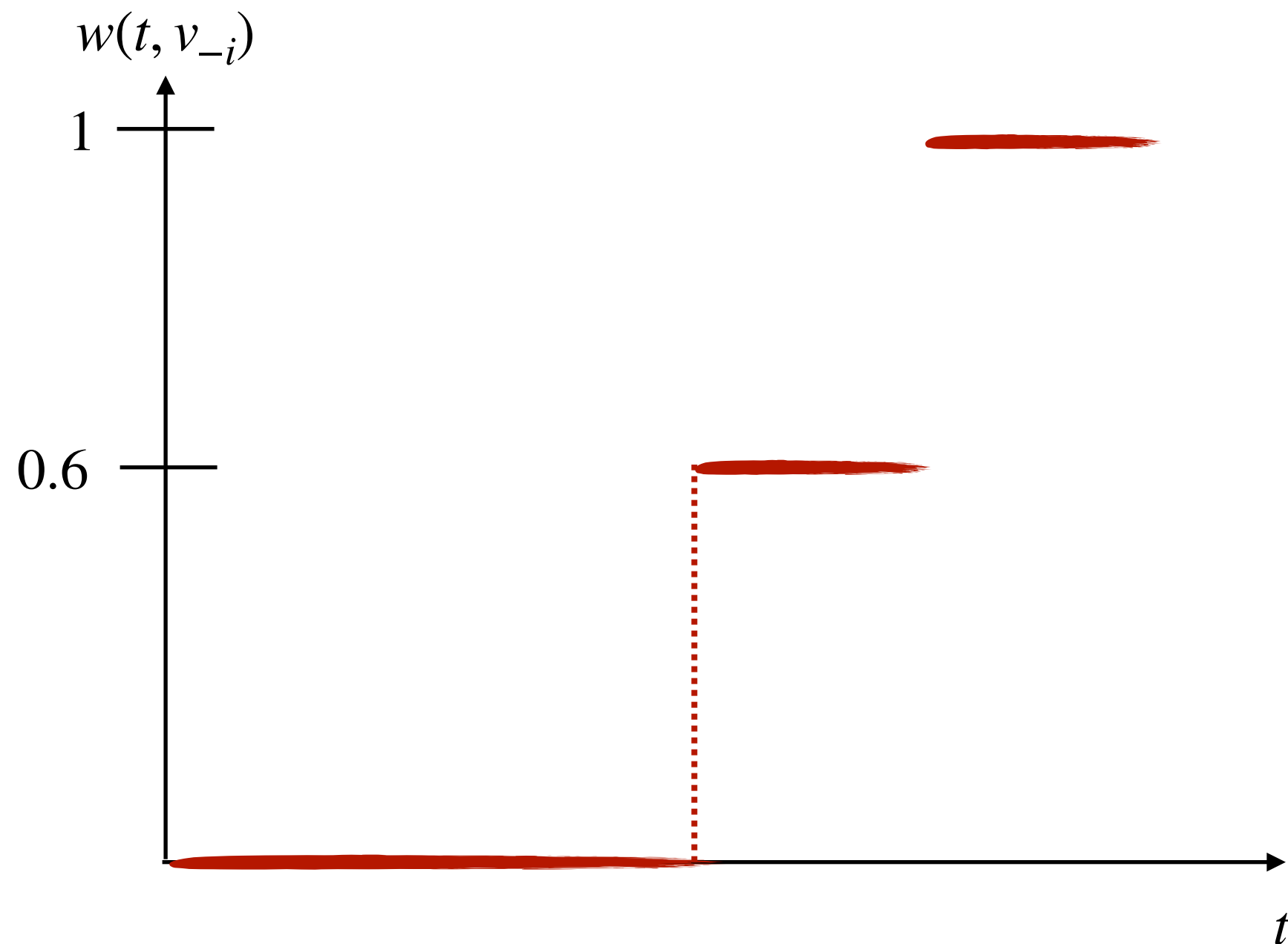
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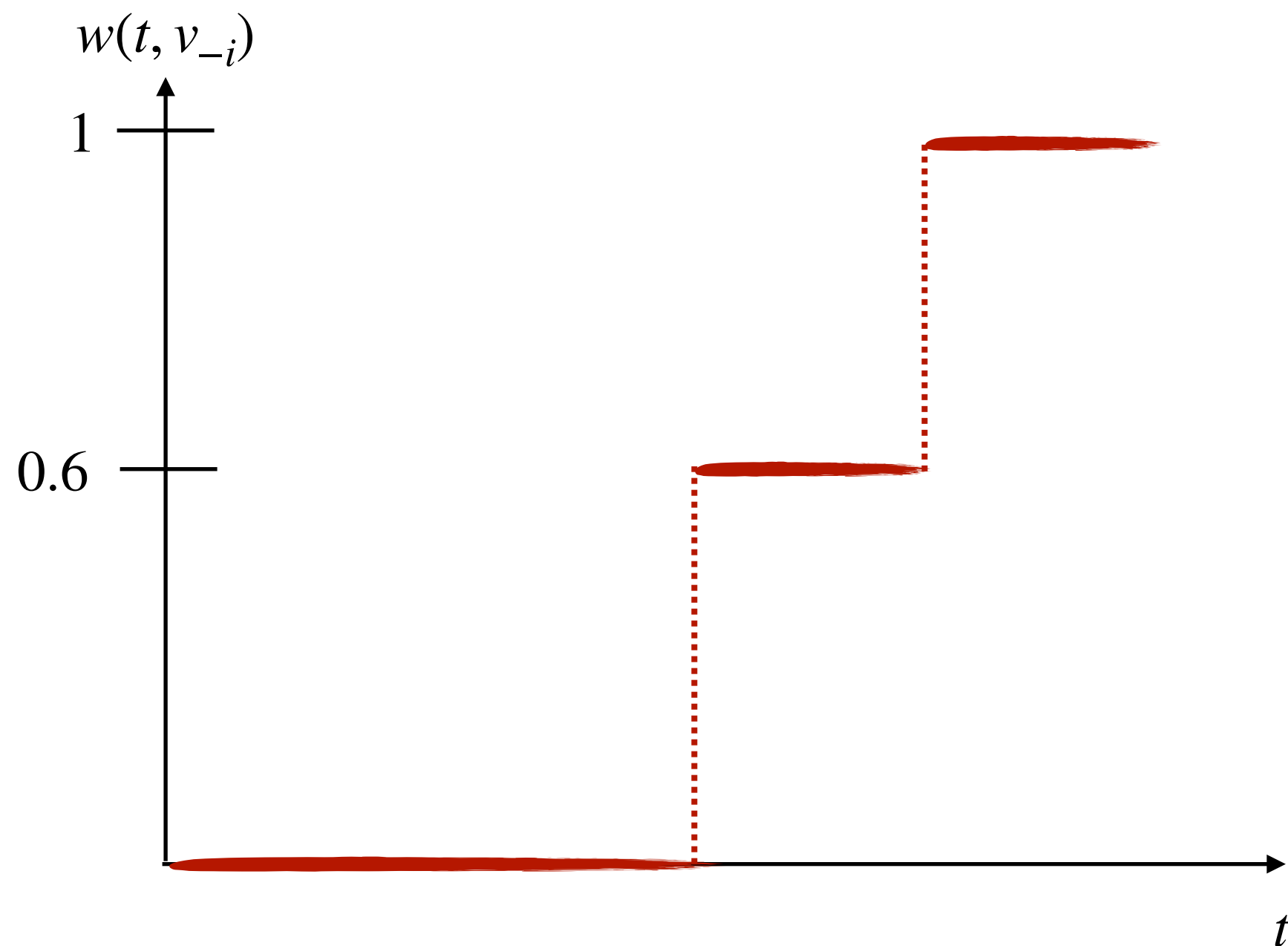
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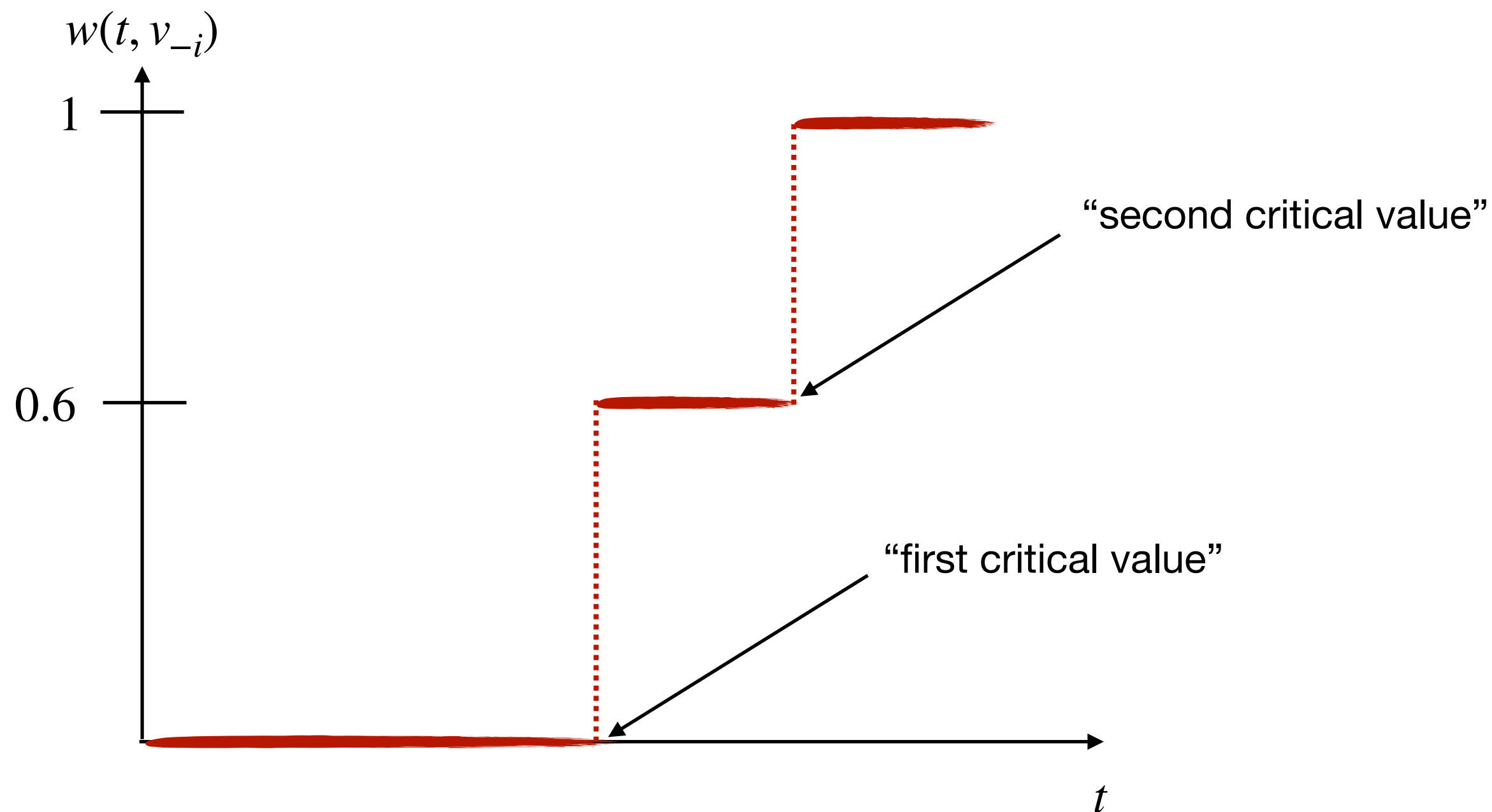
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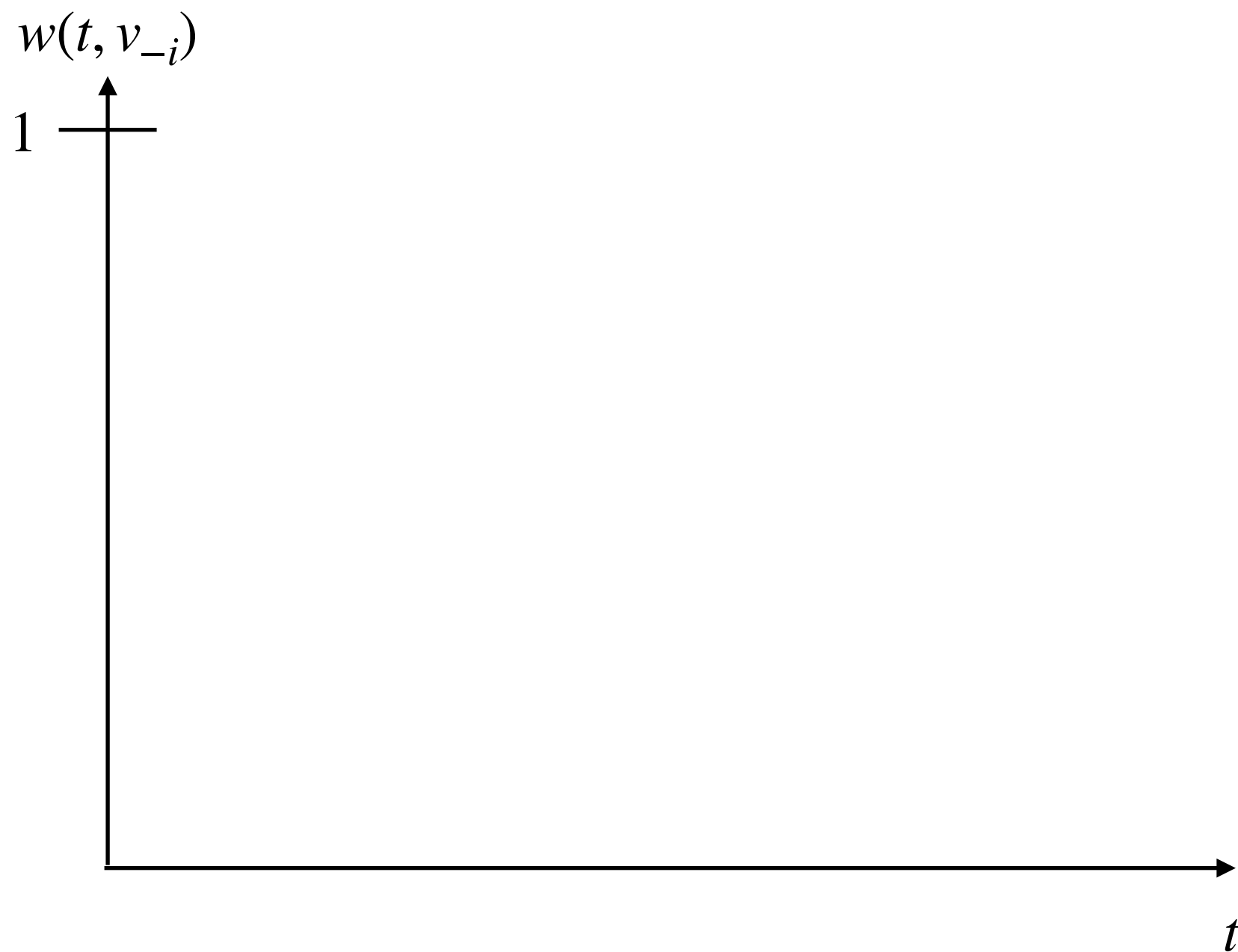
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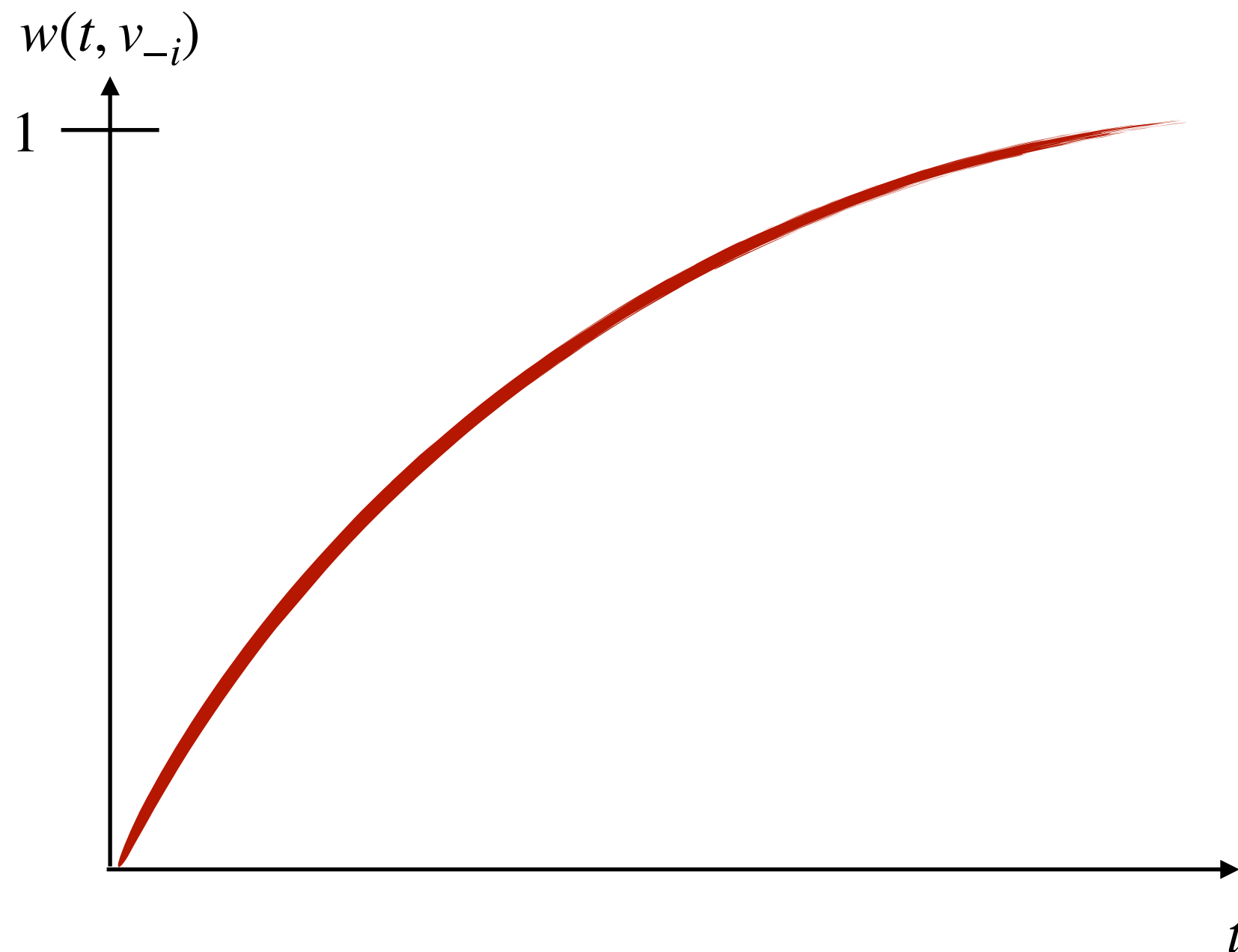
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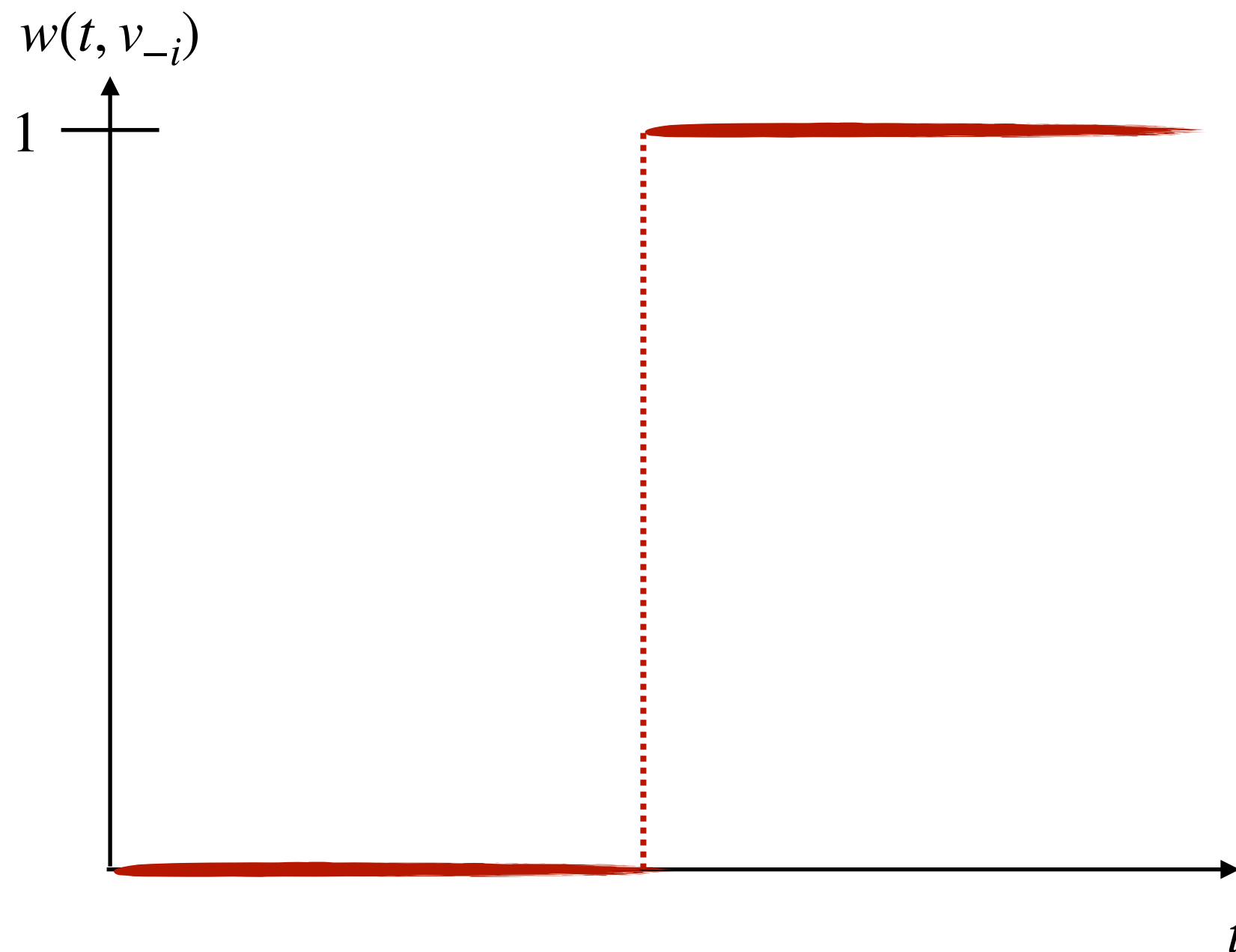
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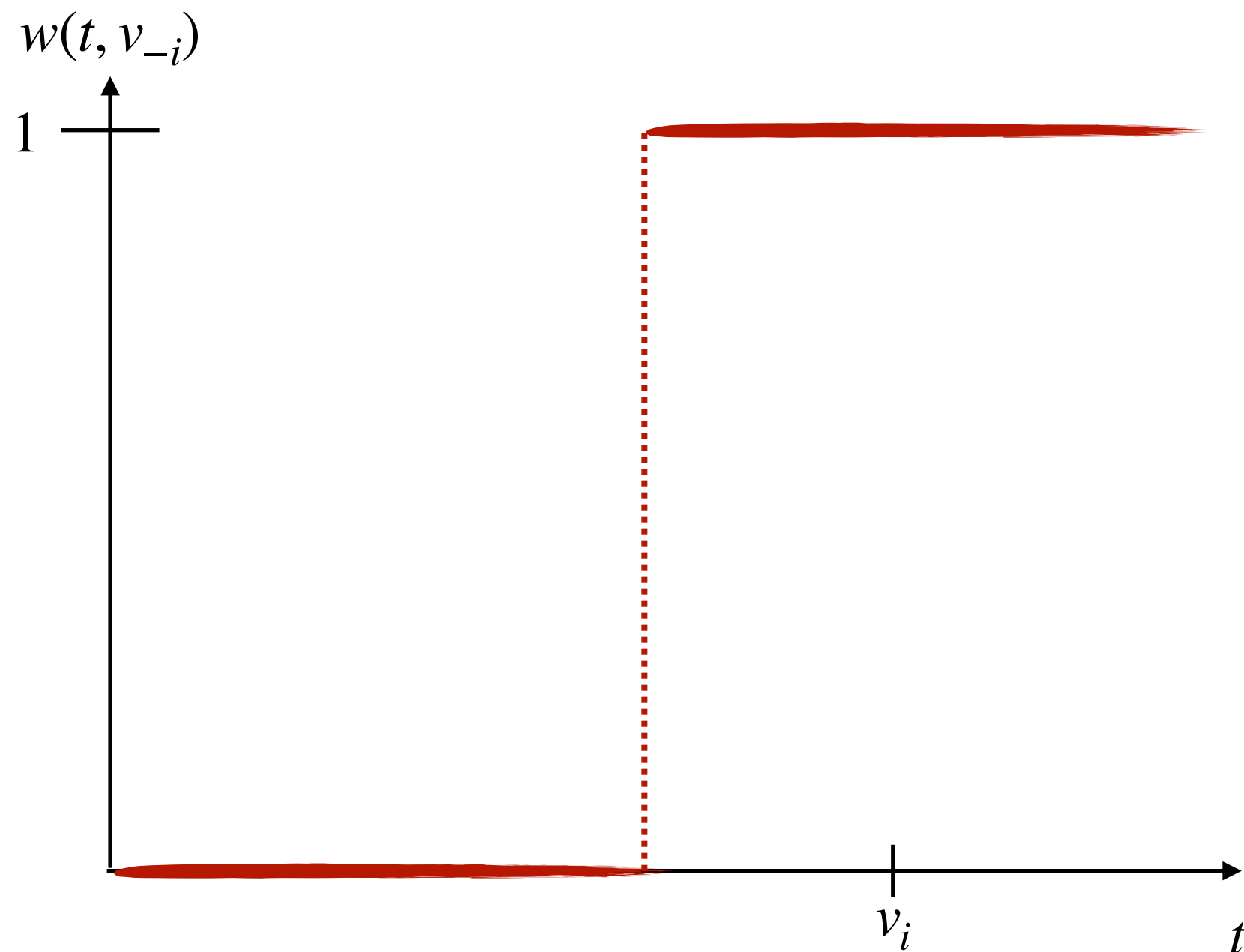
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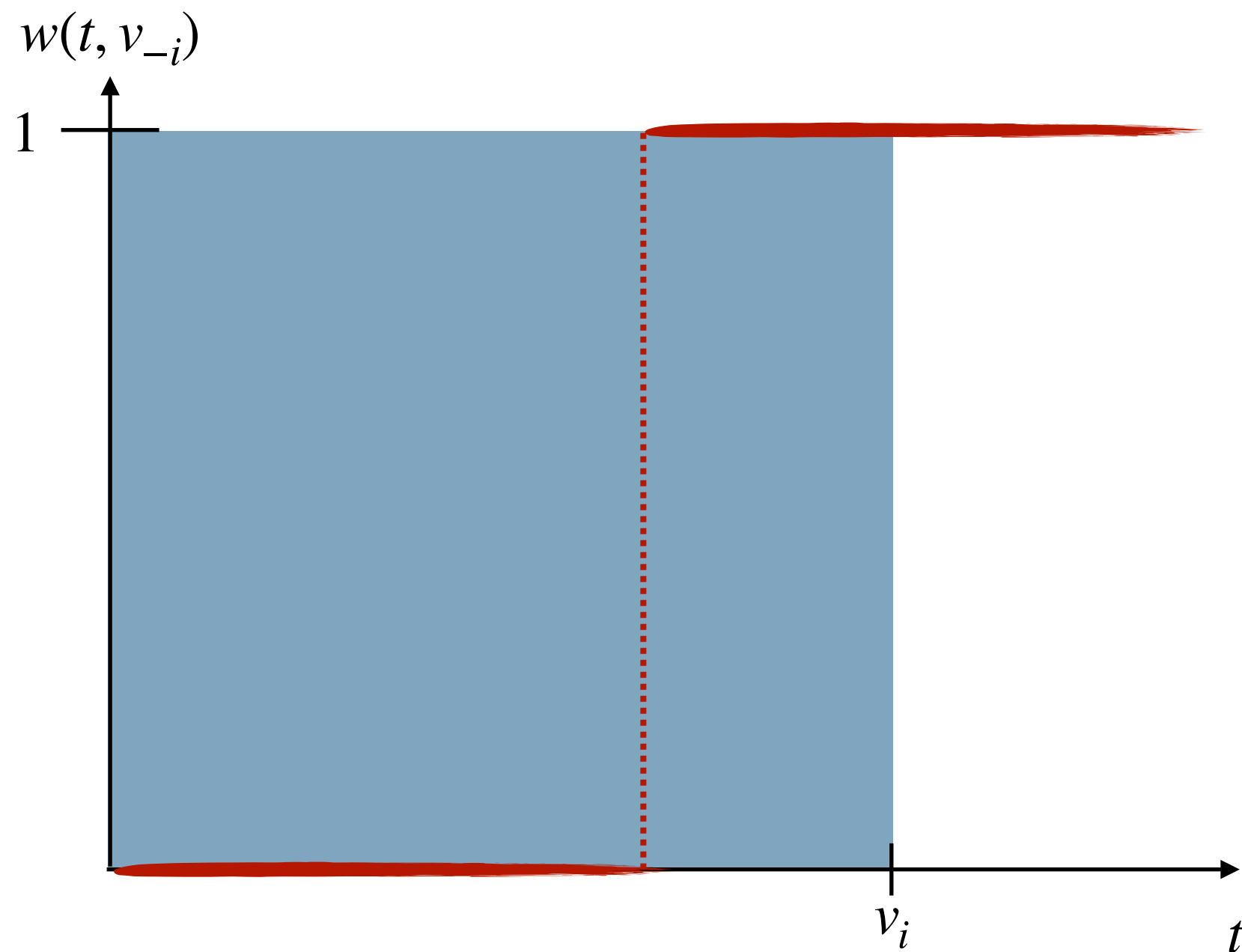
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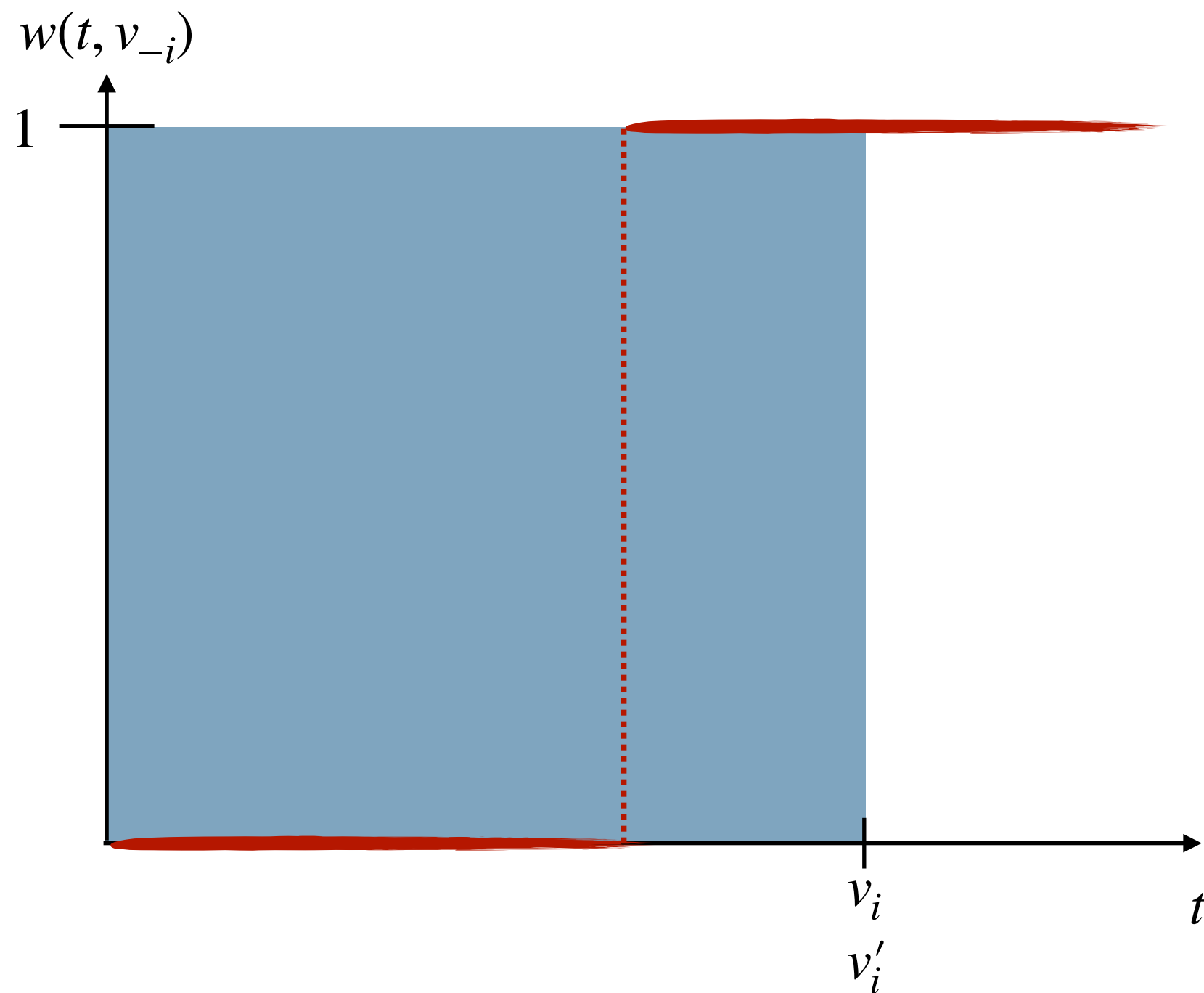
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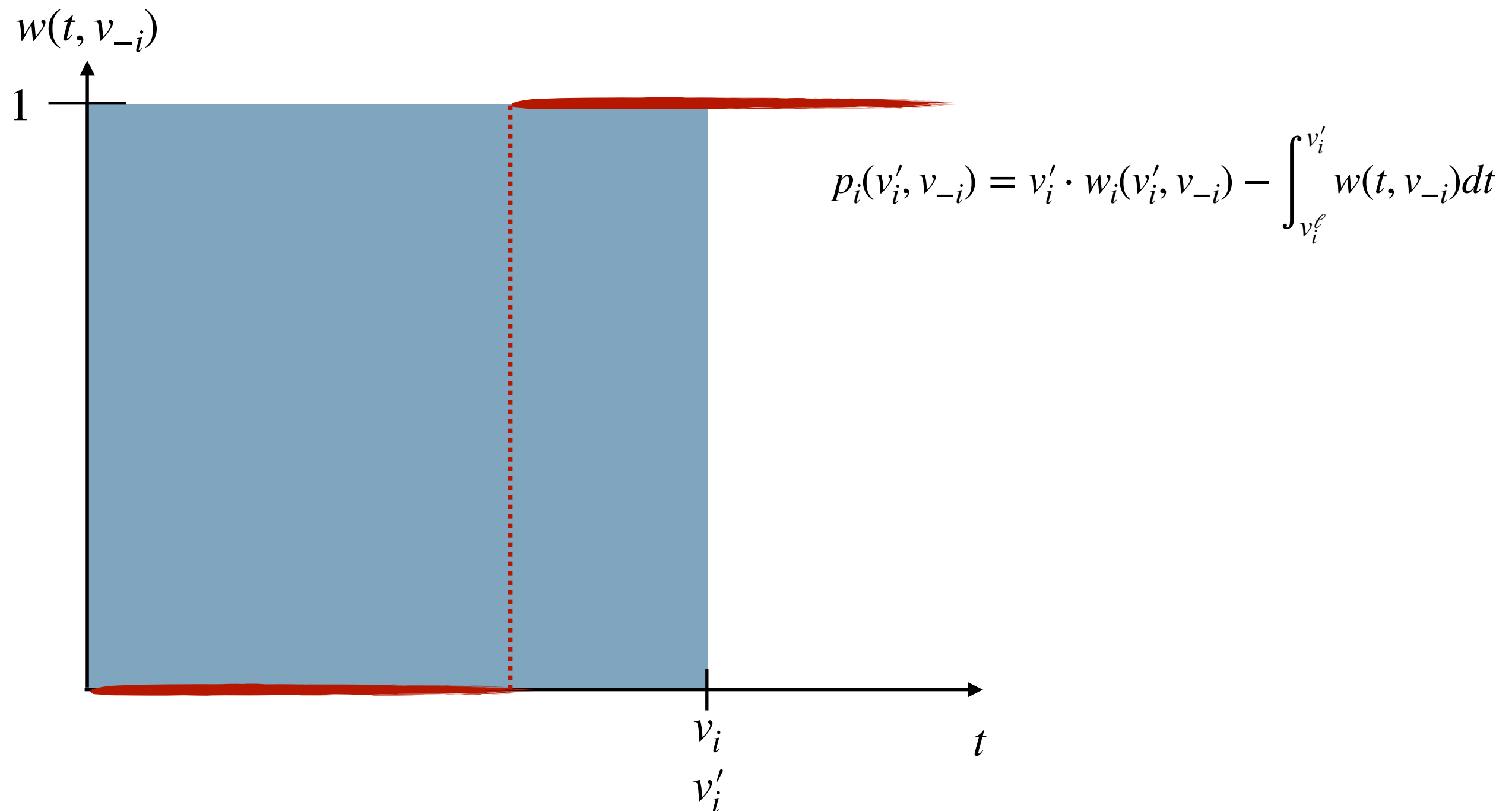
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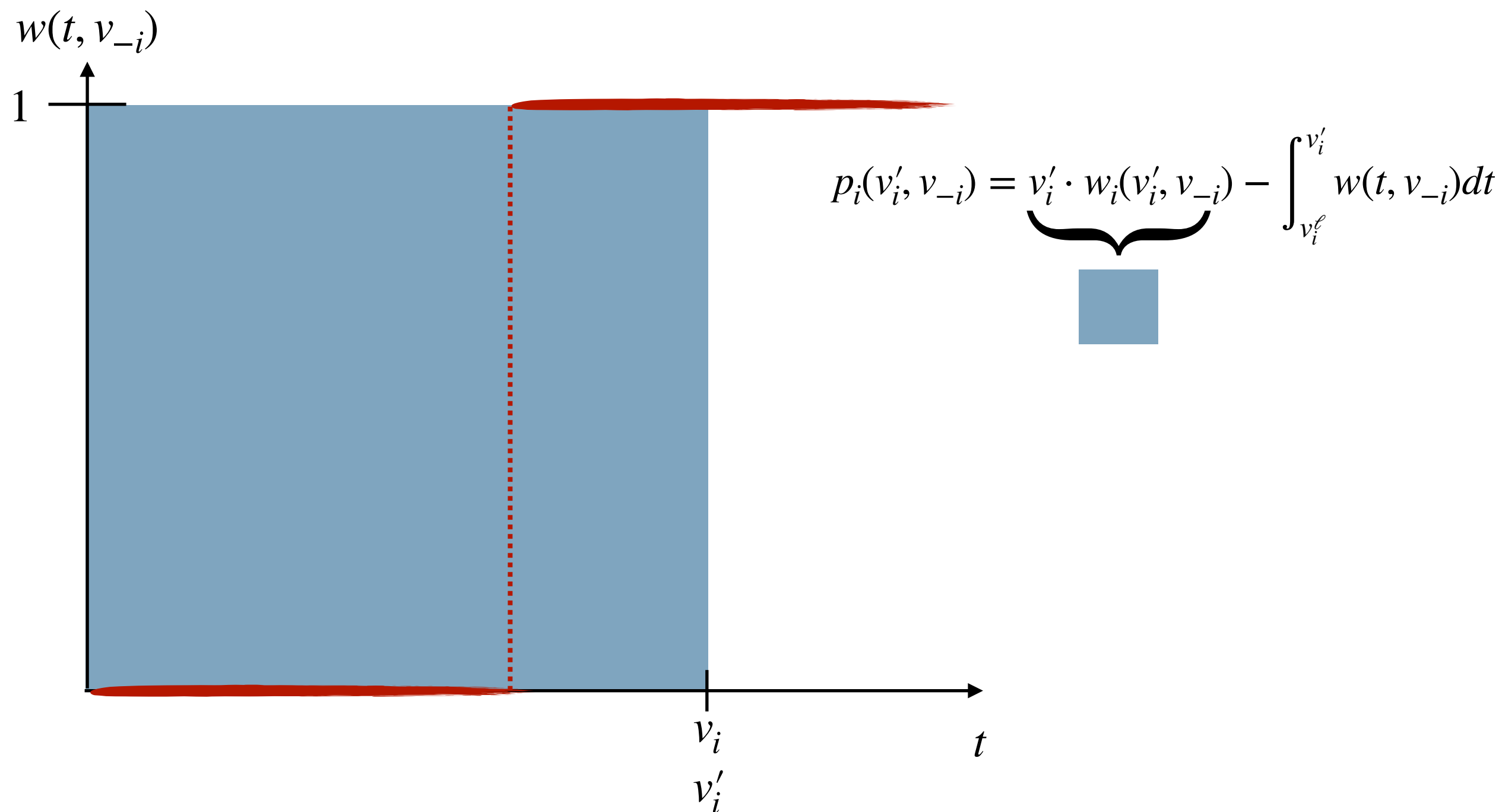
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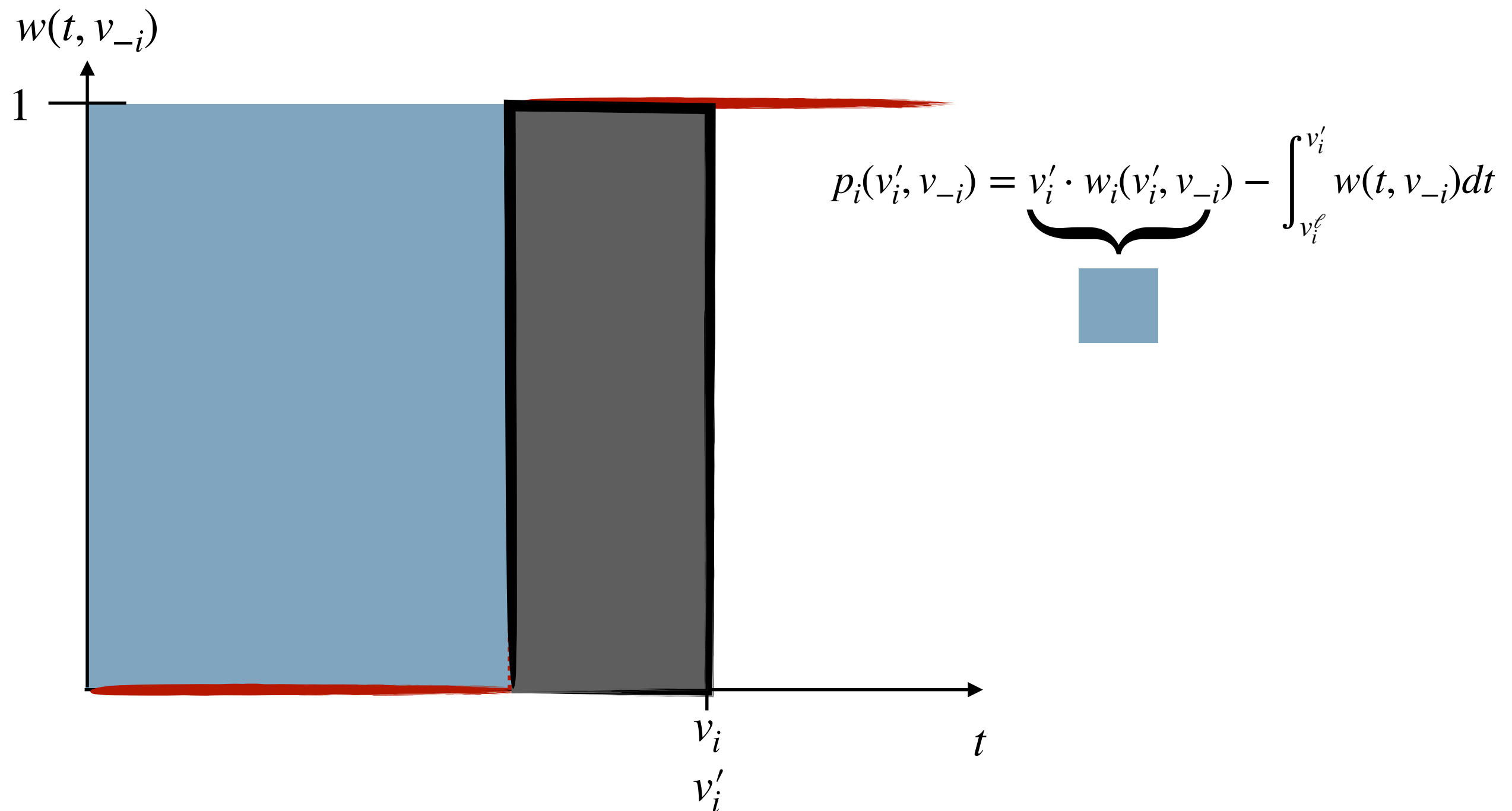
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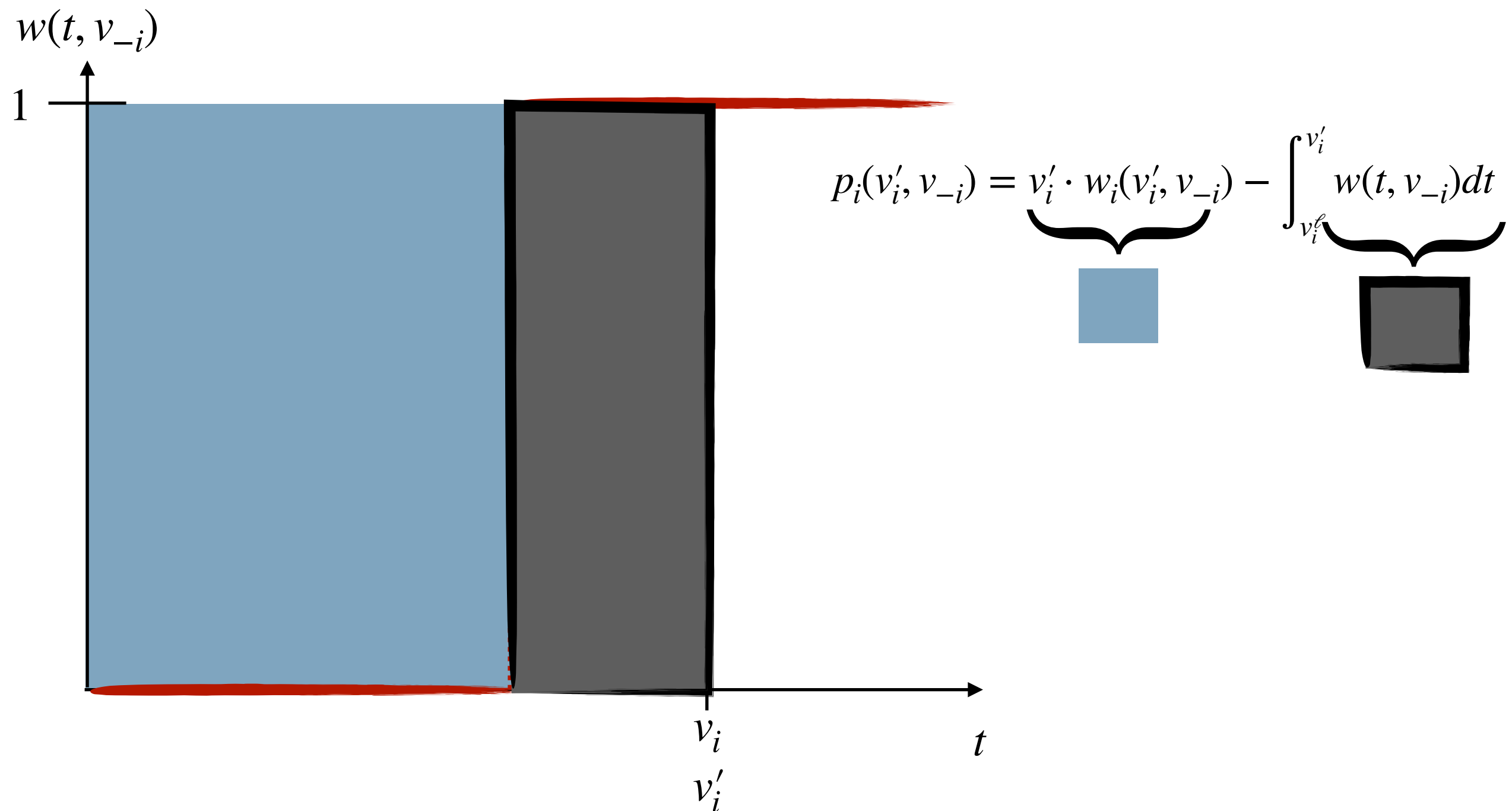
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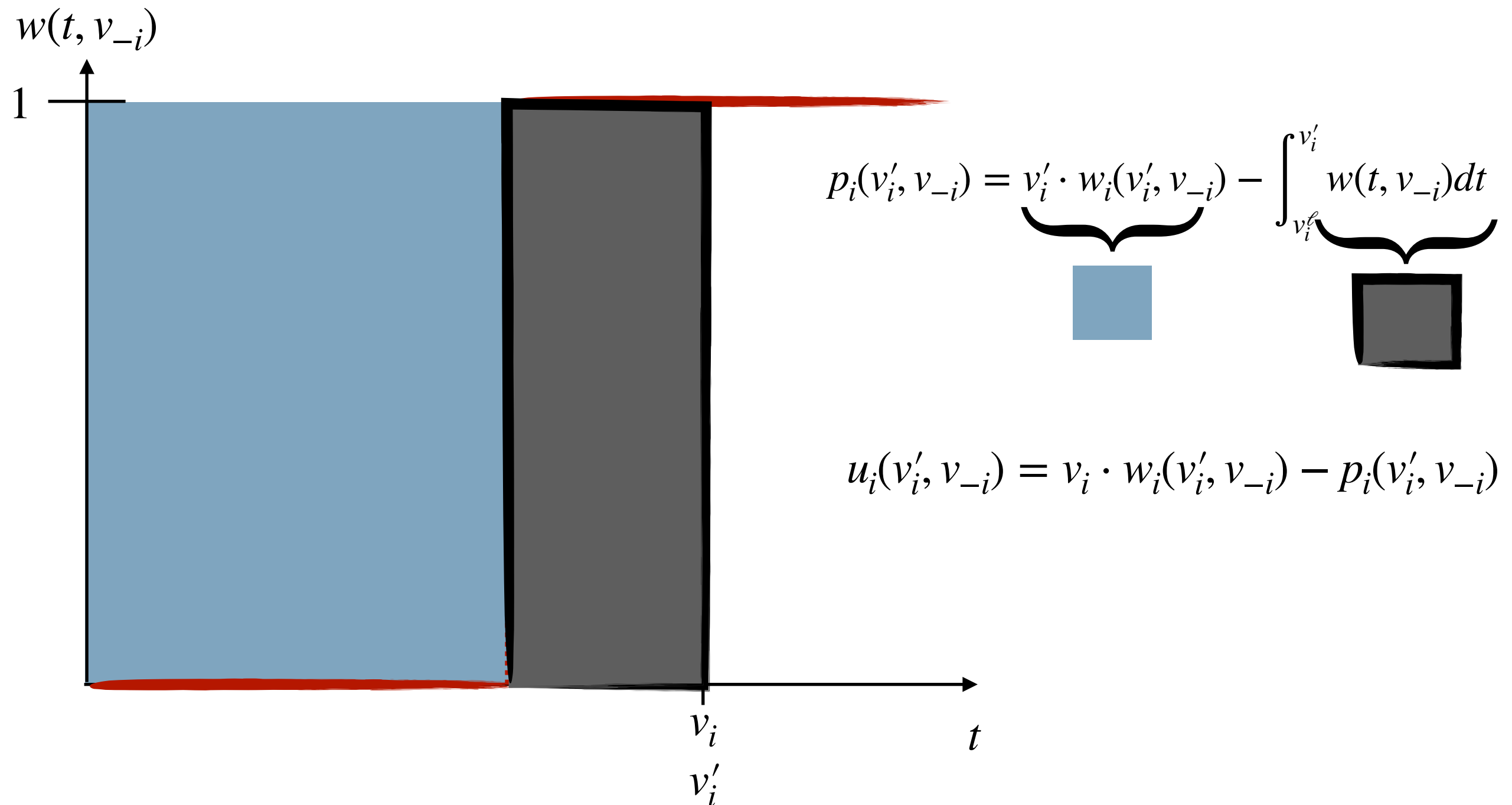
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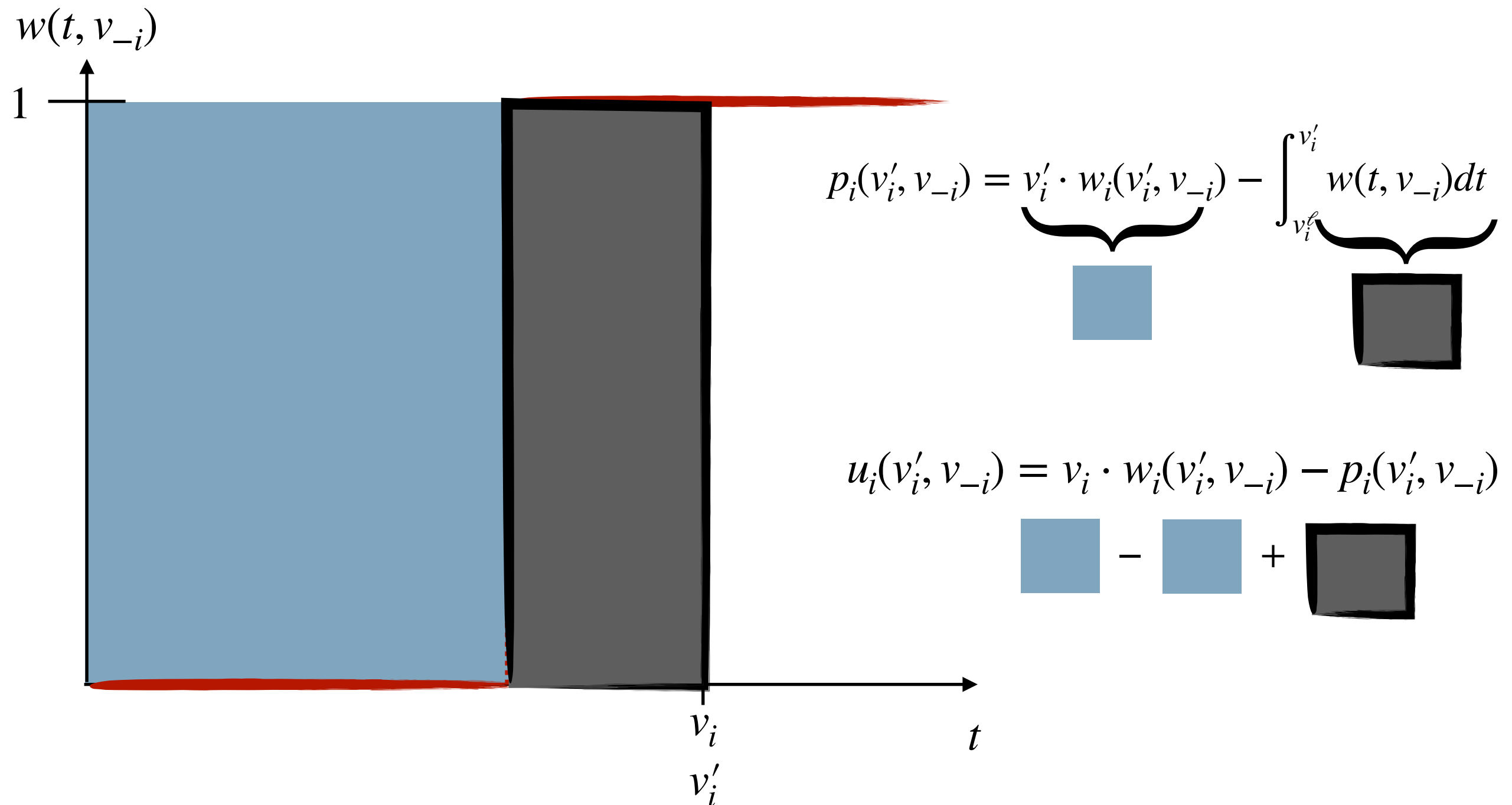
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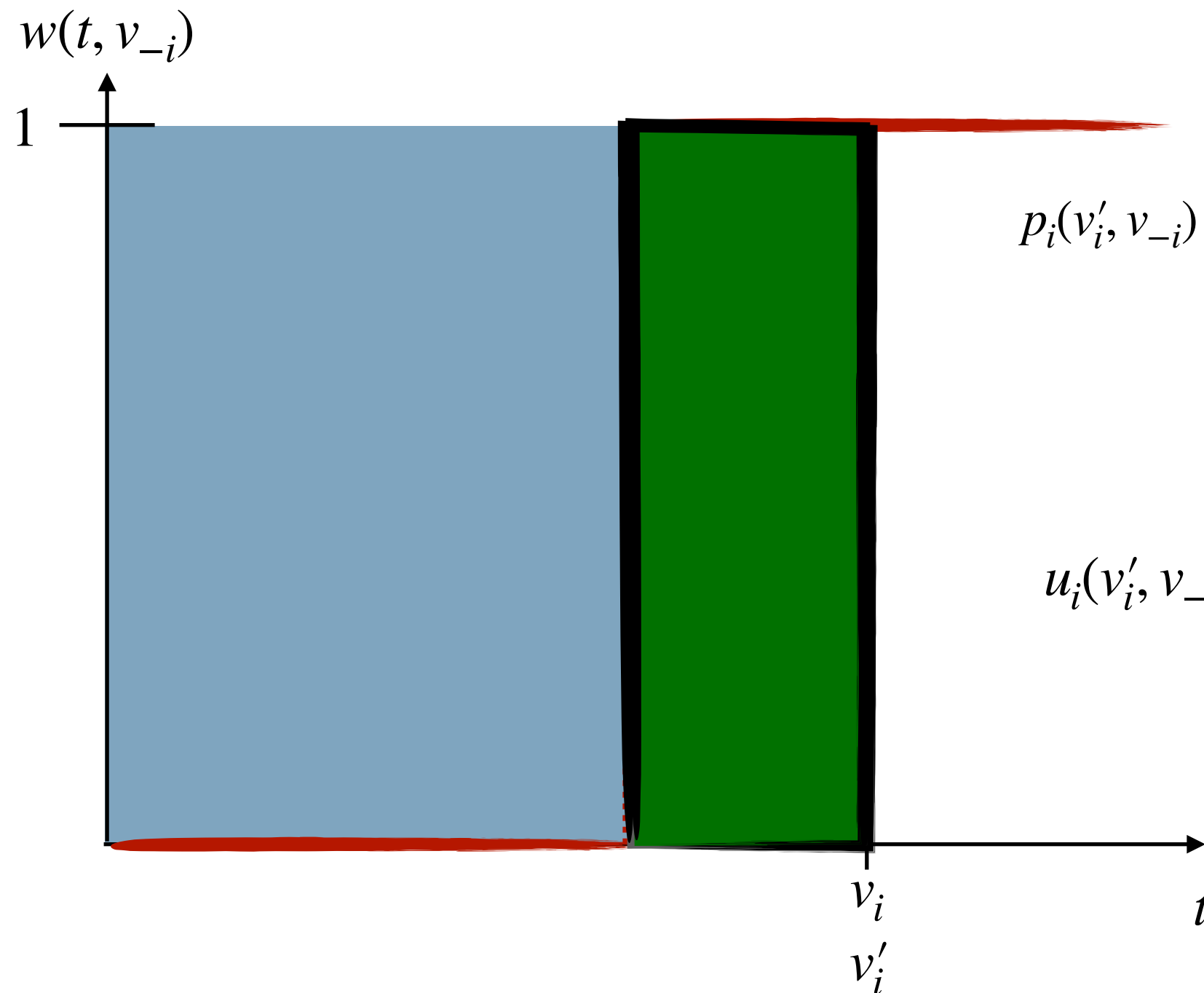
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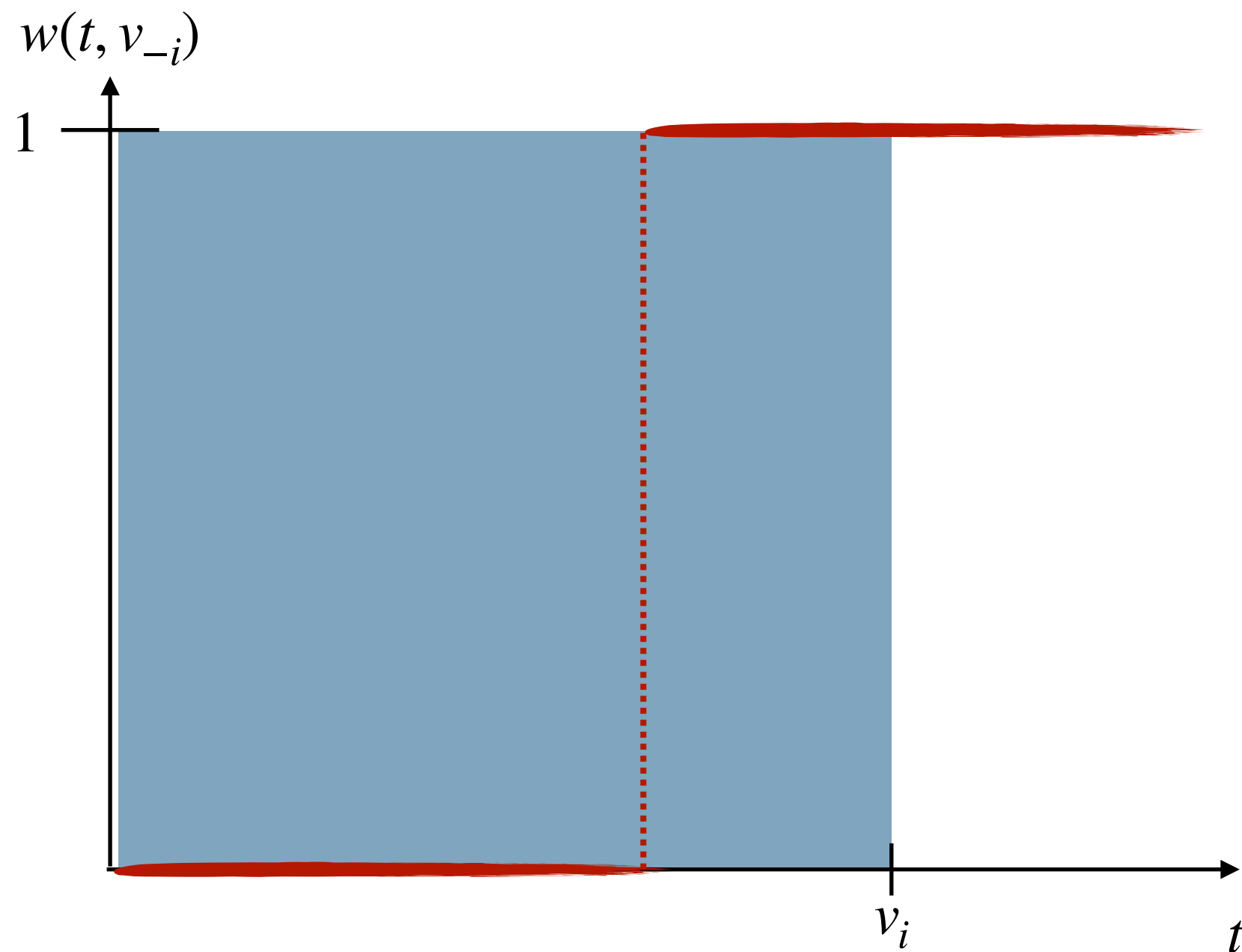
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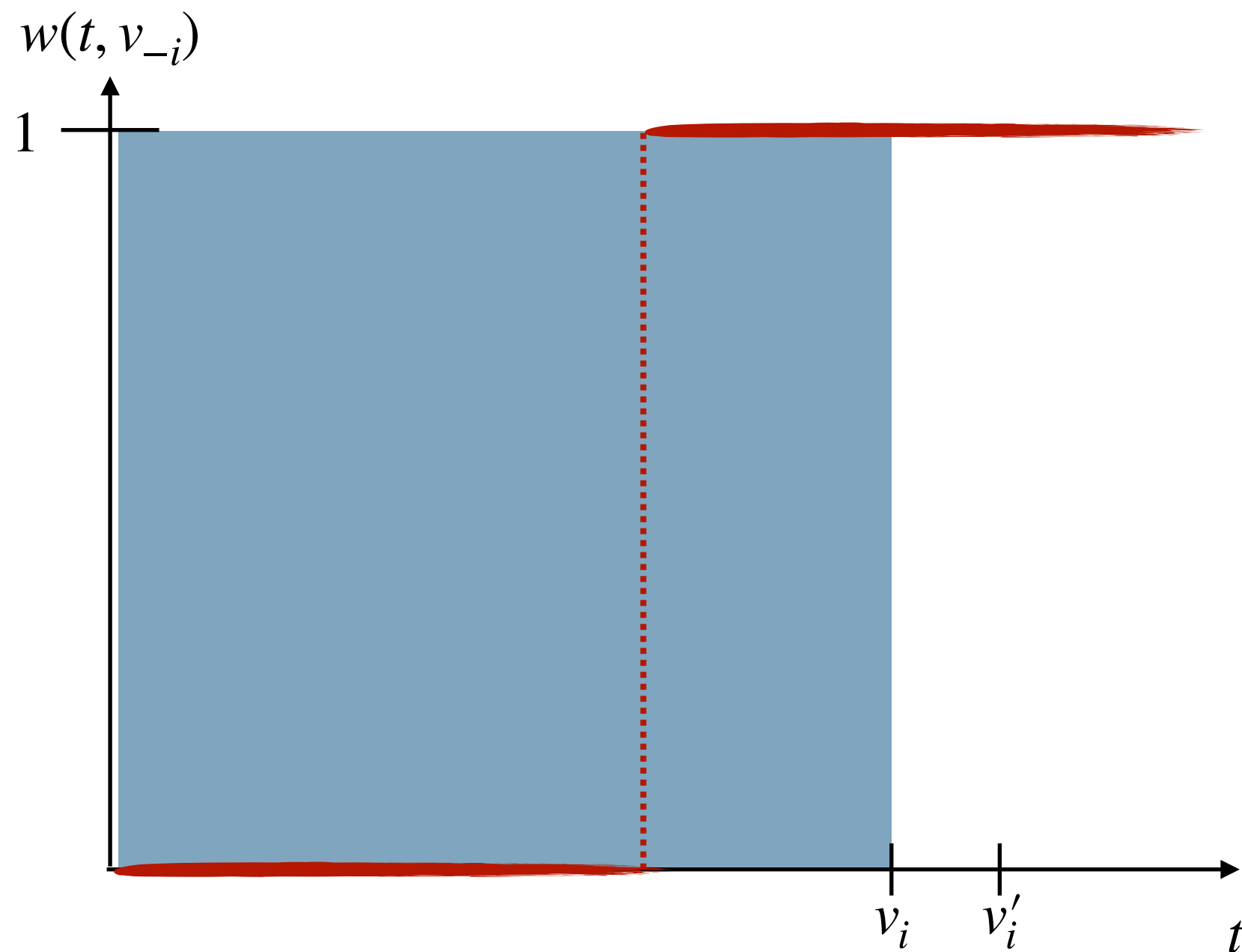
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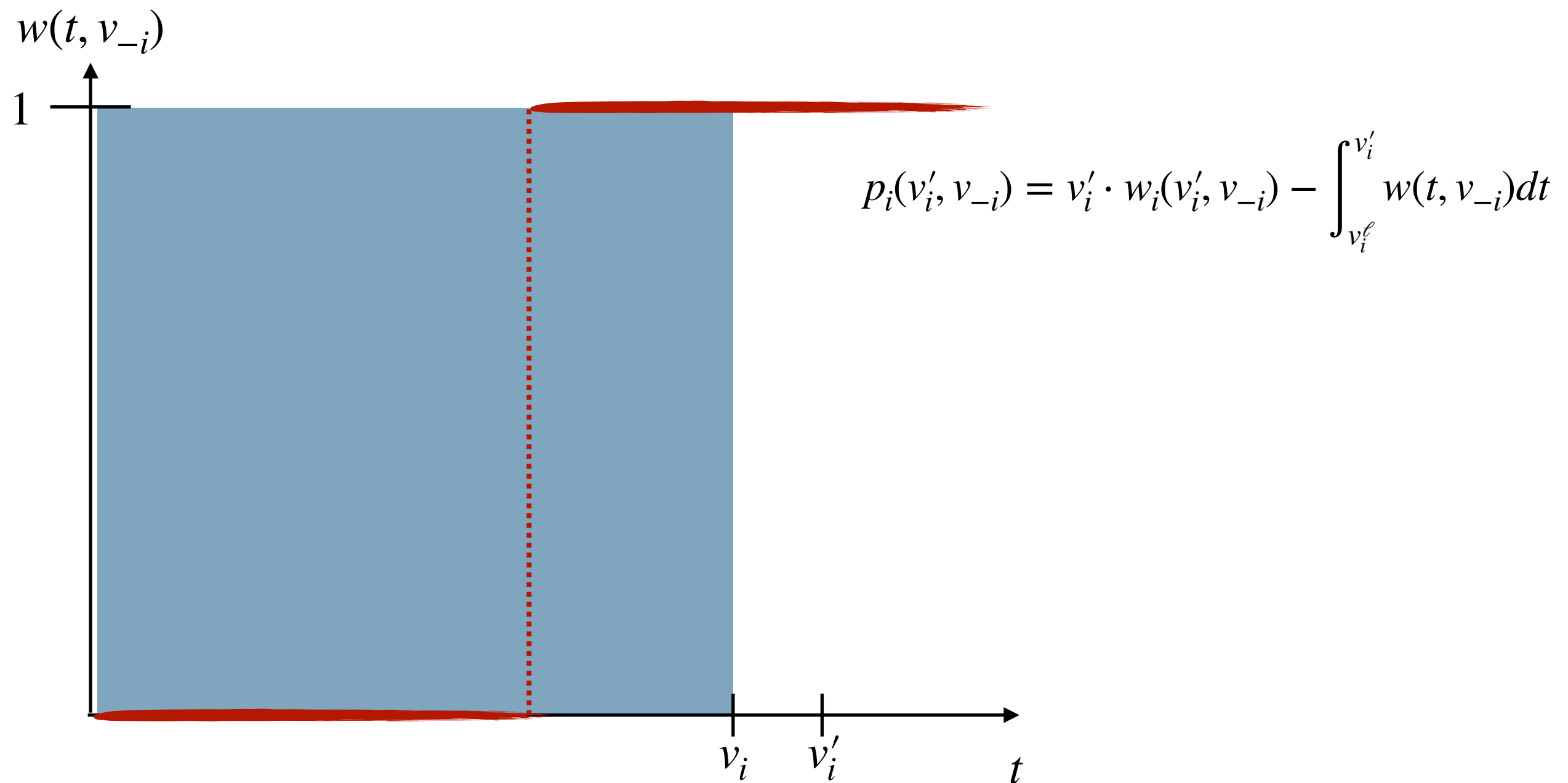
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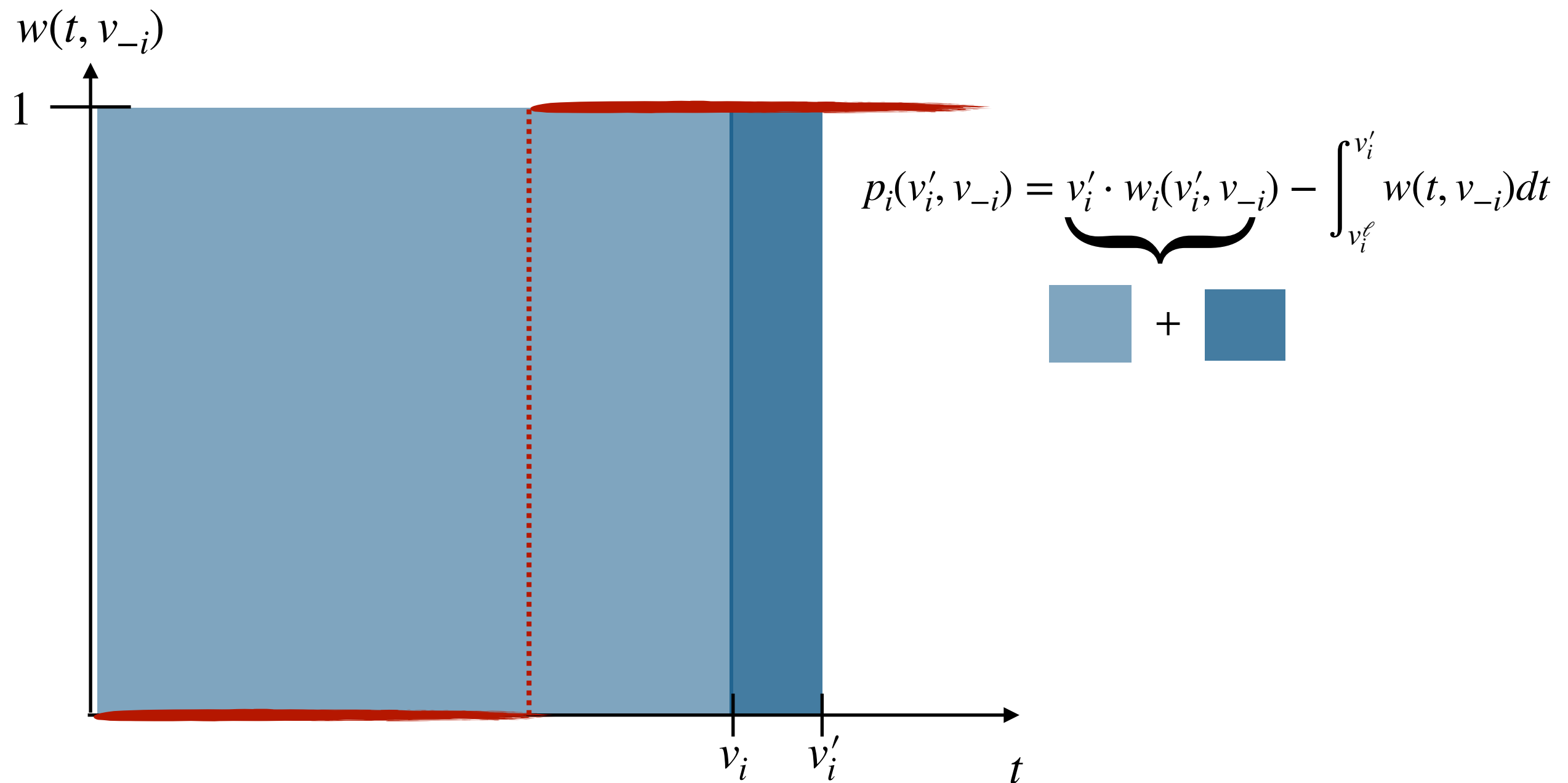
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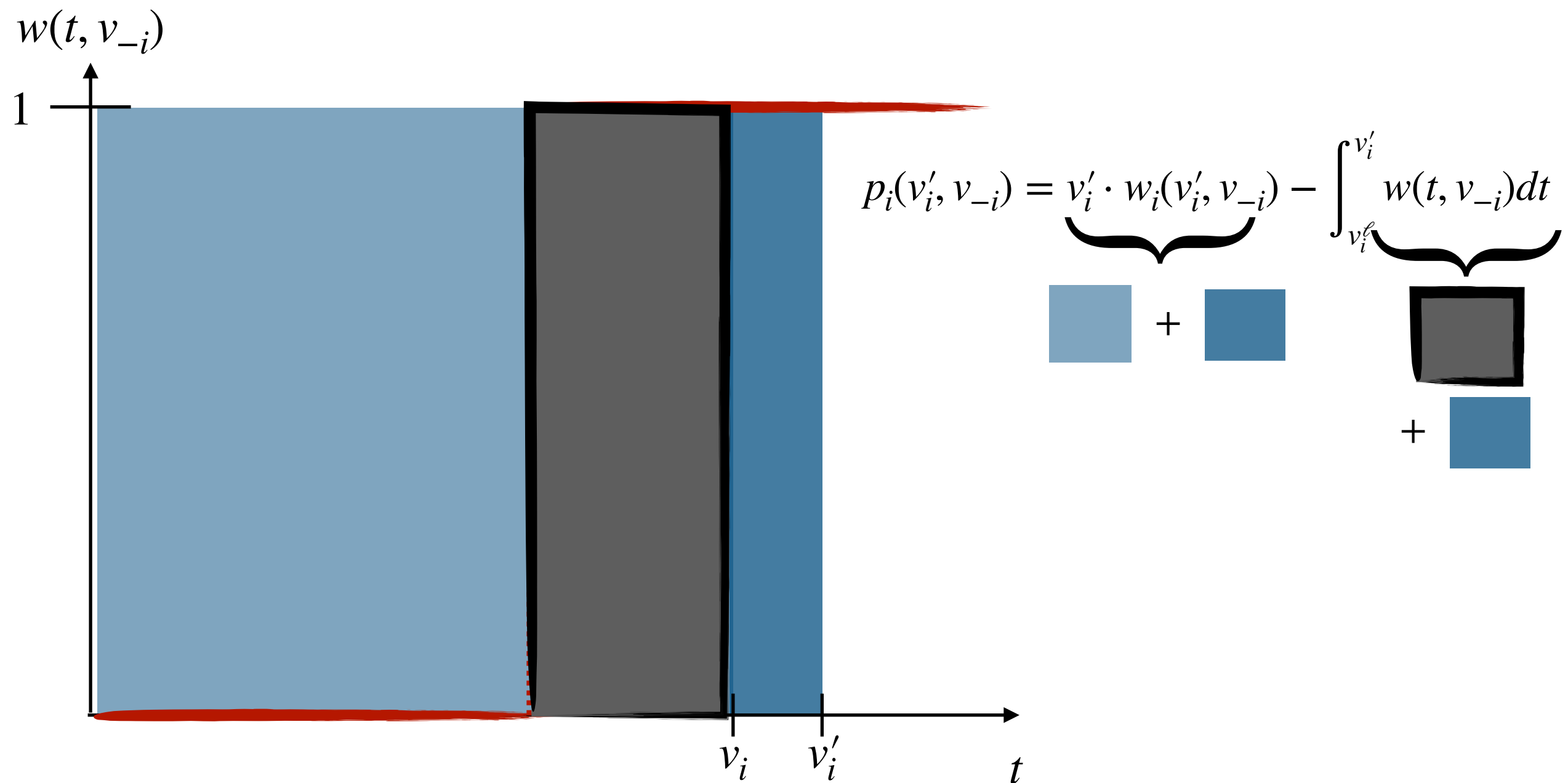
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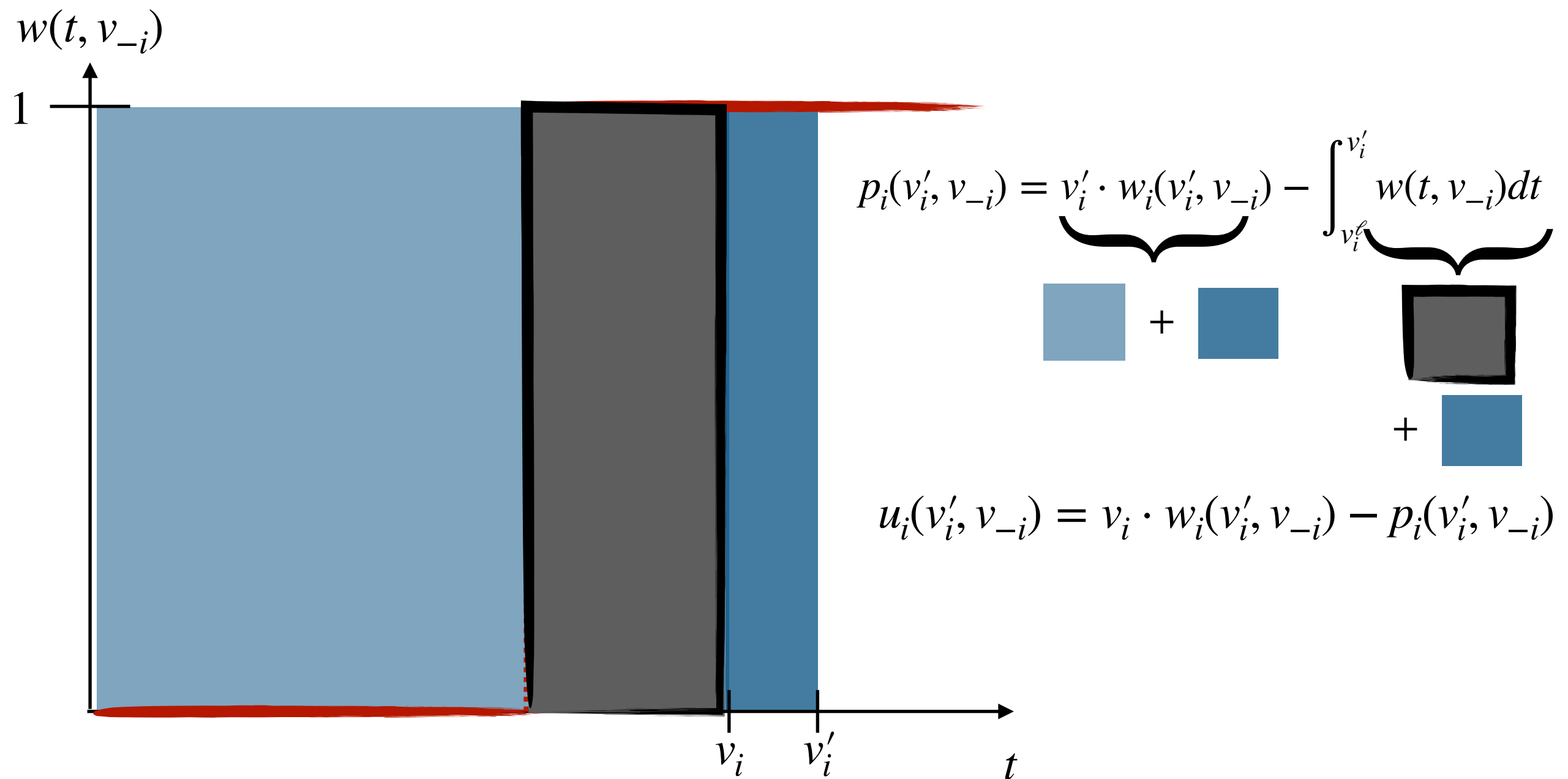
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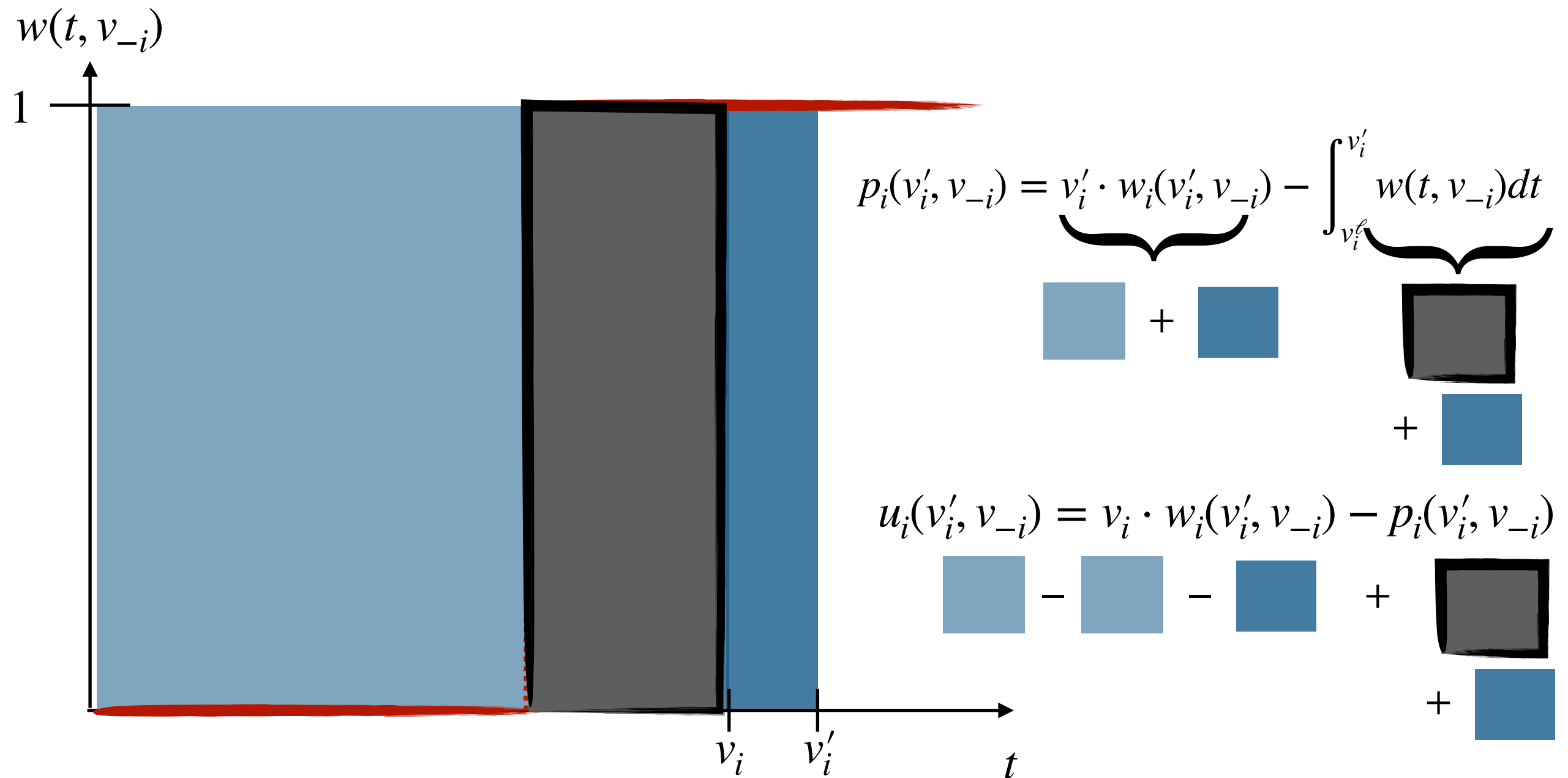
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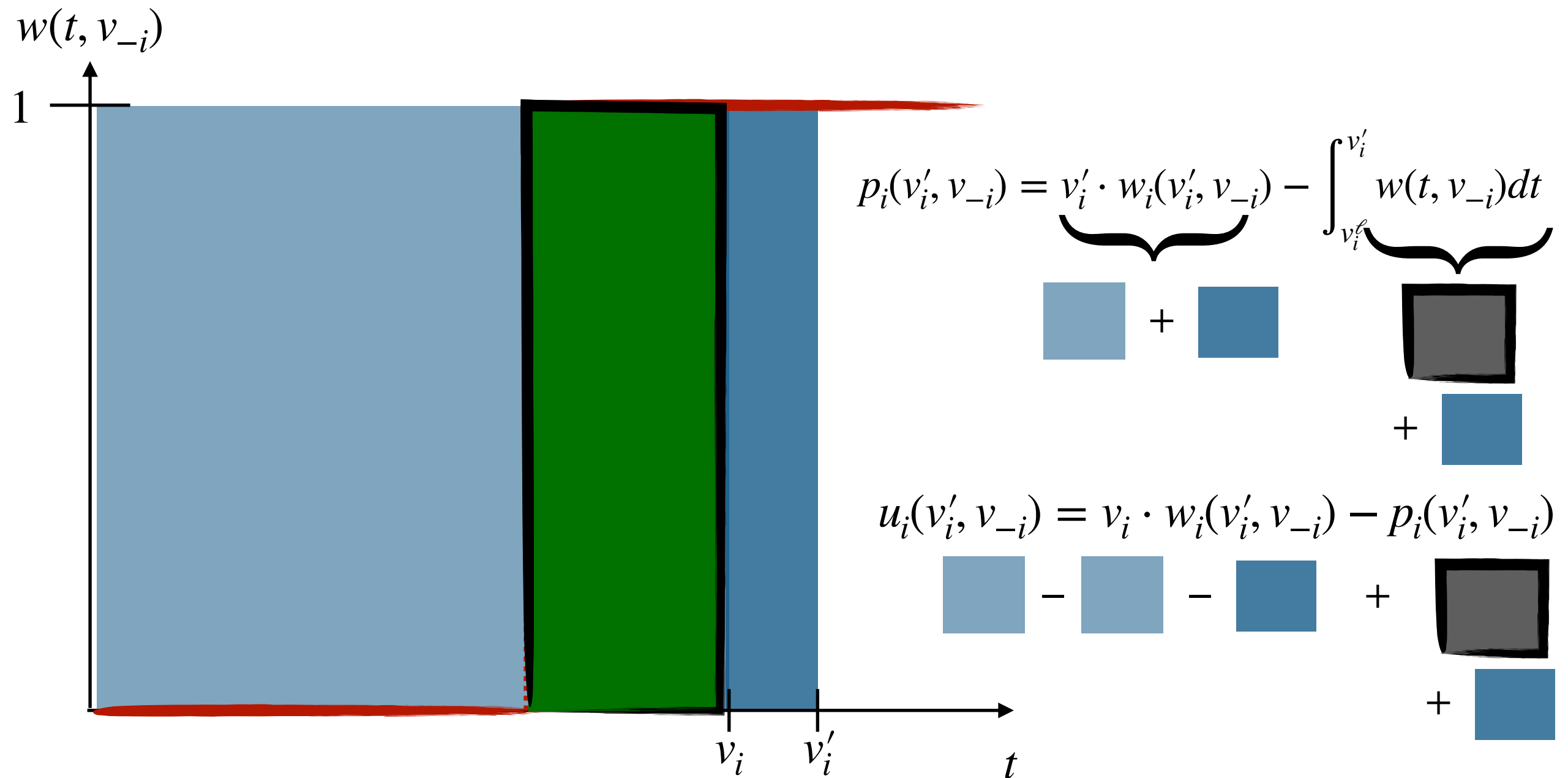
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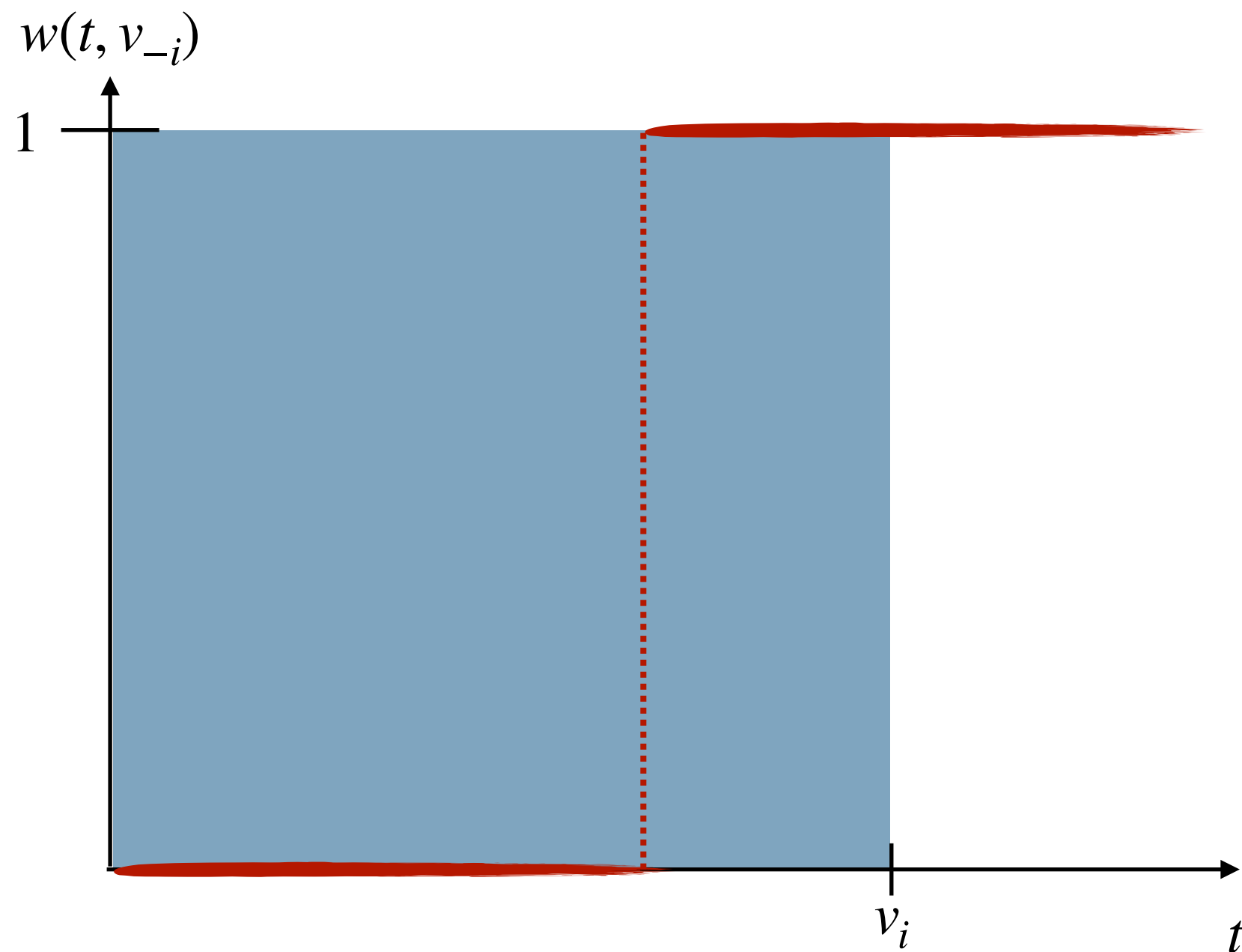
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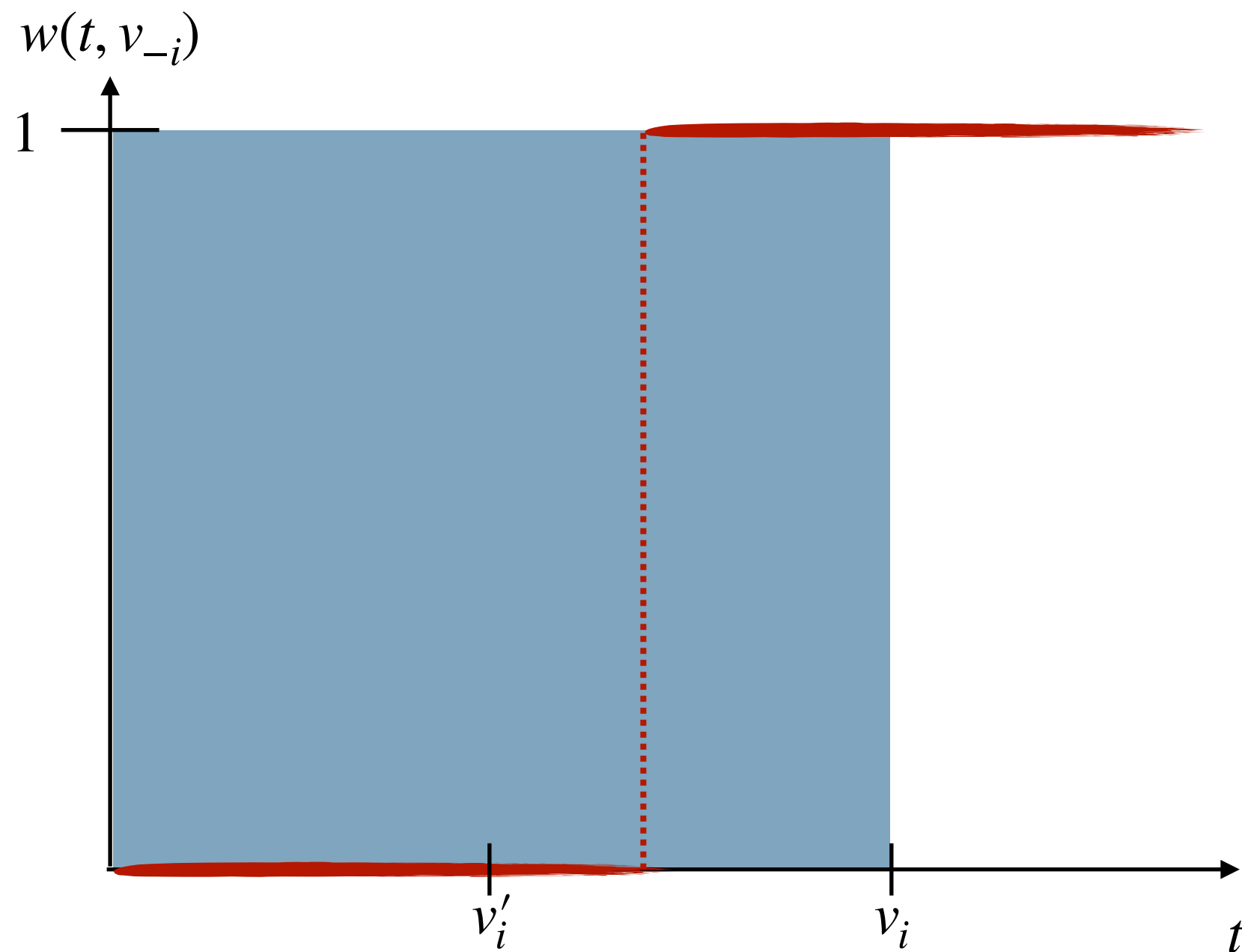
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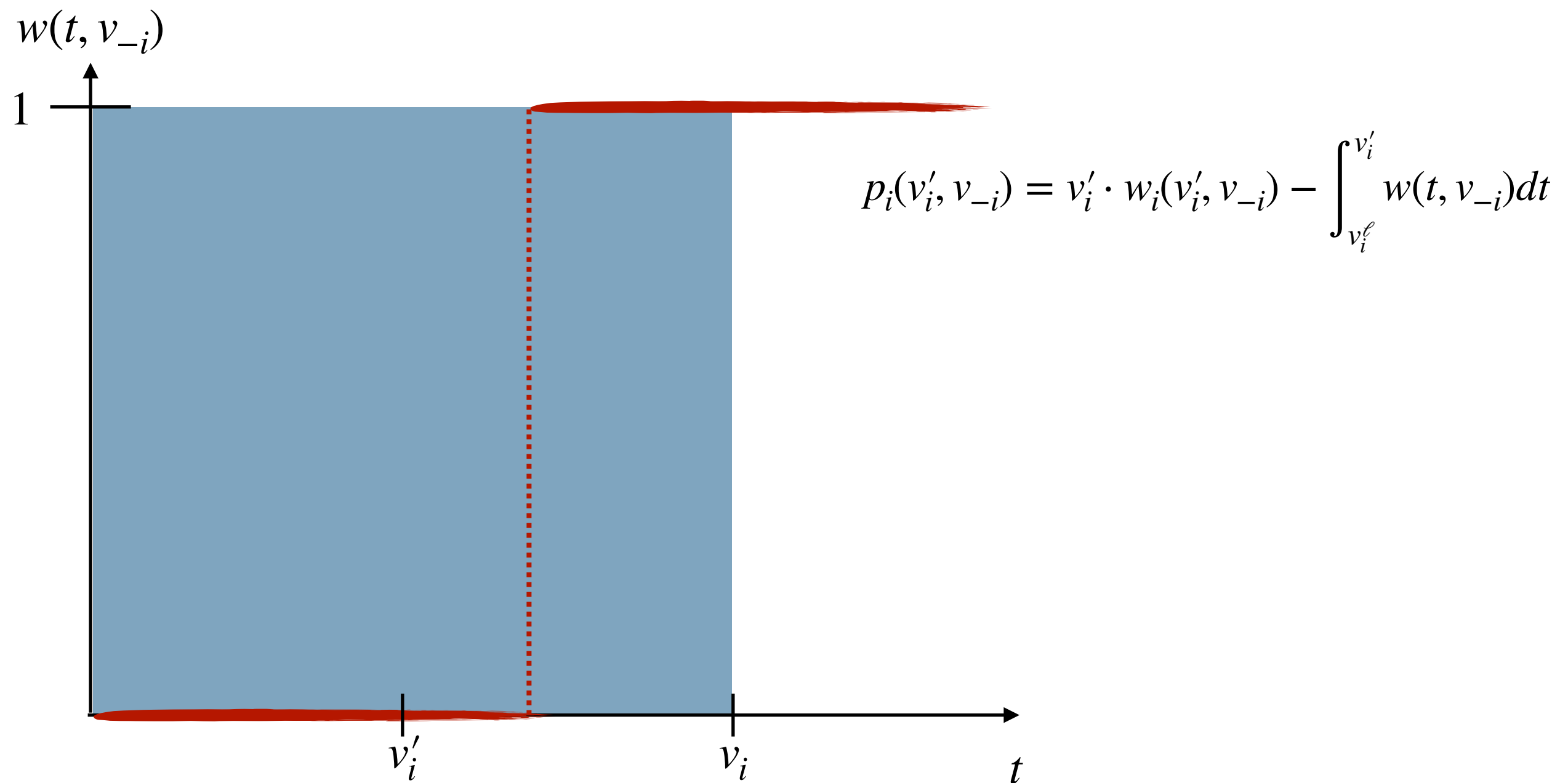
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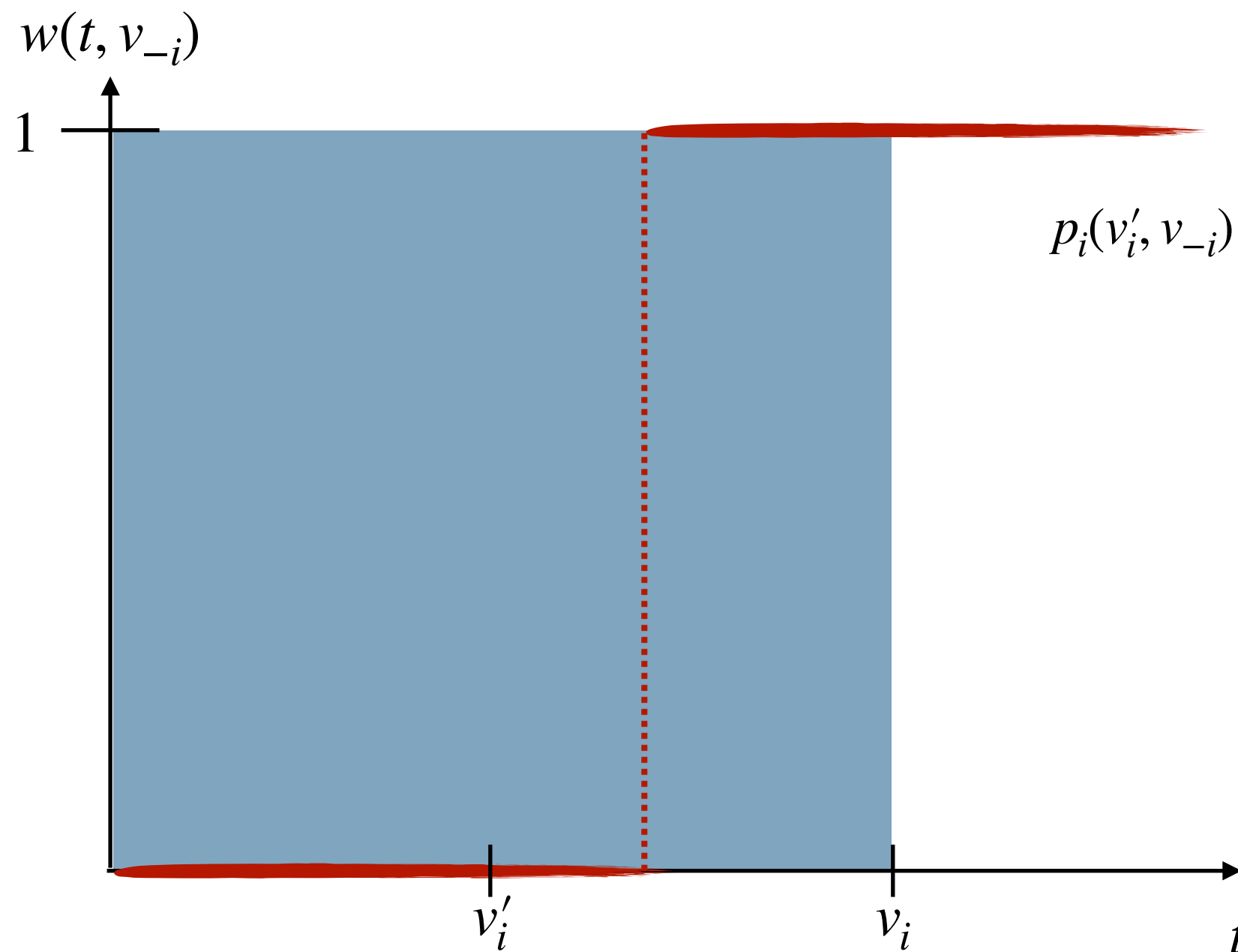
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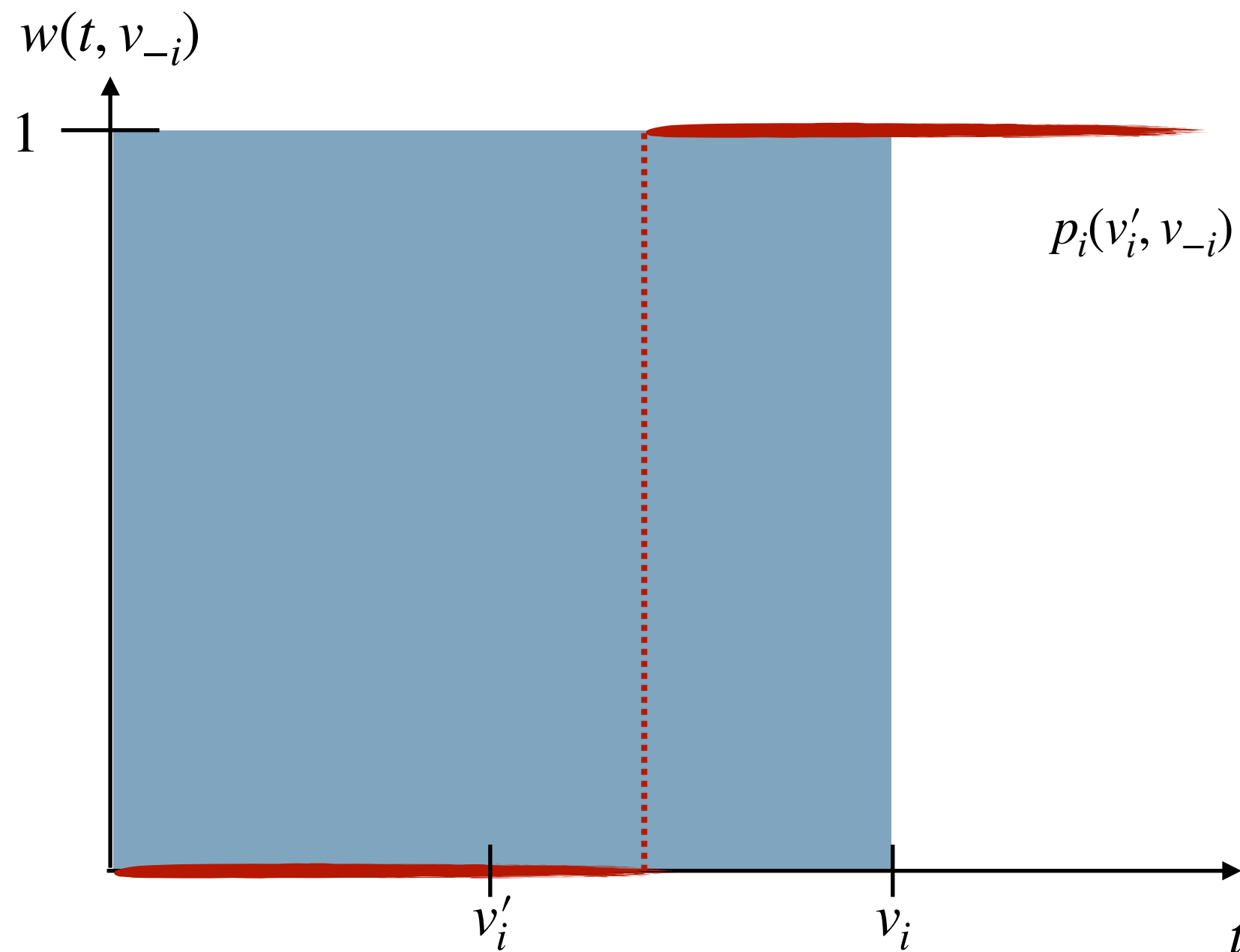


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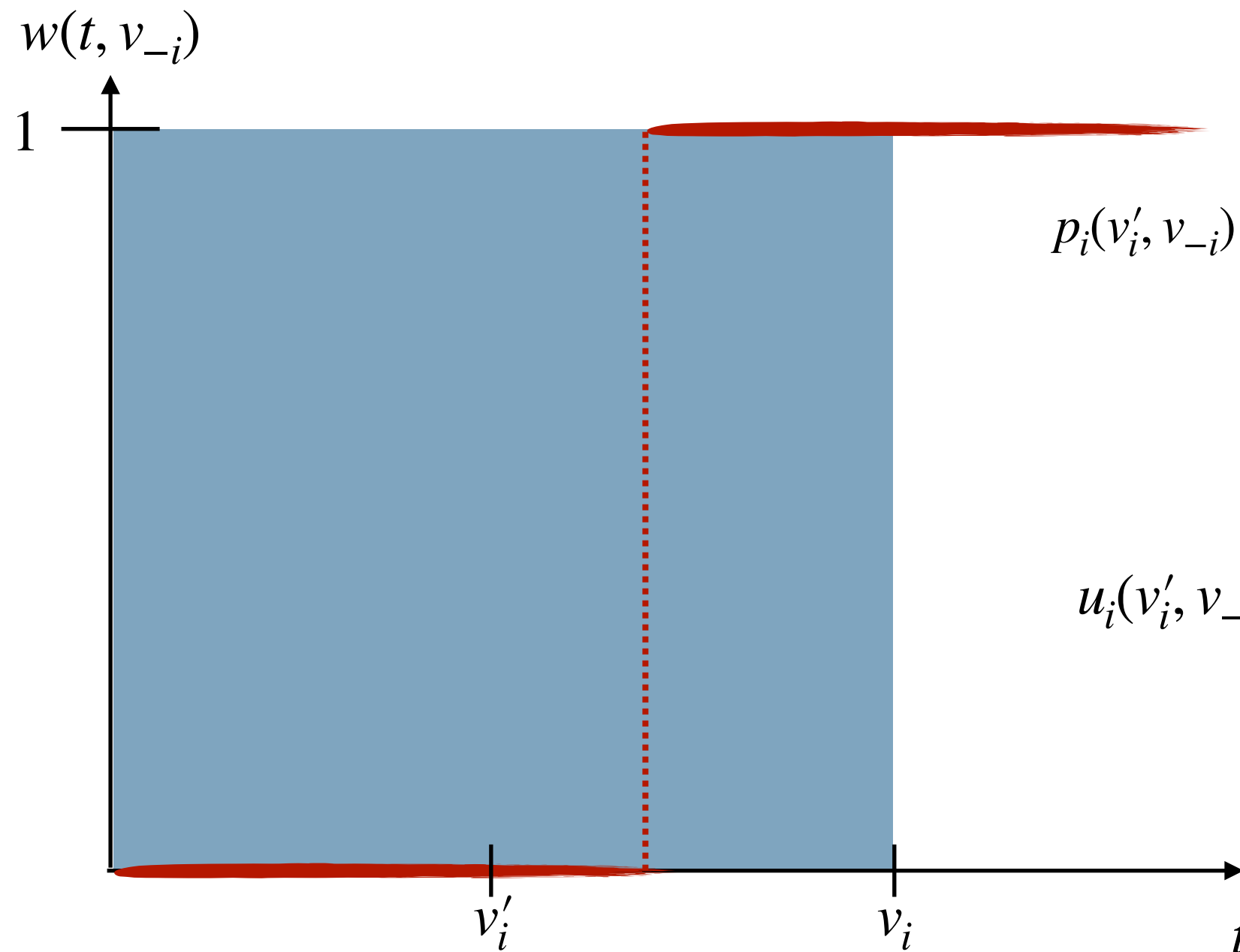
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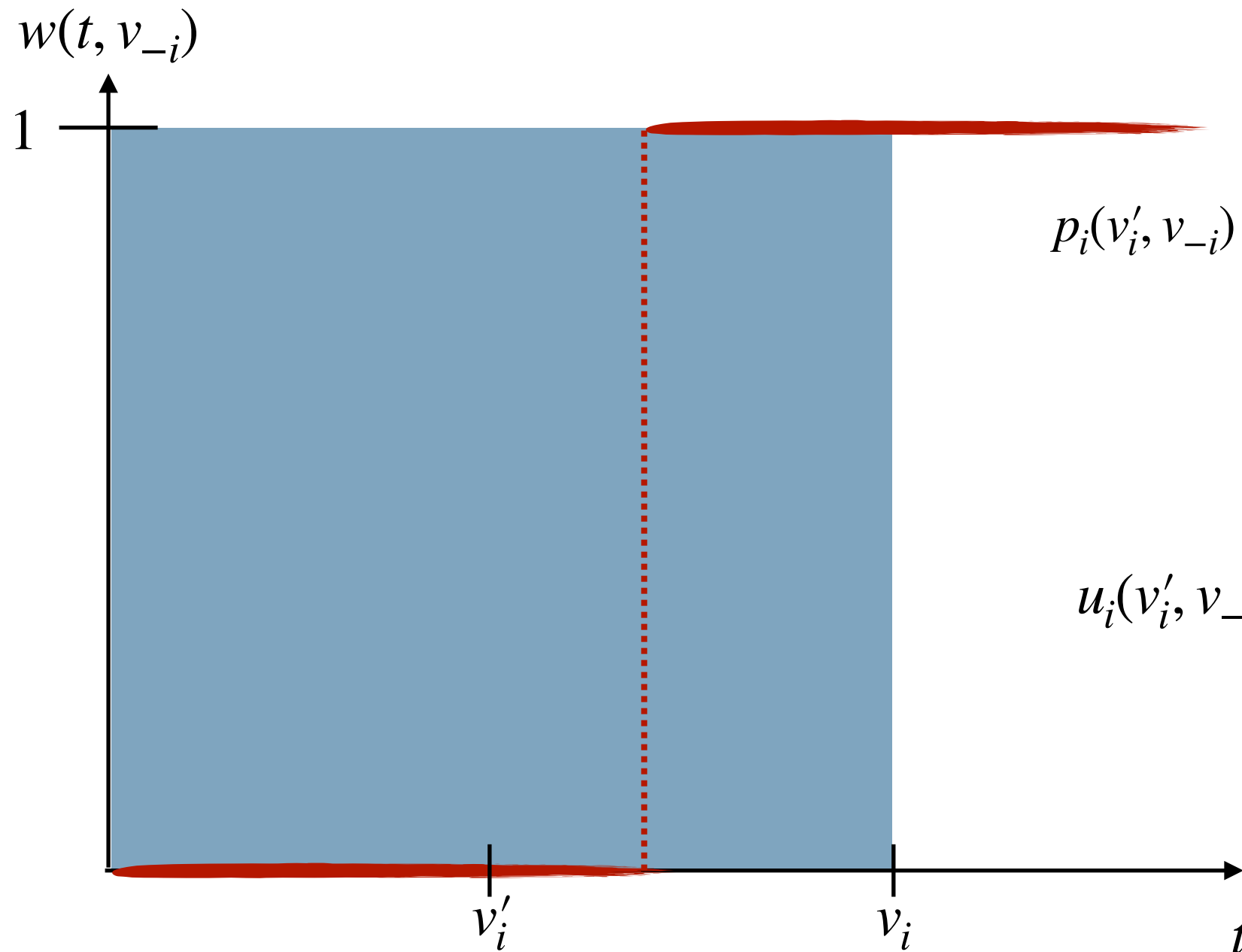
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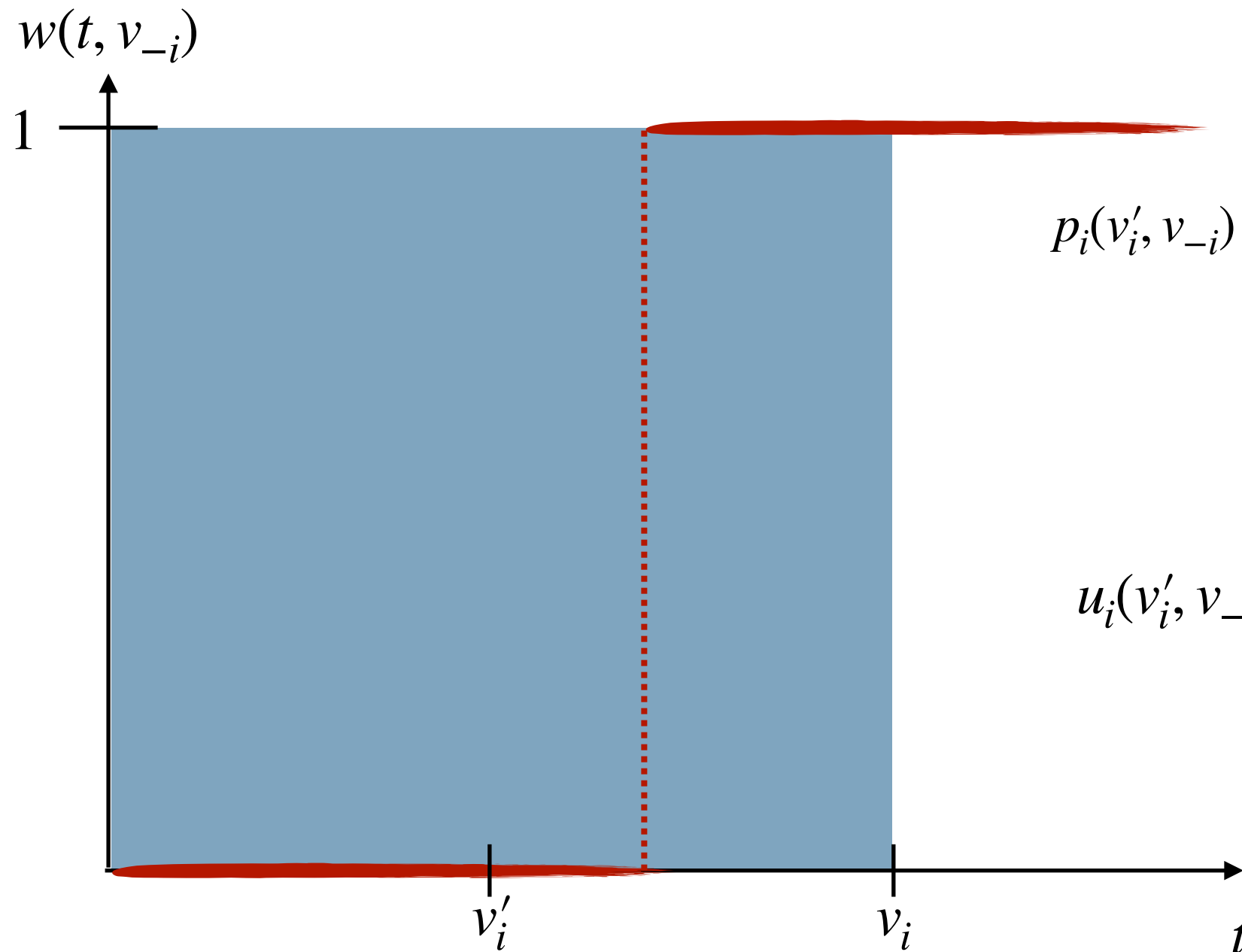
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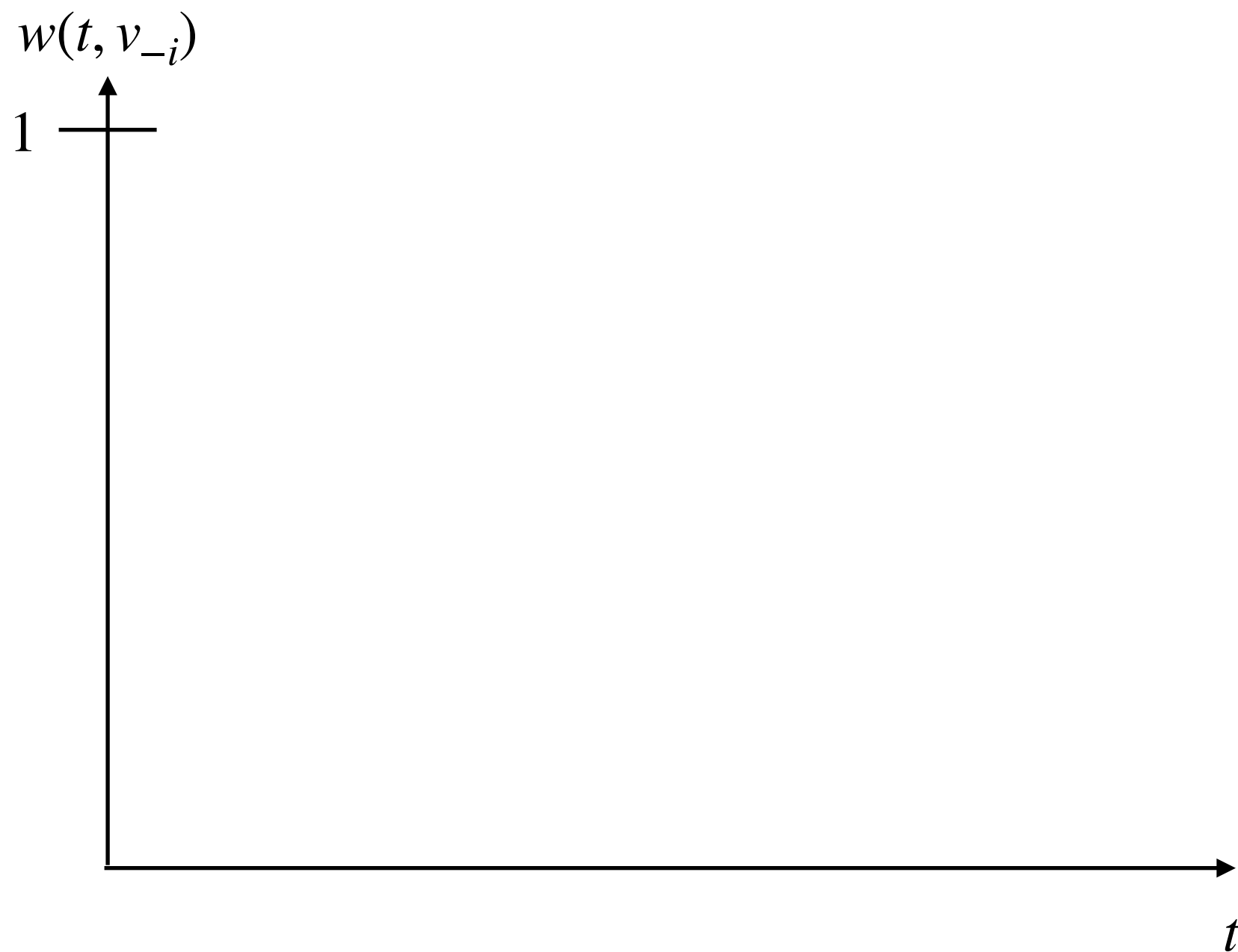
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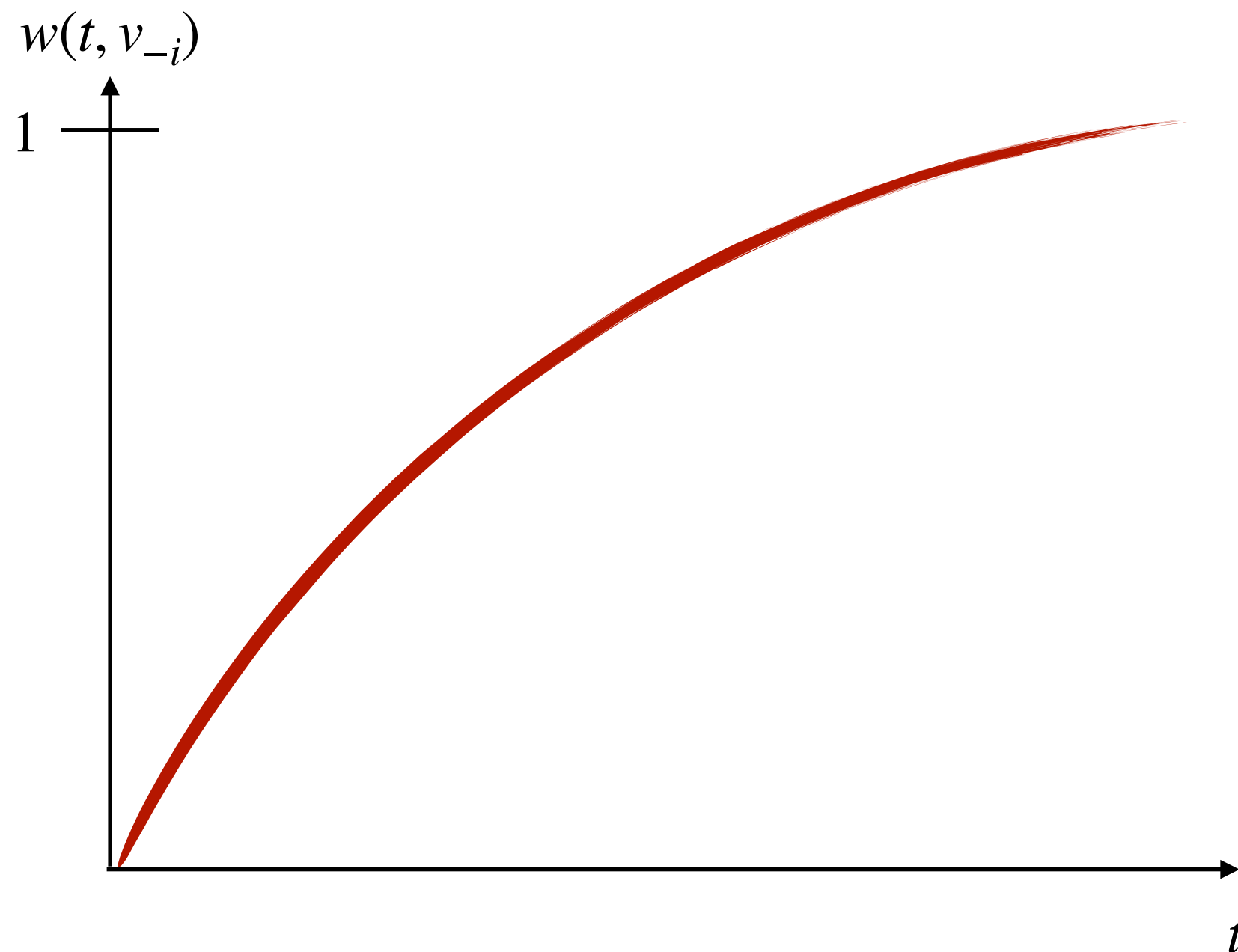
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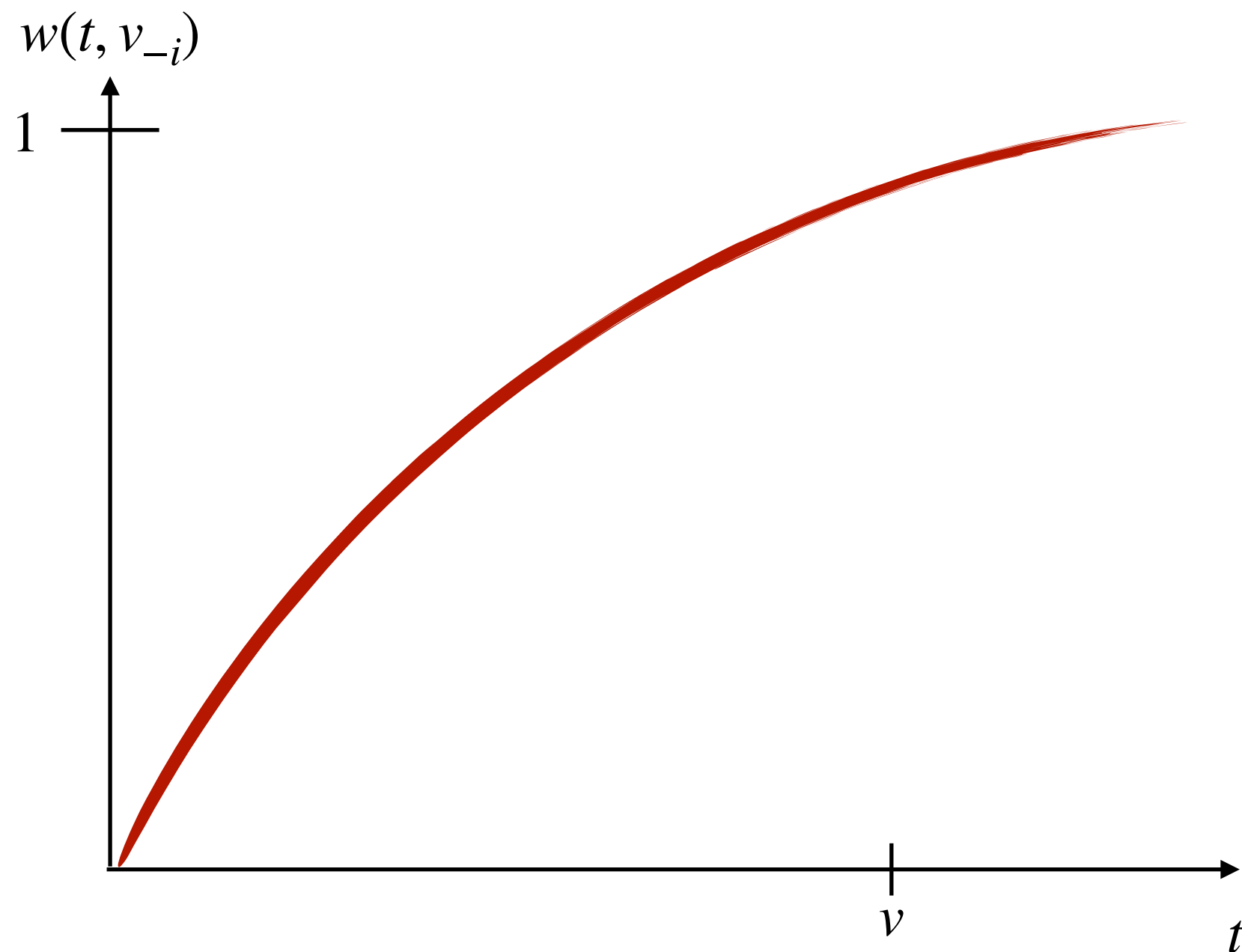
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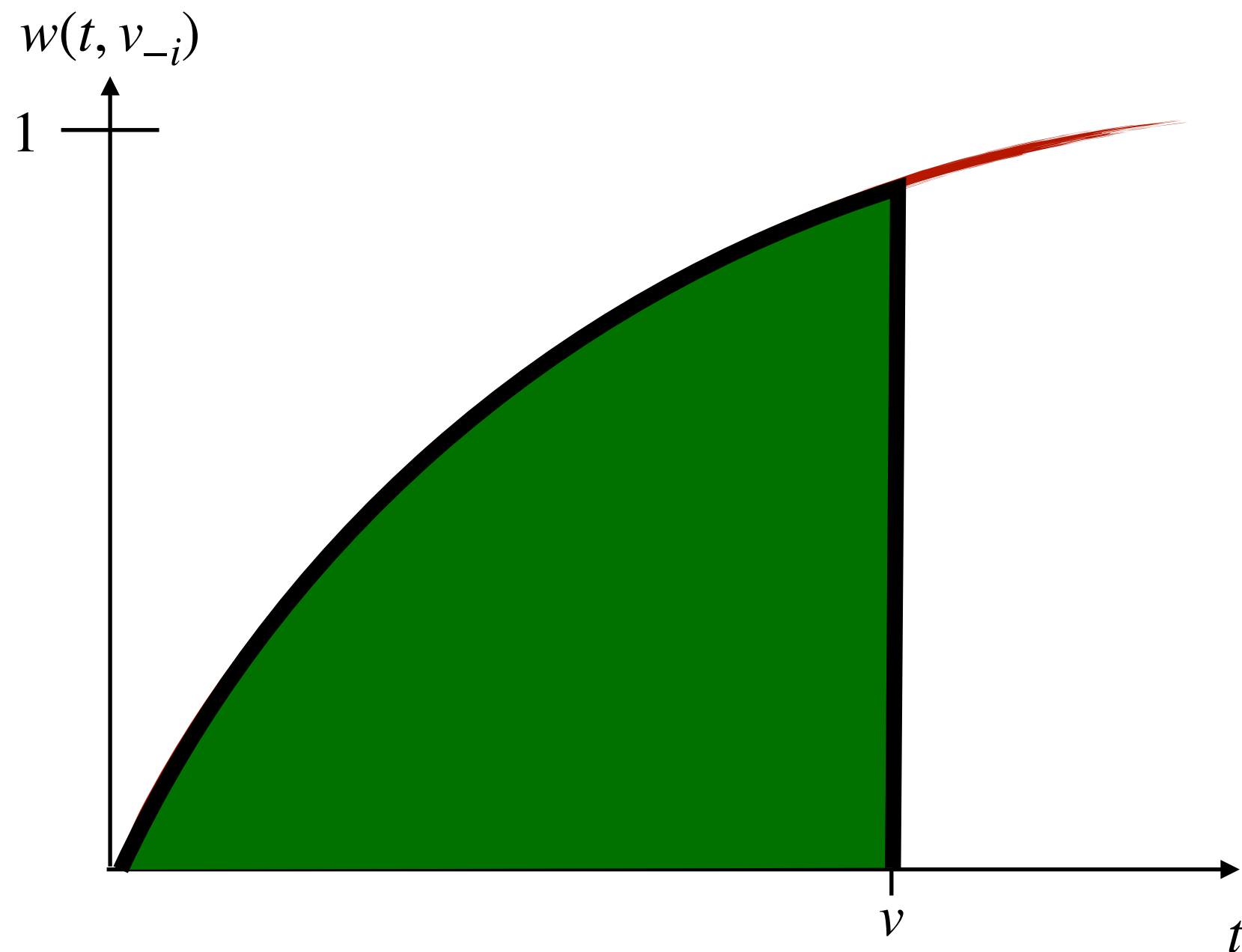
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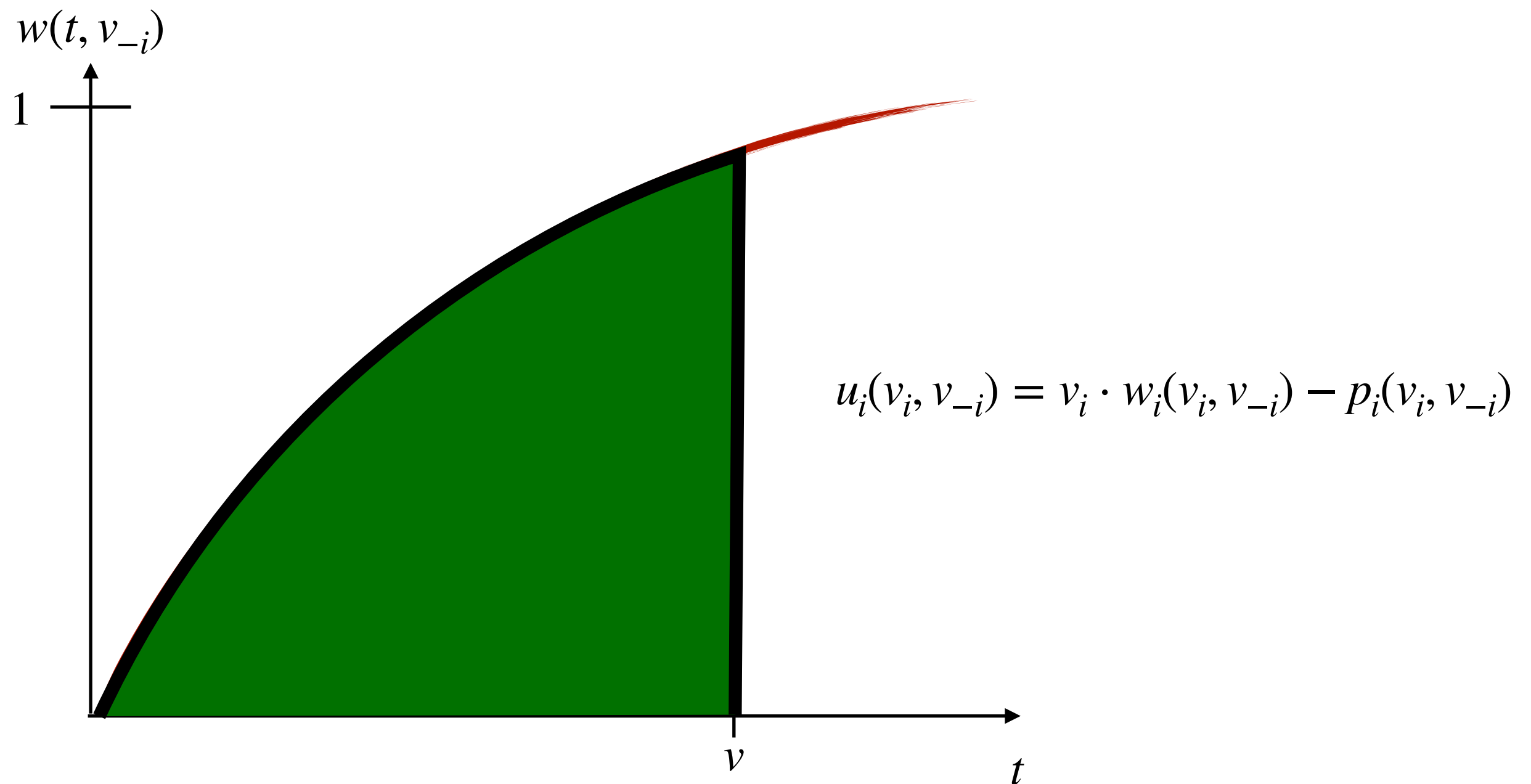
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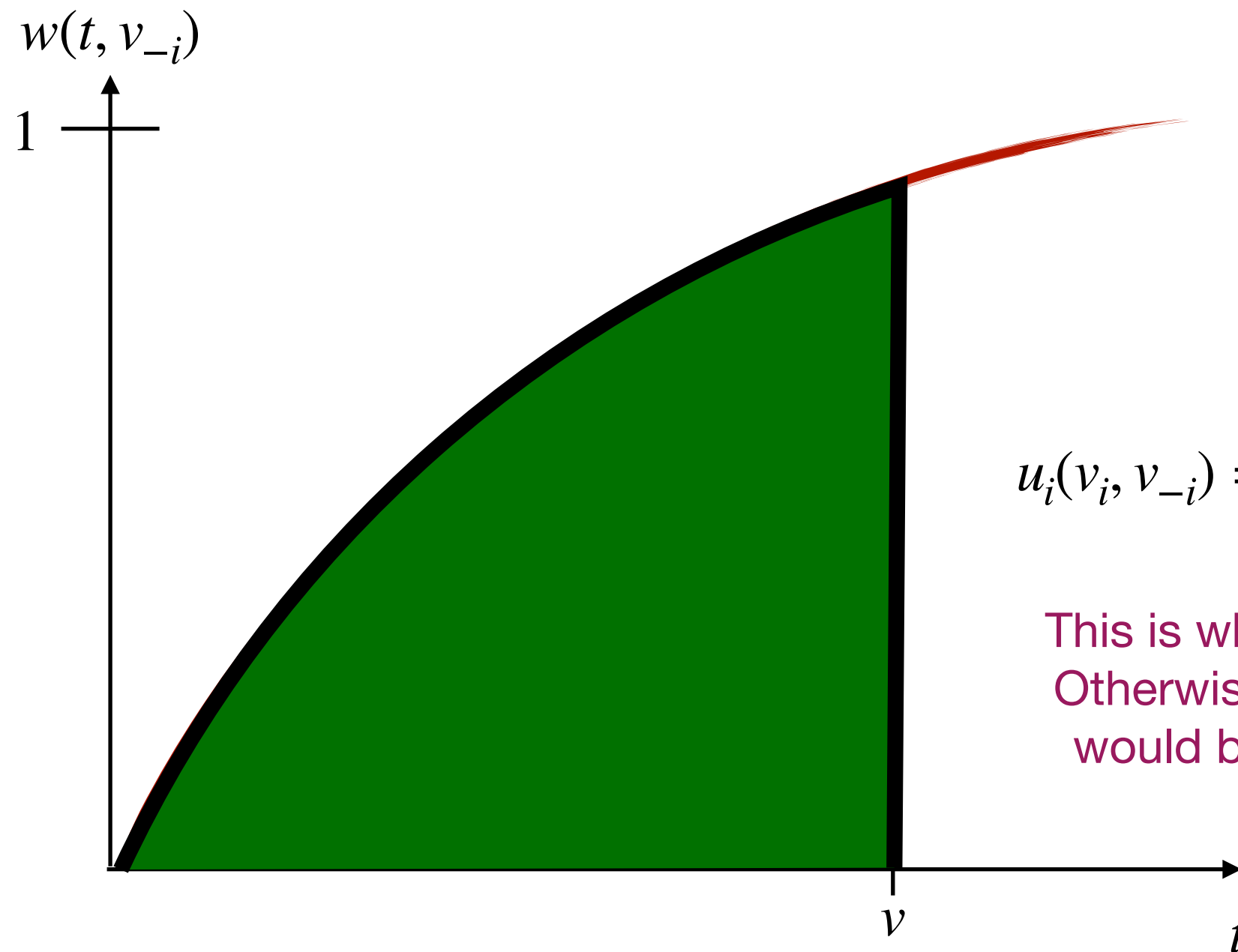
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$$u_i(v_i, v_{-i}) = v_i \cdot w_i(v_i, v_{-i}) - p_i(v_i, v_{-i})$$

This is why monotonicity is important.
Otherwise the area under the integral
would be larger when misreporting!

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We could have more complicated mechanisms which e.g., interact with the agents in rounds, ask them questions, present them with tasks etc?

More general mechanisms?

All the mechanisms that we have seen so far are of the following form:

- The agents declare their values/preferences up front,
- The mechanism chooses an outcome (and payments) based on this declarations.

These mechanisms are called *direct revelation* mechanisms.

We could have more complicated mechanisms which e.g., interact with the agents in rounds, ask them questions, present them with tasks etc?

Crucially, could these mechanisms achieve things that *truthful direct revelation mechanisms* cannot?

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- (1) For every valuation/preference profile, the corresponding game has a **dominant strategy equilibrium (DSE)**.
- (2) At this DSE, every agent **truthfully** reports her **true** value.

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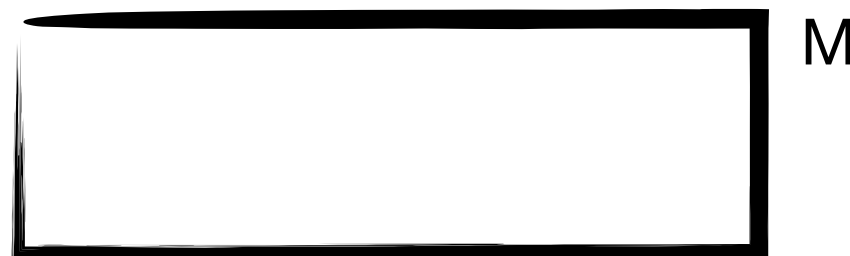
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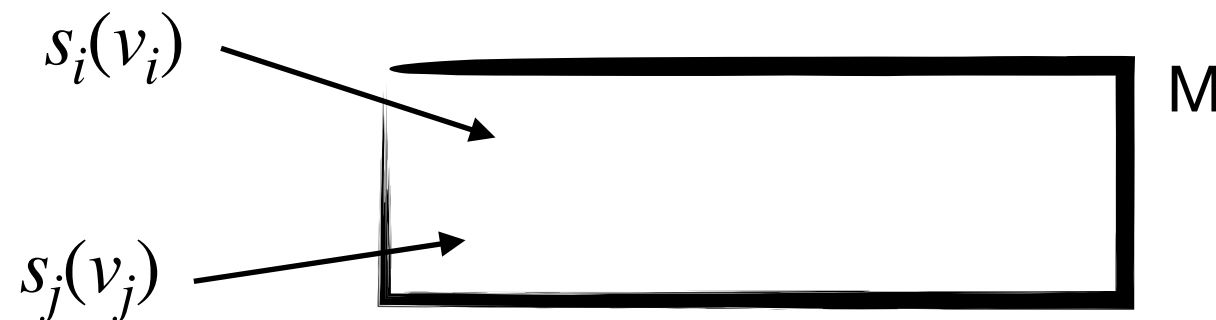
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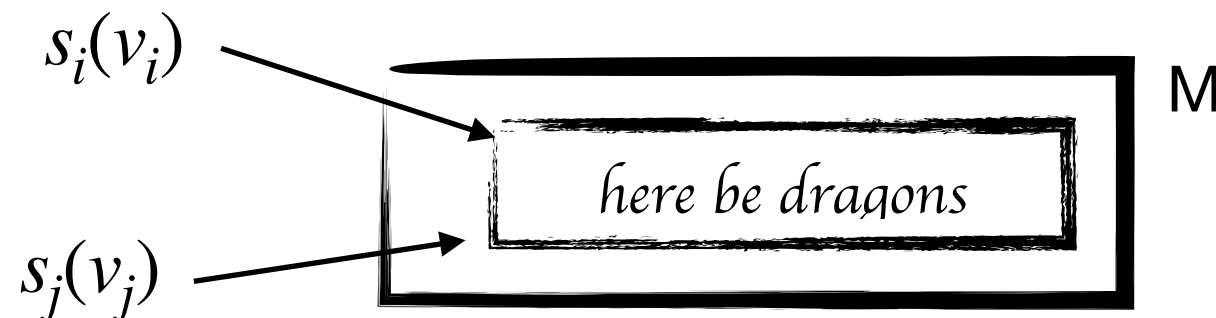
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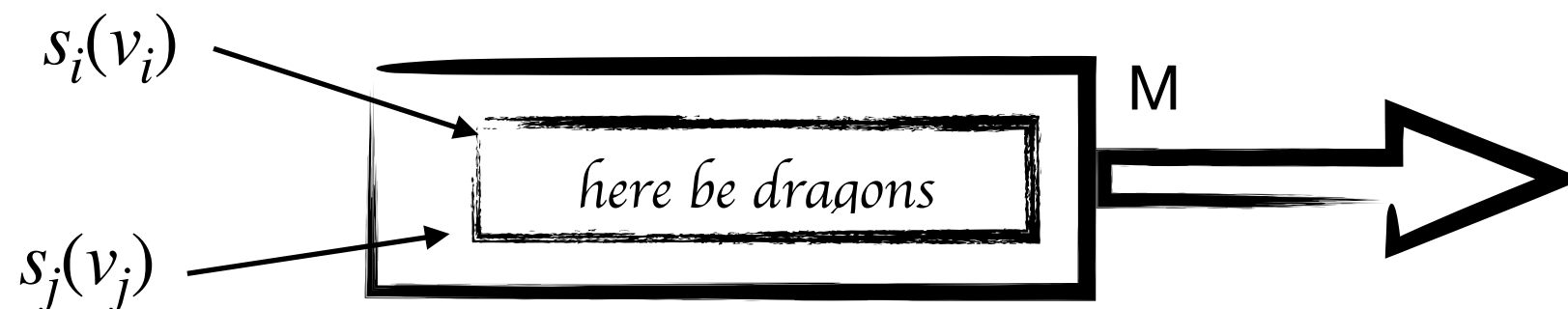
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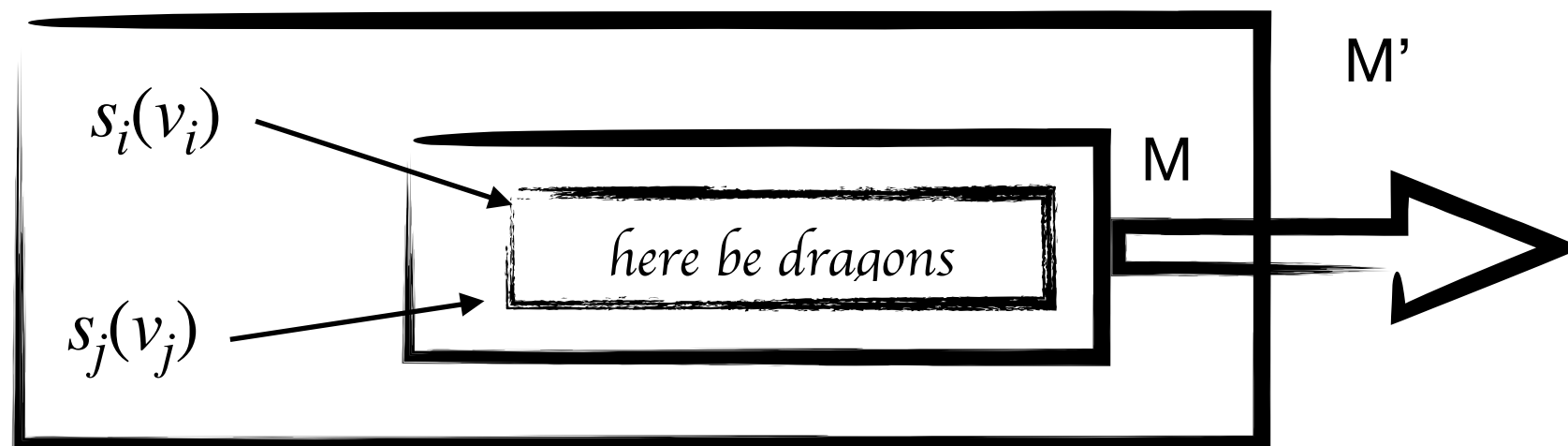
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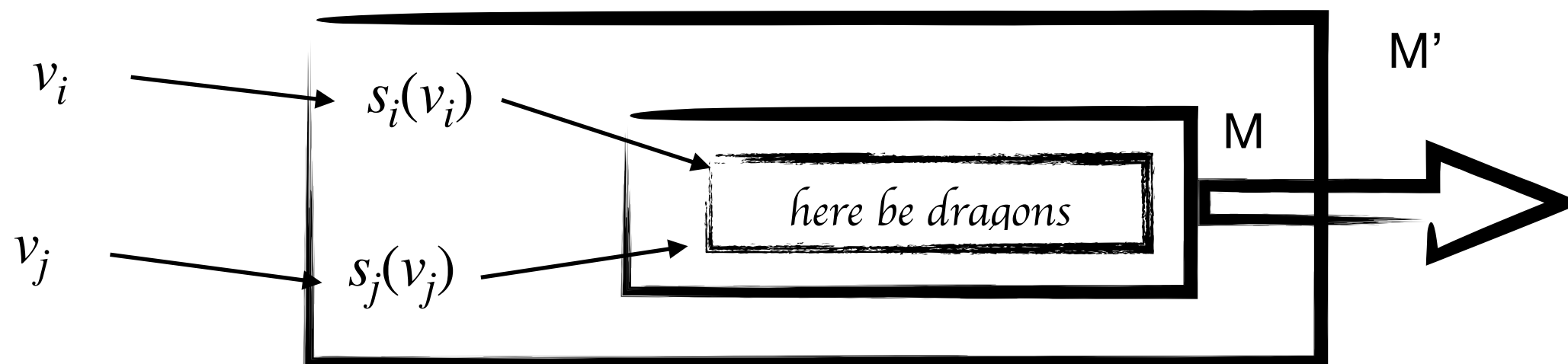
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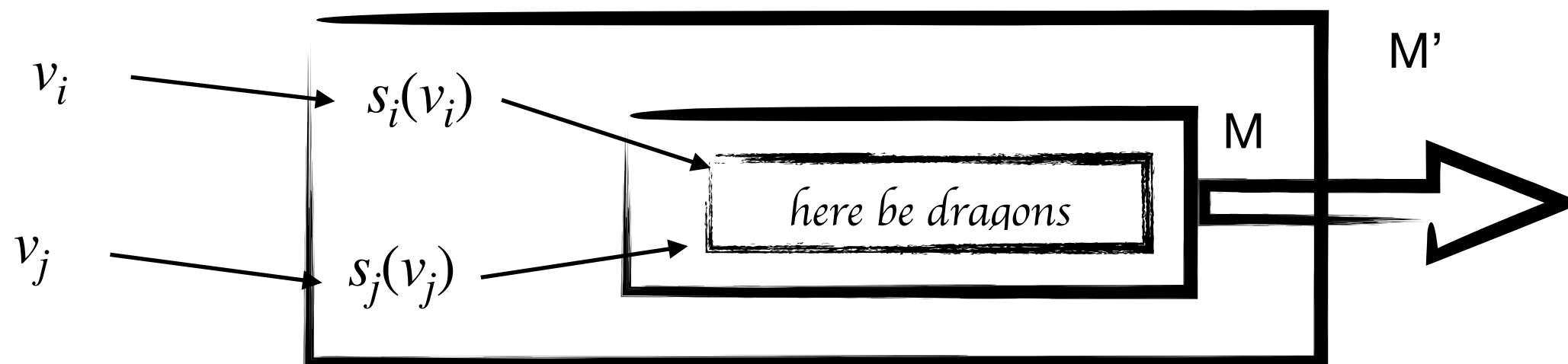
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We would still like our mechanisms to have DSE. What is we relax this requirement?

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