Algorithmic Game Theory and Applications

Optimal Auctions

Single-Item Auctions: Quick Refresher

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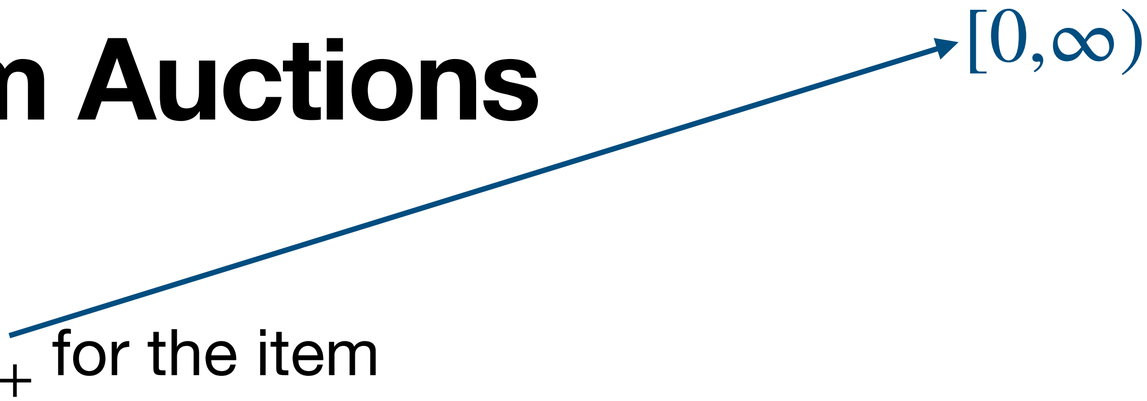

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  + Individual rationality (IR): $u_i(v_i, \boldsymbol{b}_{-i}; v_i) \geq 0 \quad \forall i \; \forall \boldsymbol{b}_{-i} \; \forall v_i$

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Roger Myerson (1951 -)

Nobel prize in Economics
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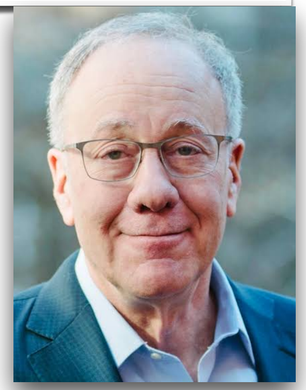

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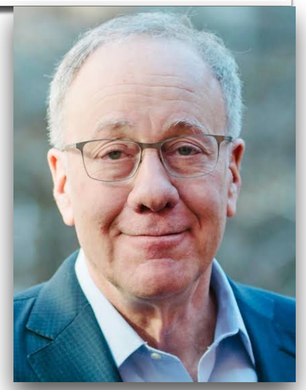

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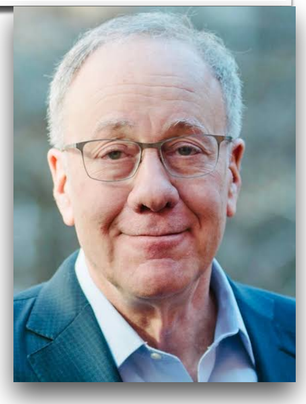

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  - Shall we still *always* sell to the *highest* bidder?

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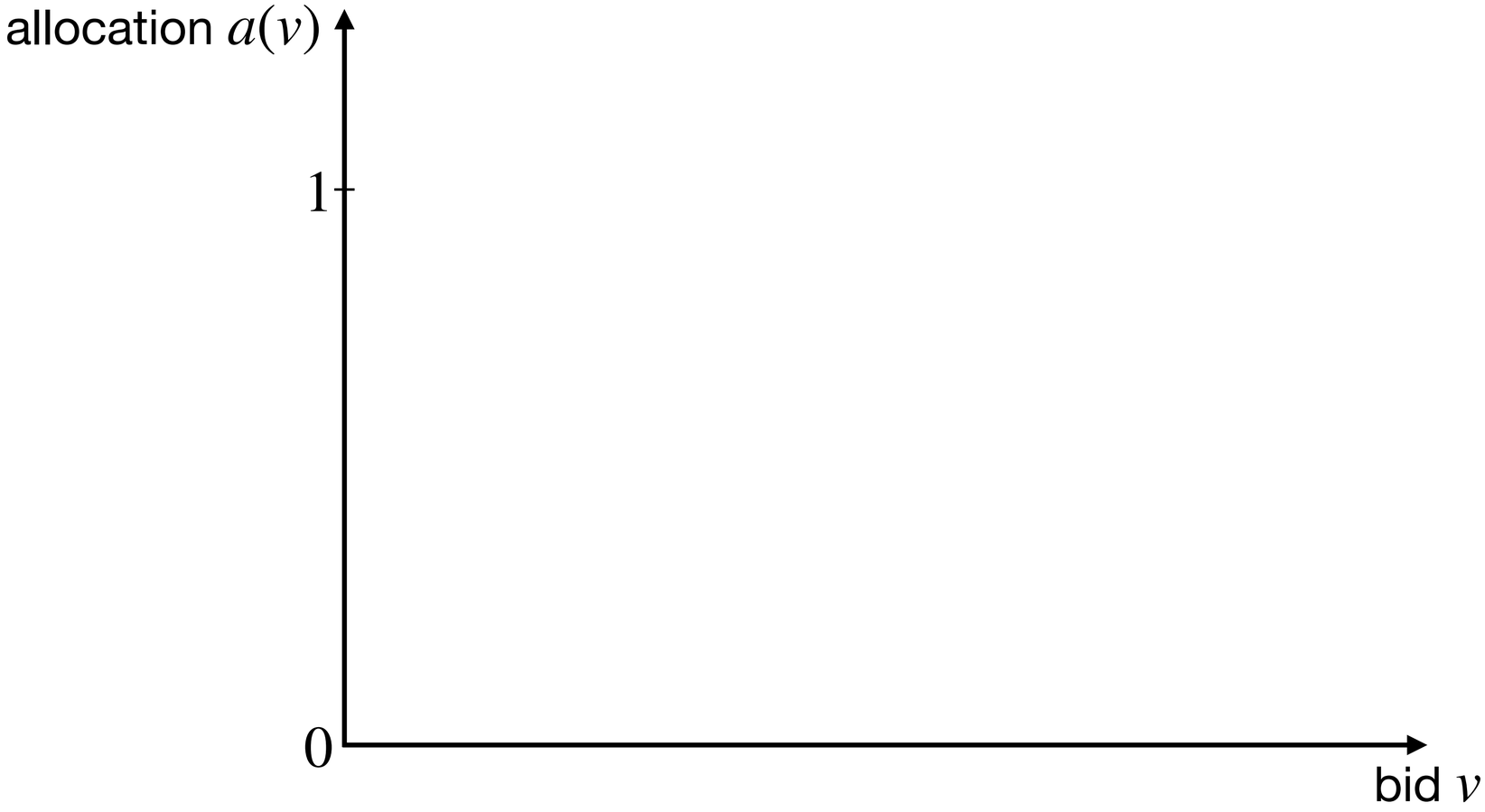

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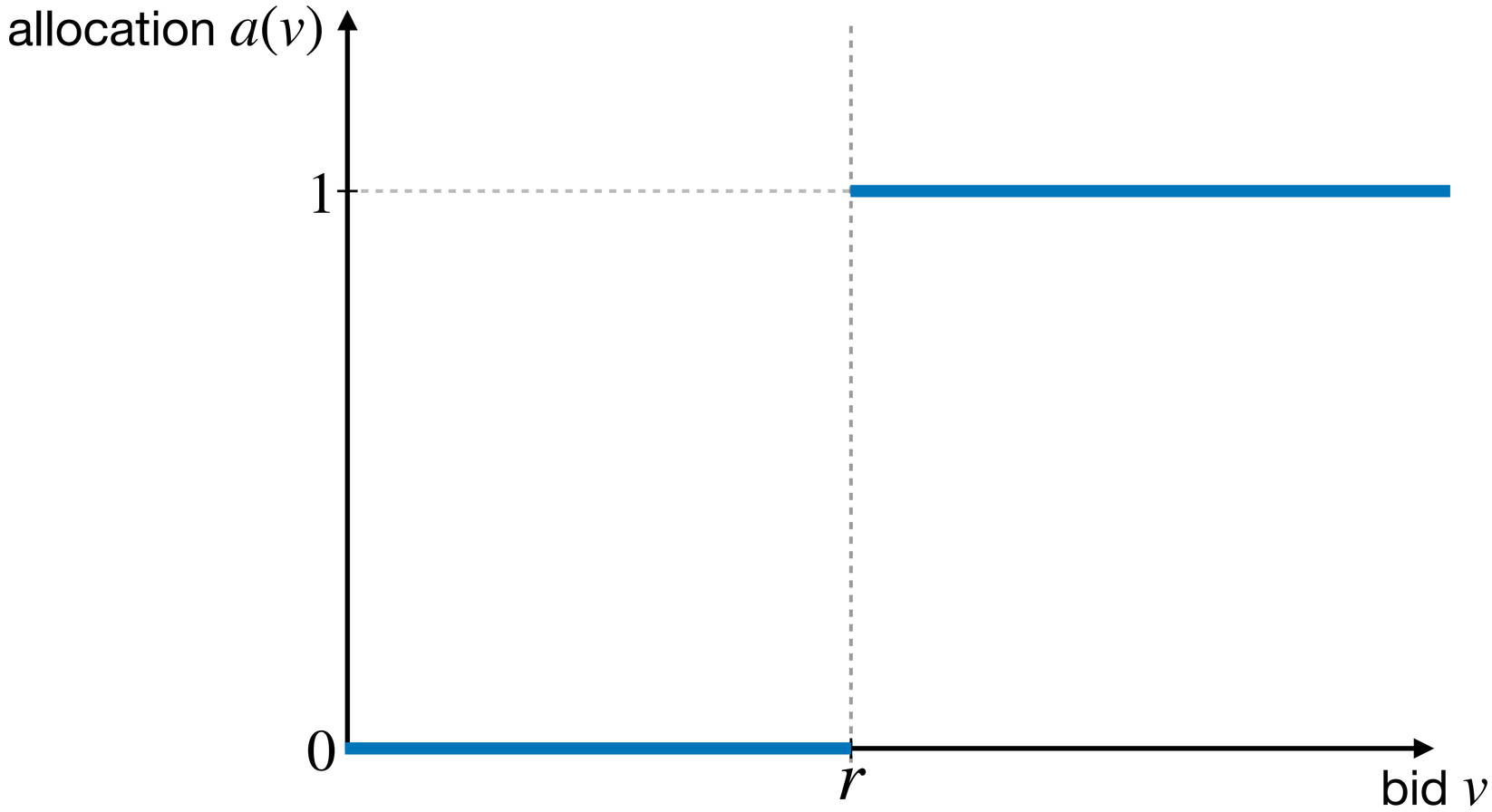

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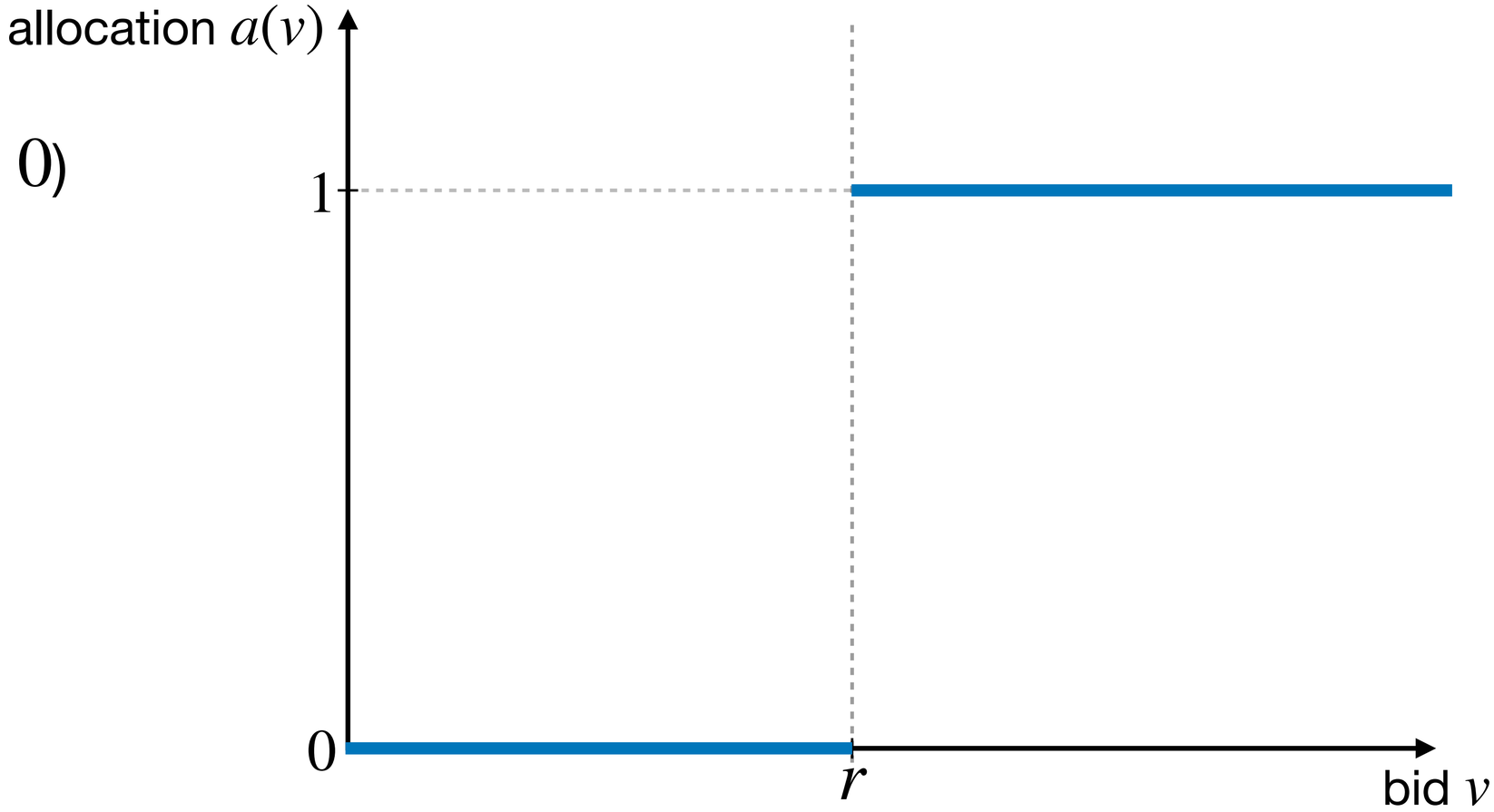

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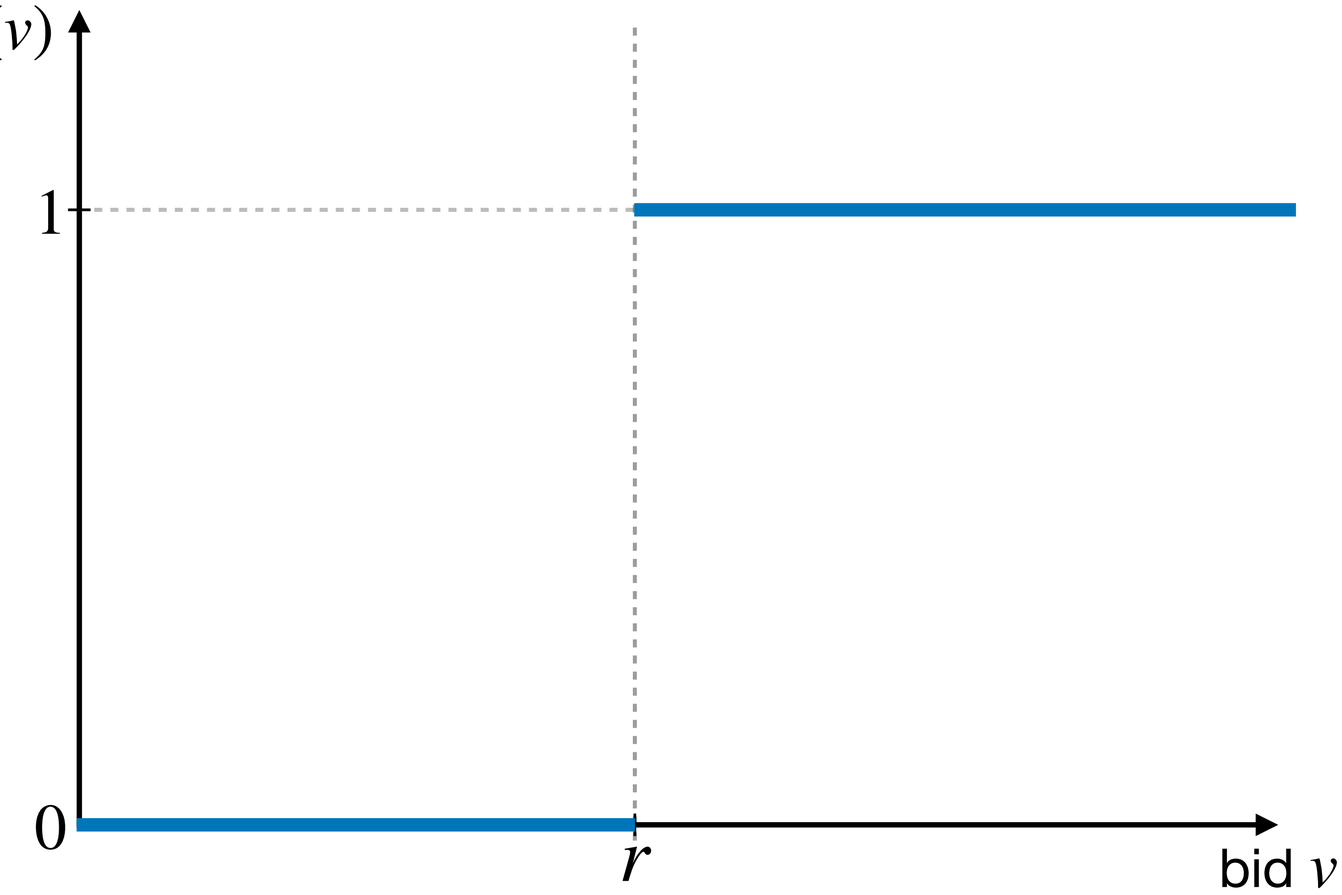

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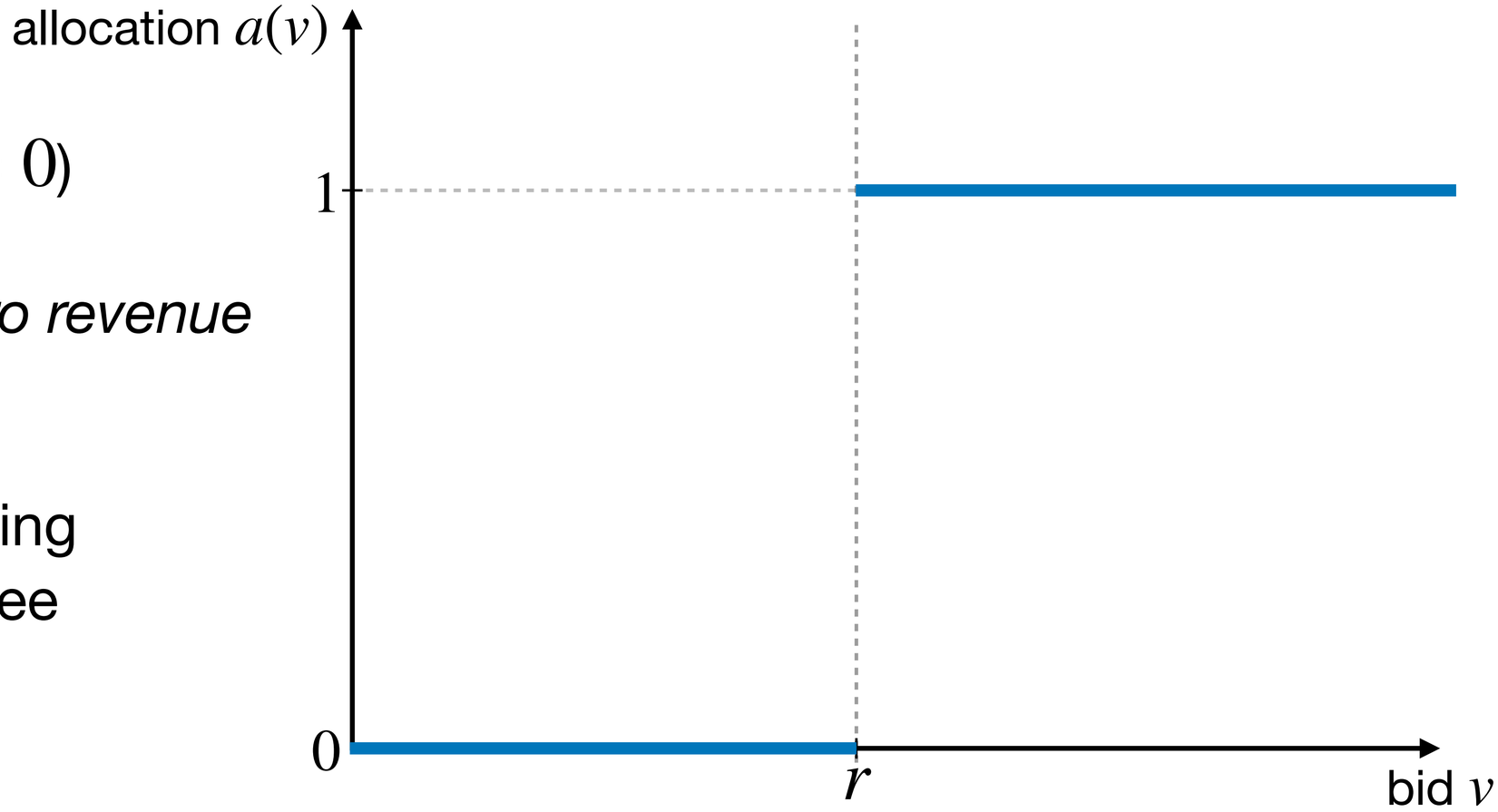

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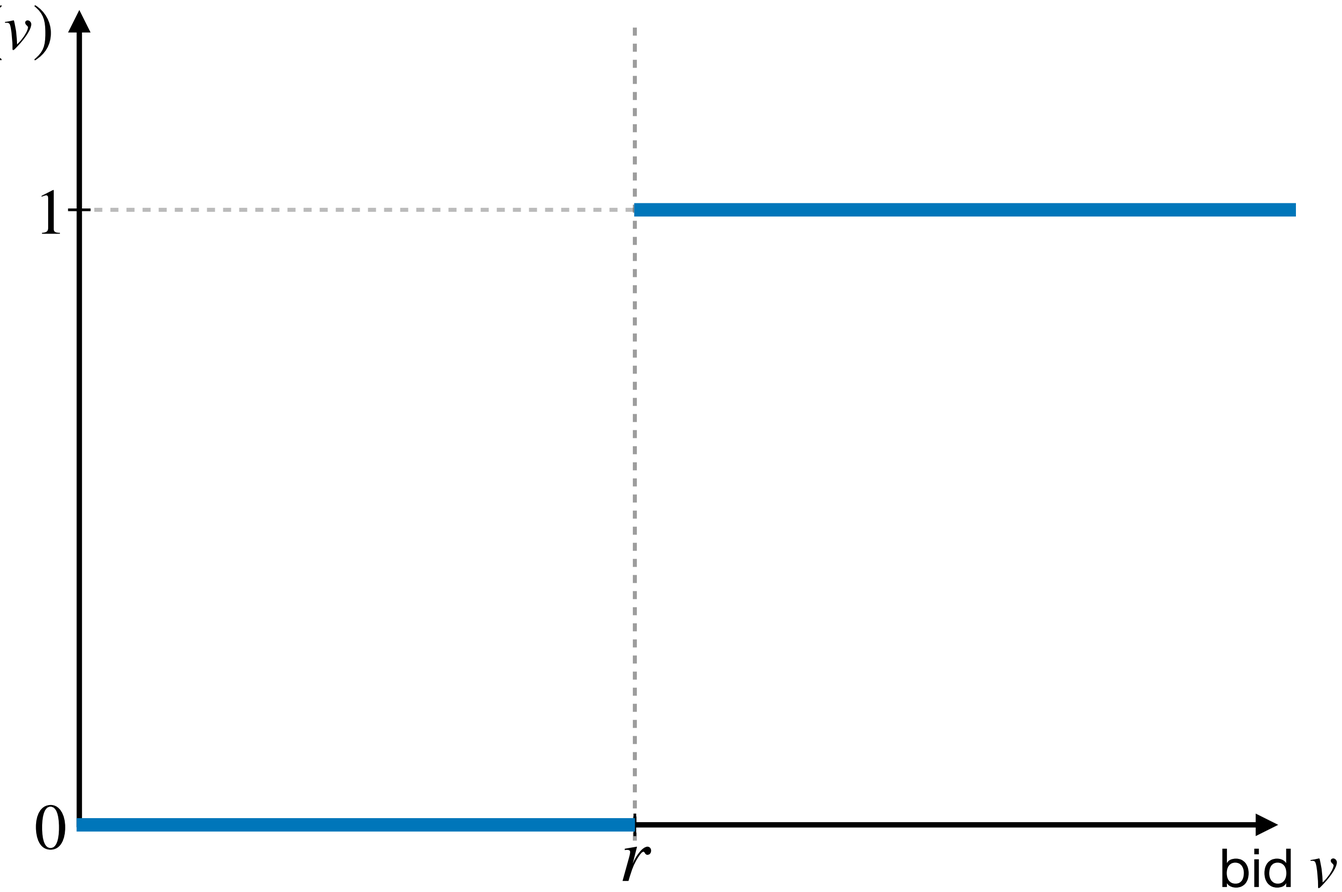
  - *Highly* dependent on the (private) value v of the bidder.



Bayesian (Single-Item) Auctions

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  - drawn from **distributions** ("priors") $F_1, F_2, \ldots, F_n$ supported over $[0,1]$

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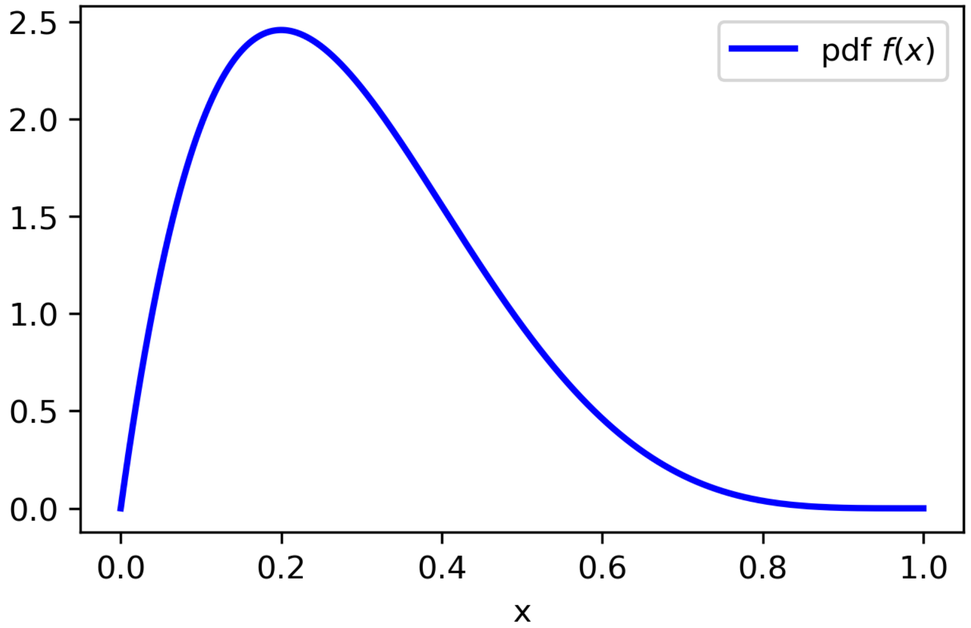

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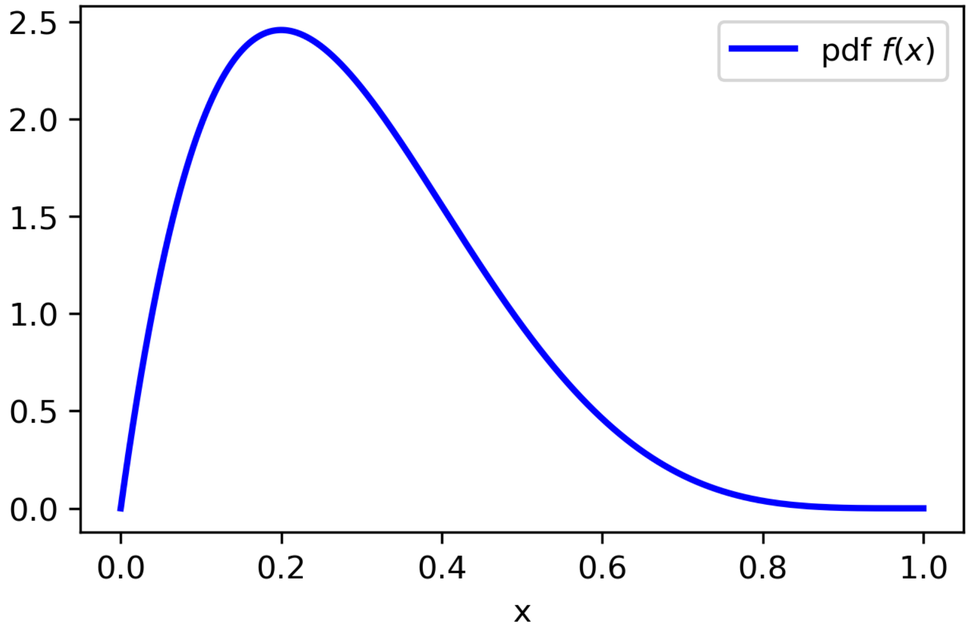

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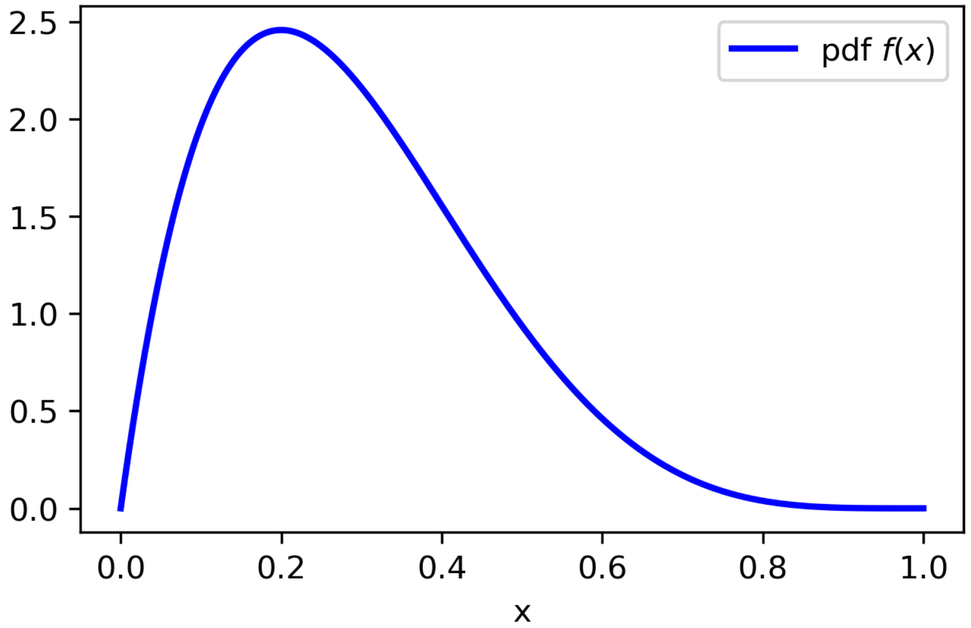

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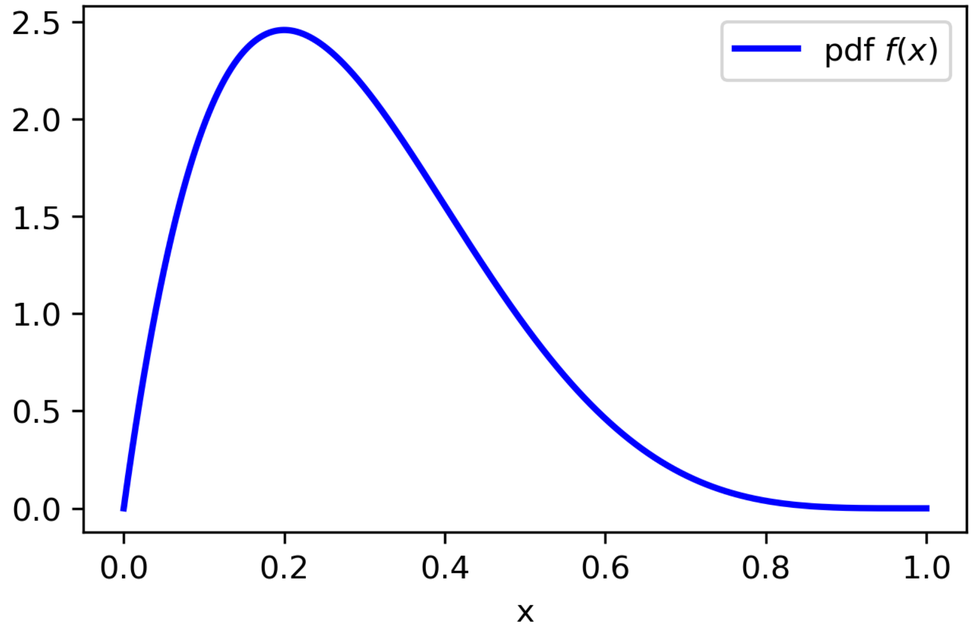

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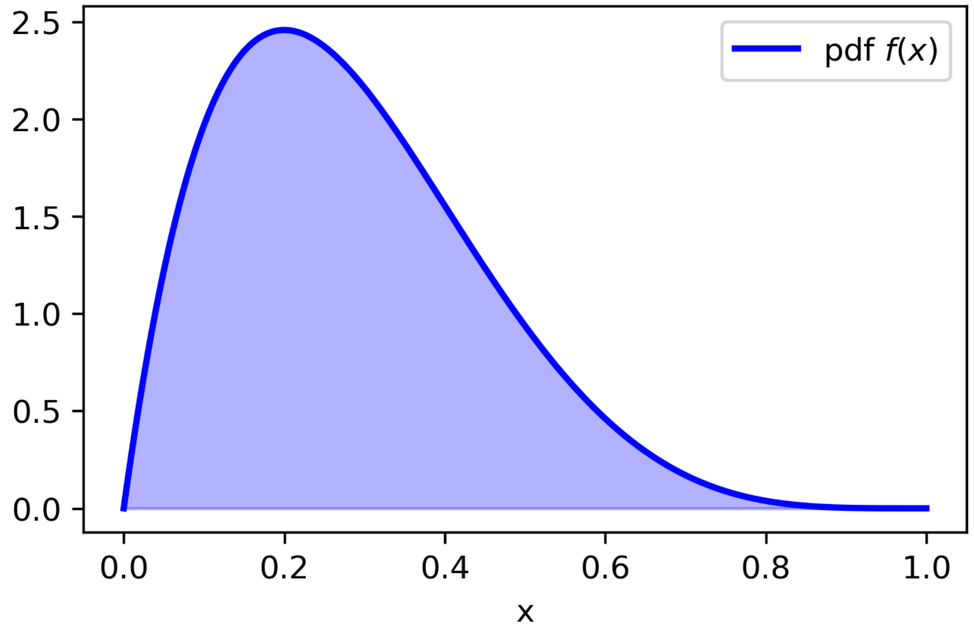

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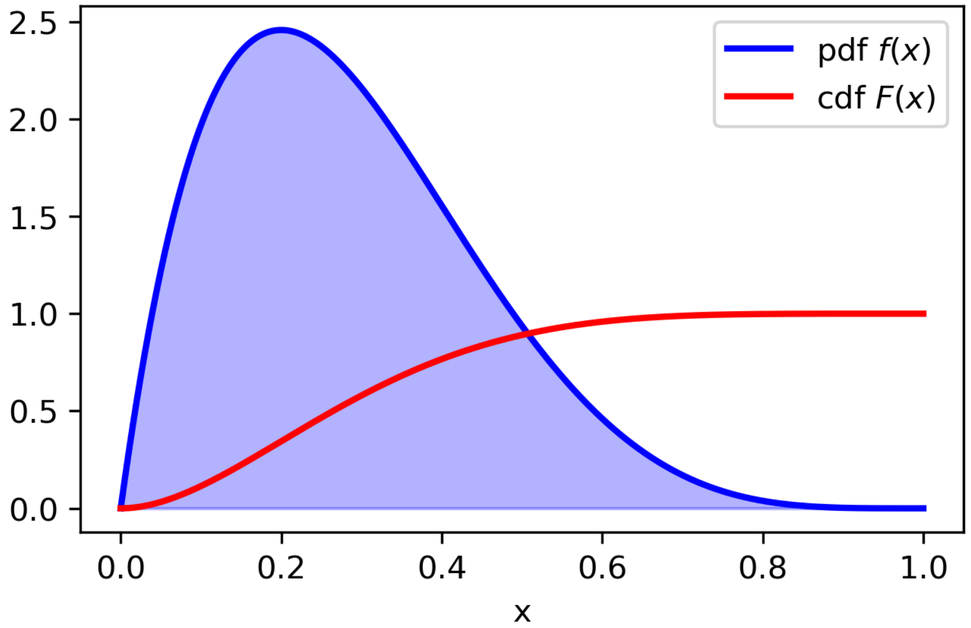

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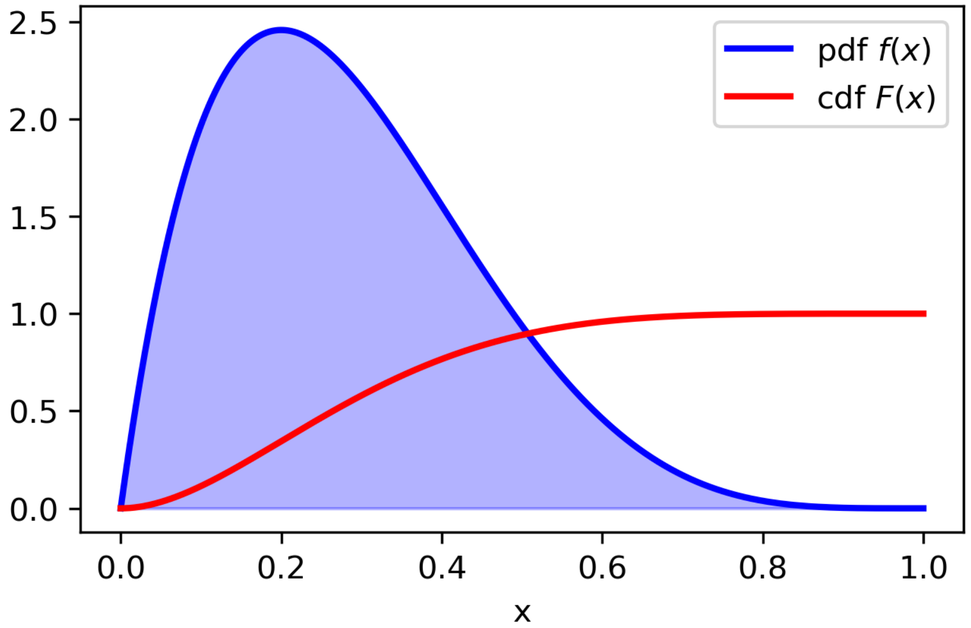

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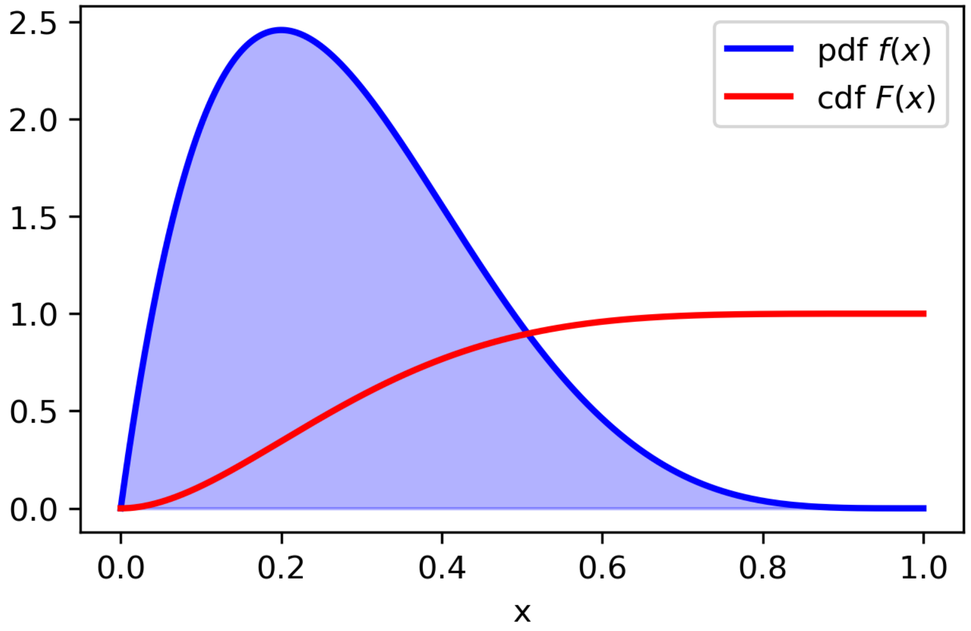

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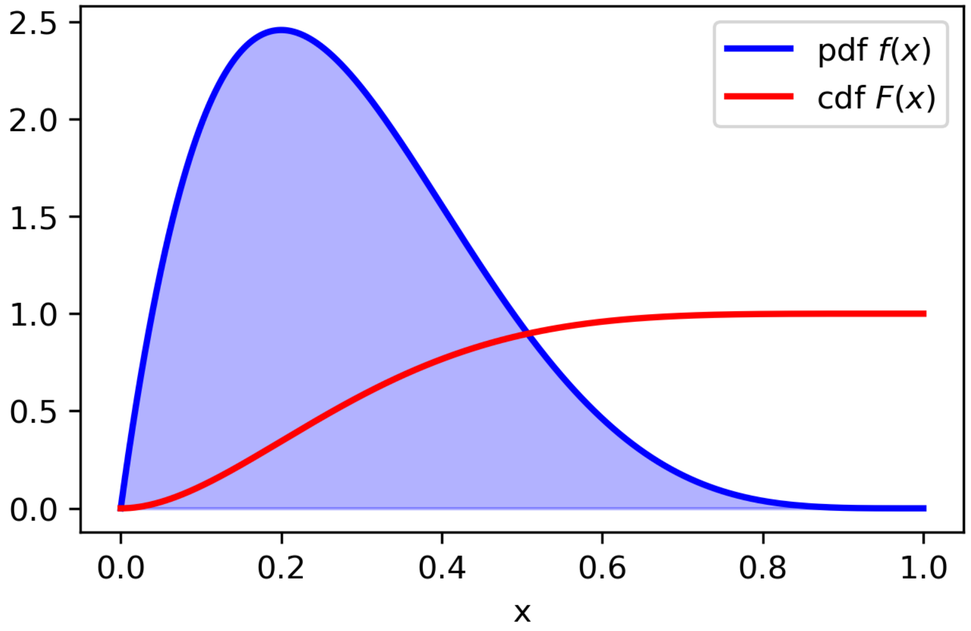

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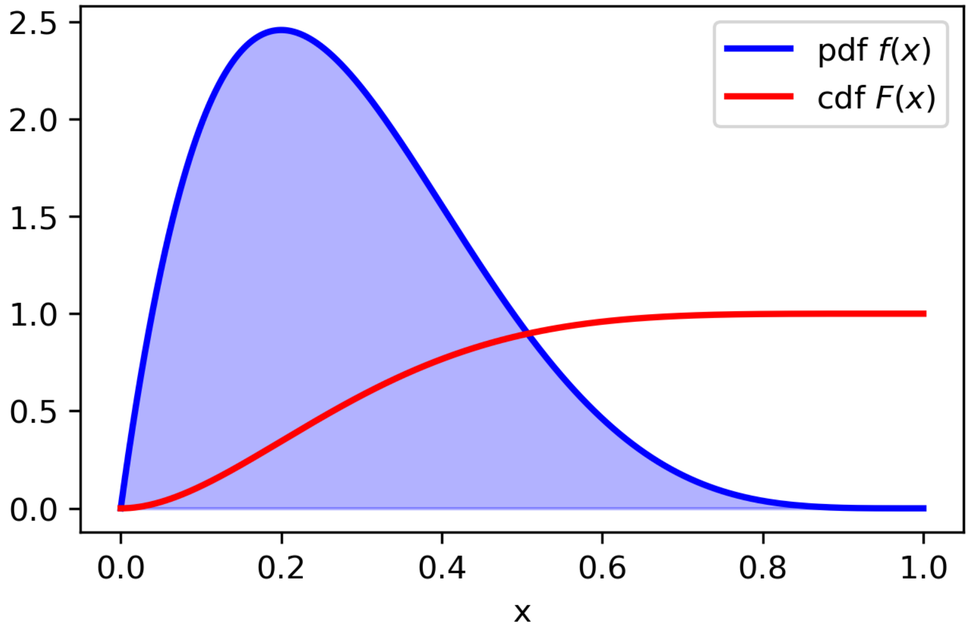

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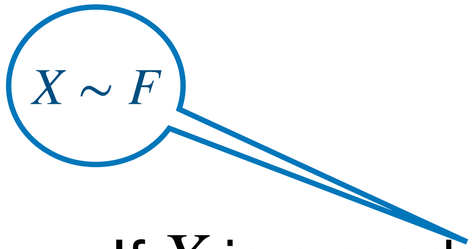

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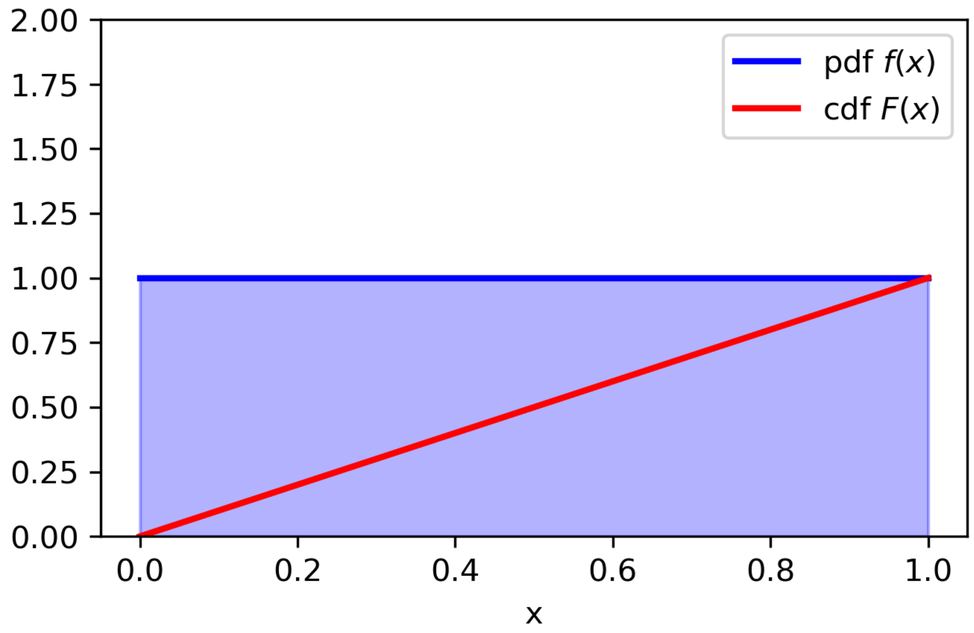

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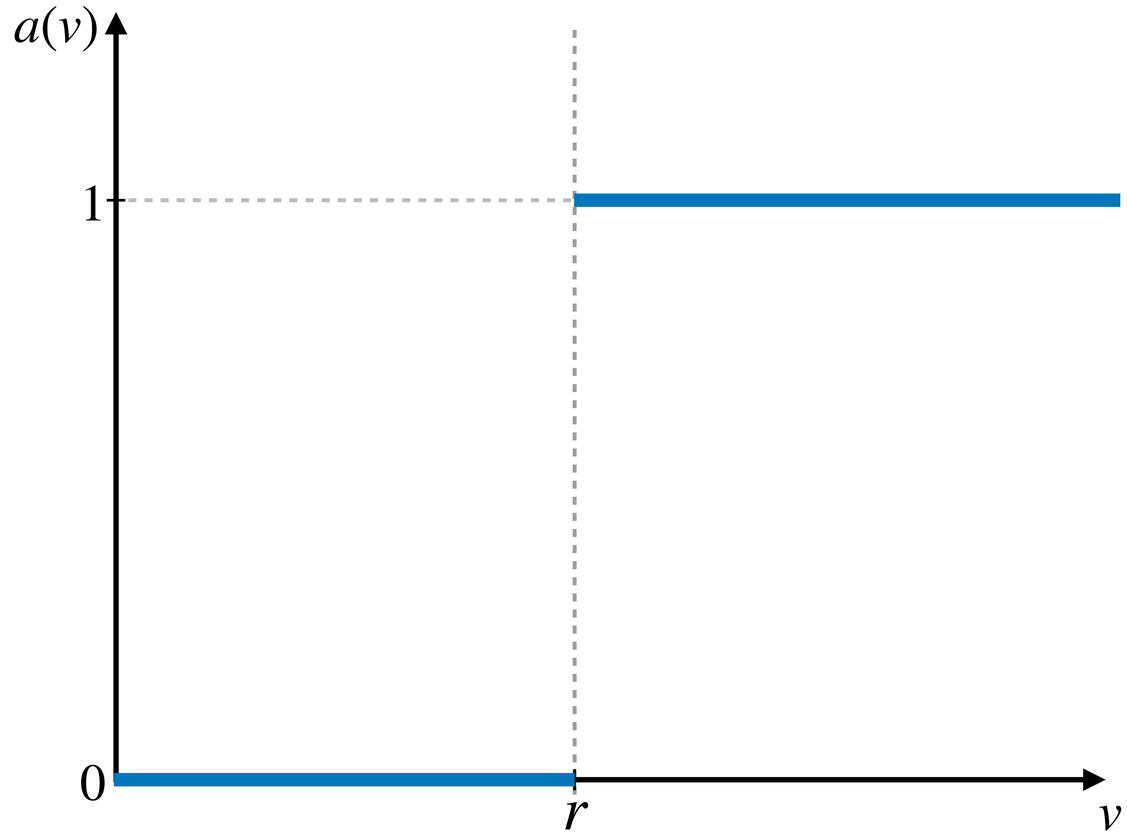

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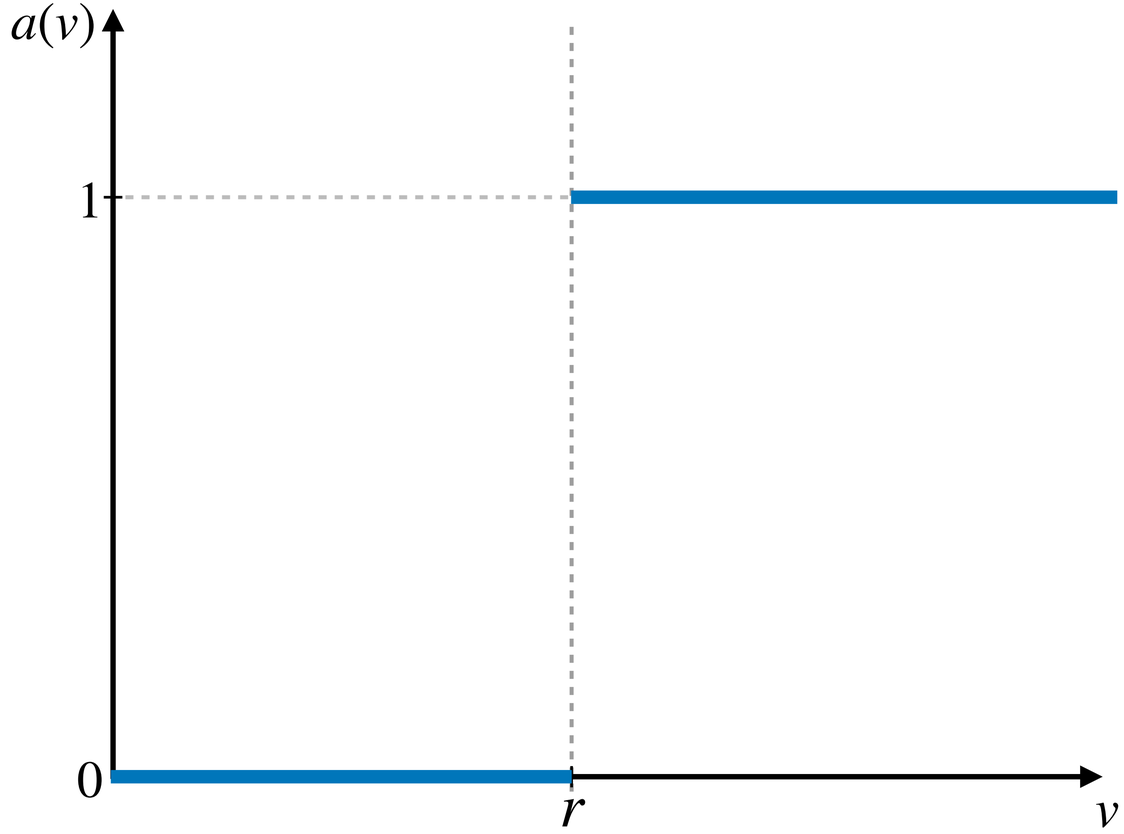

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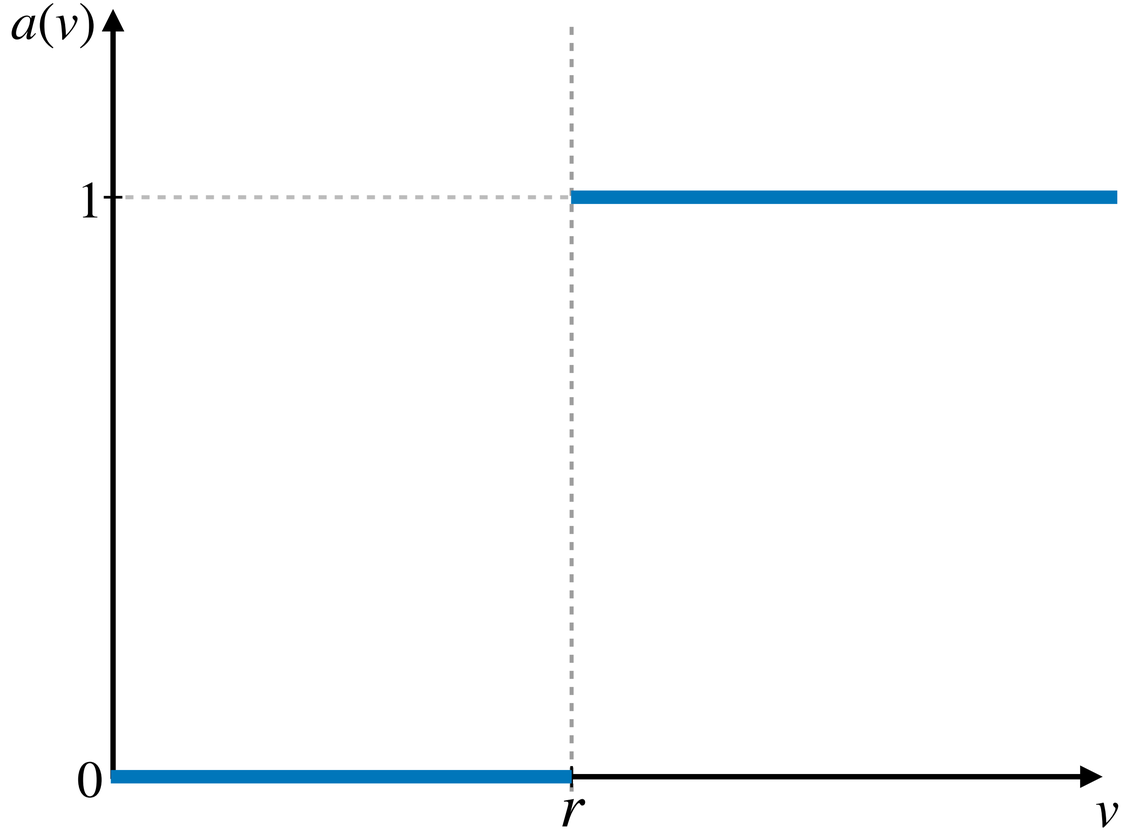

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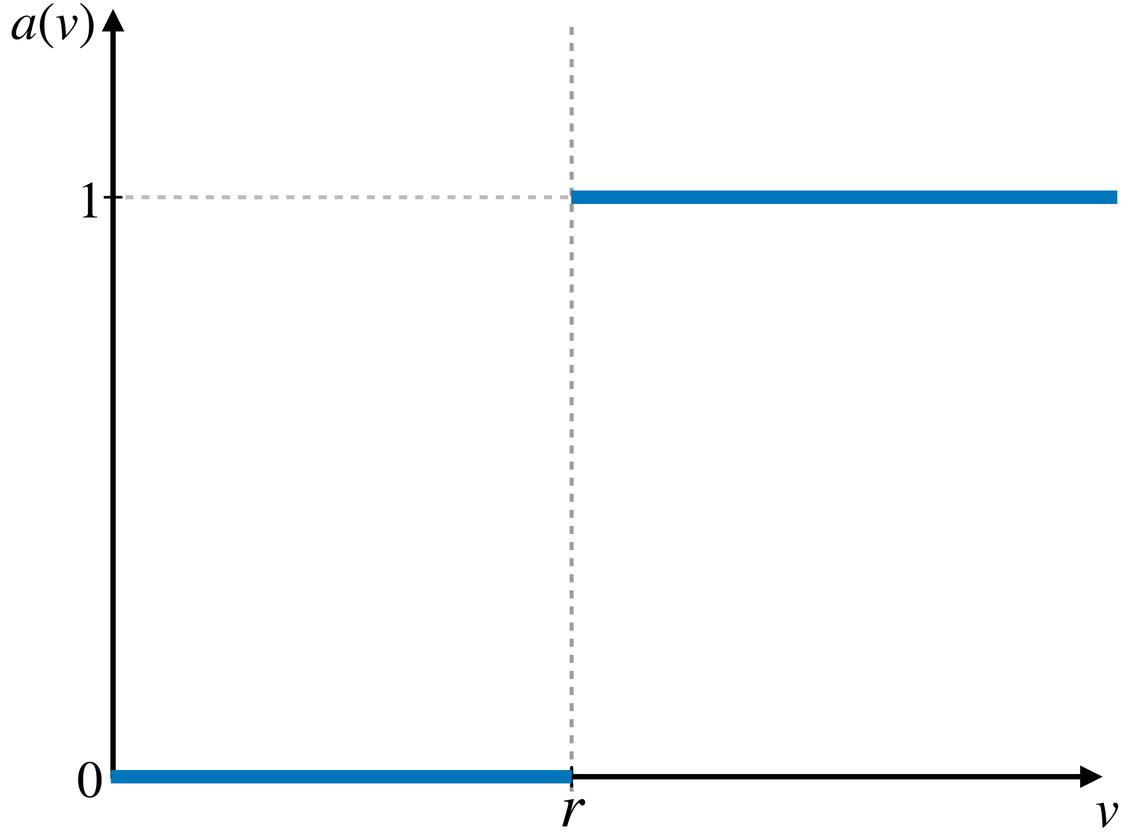

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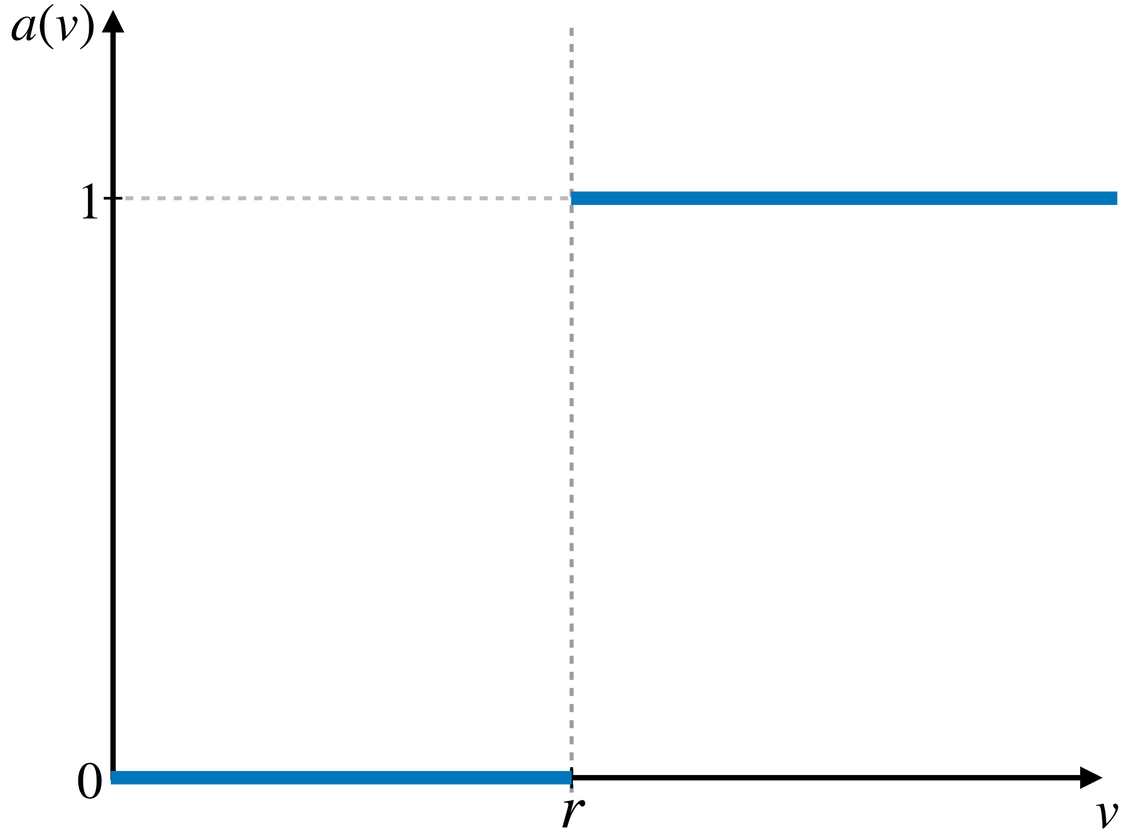

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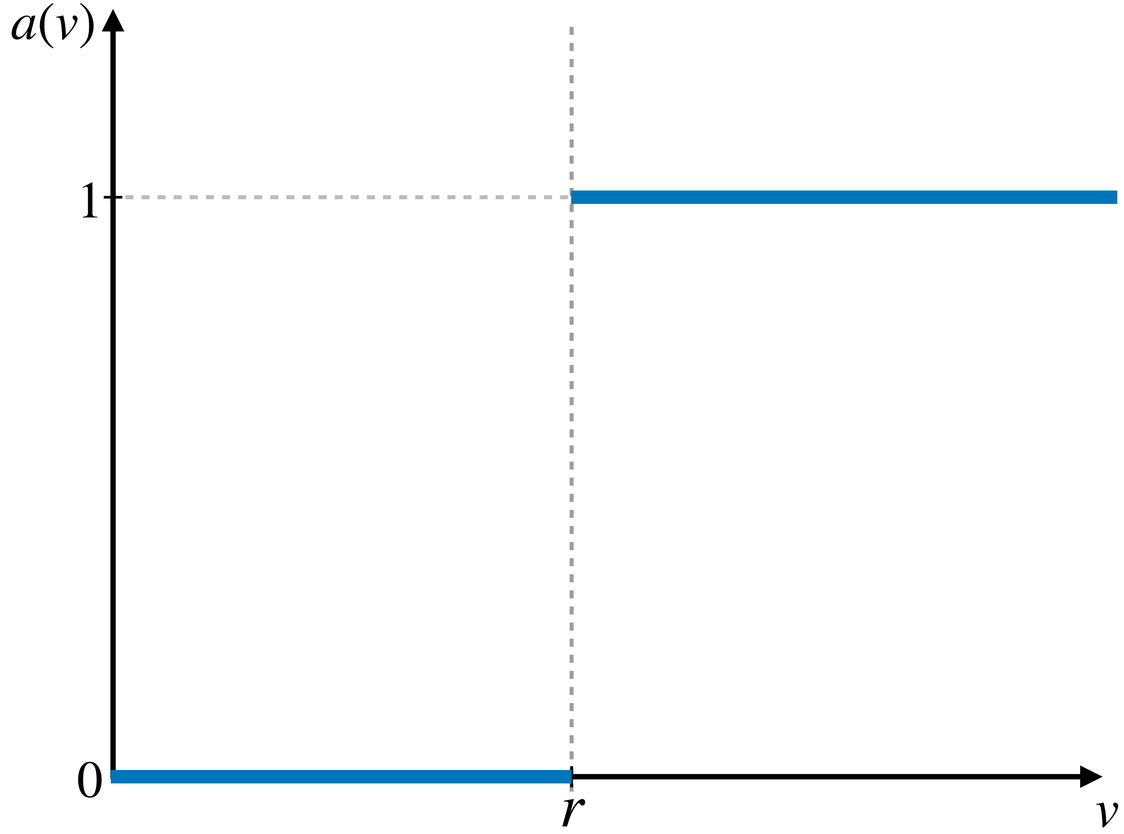

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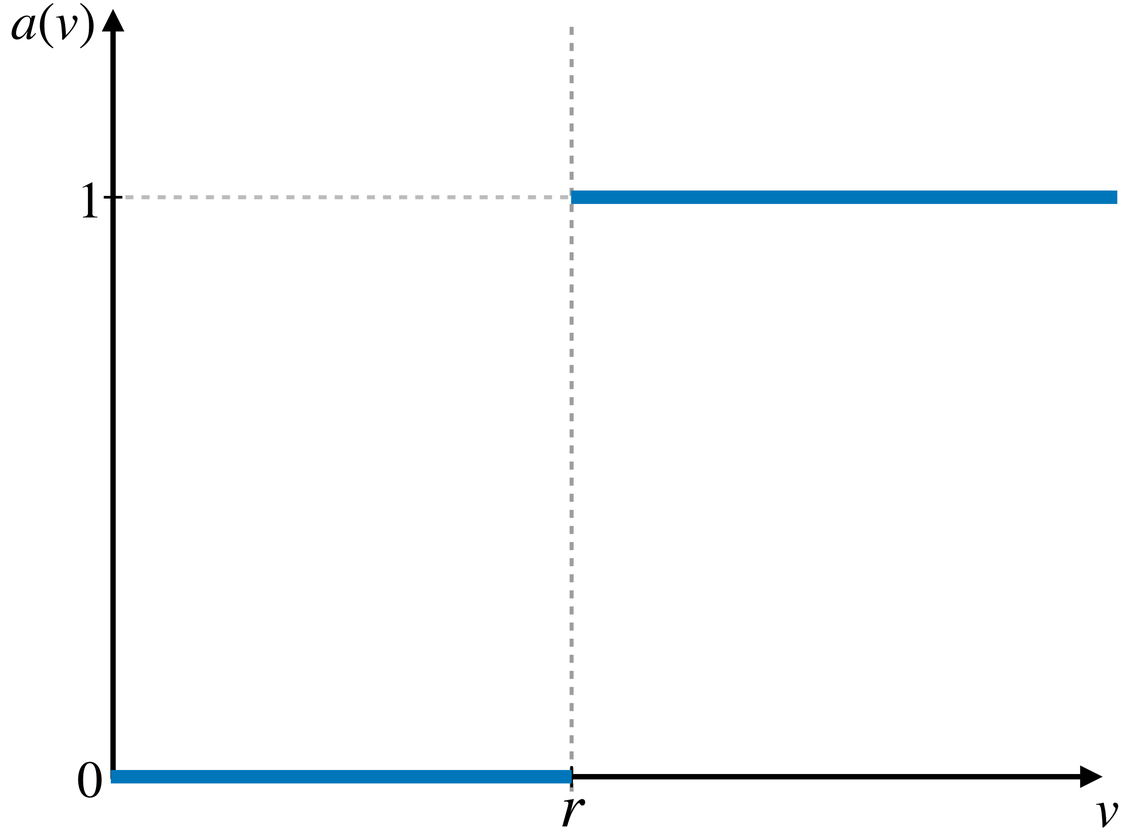

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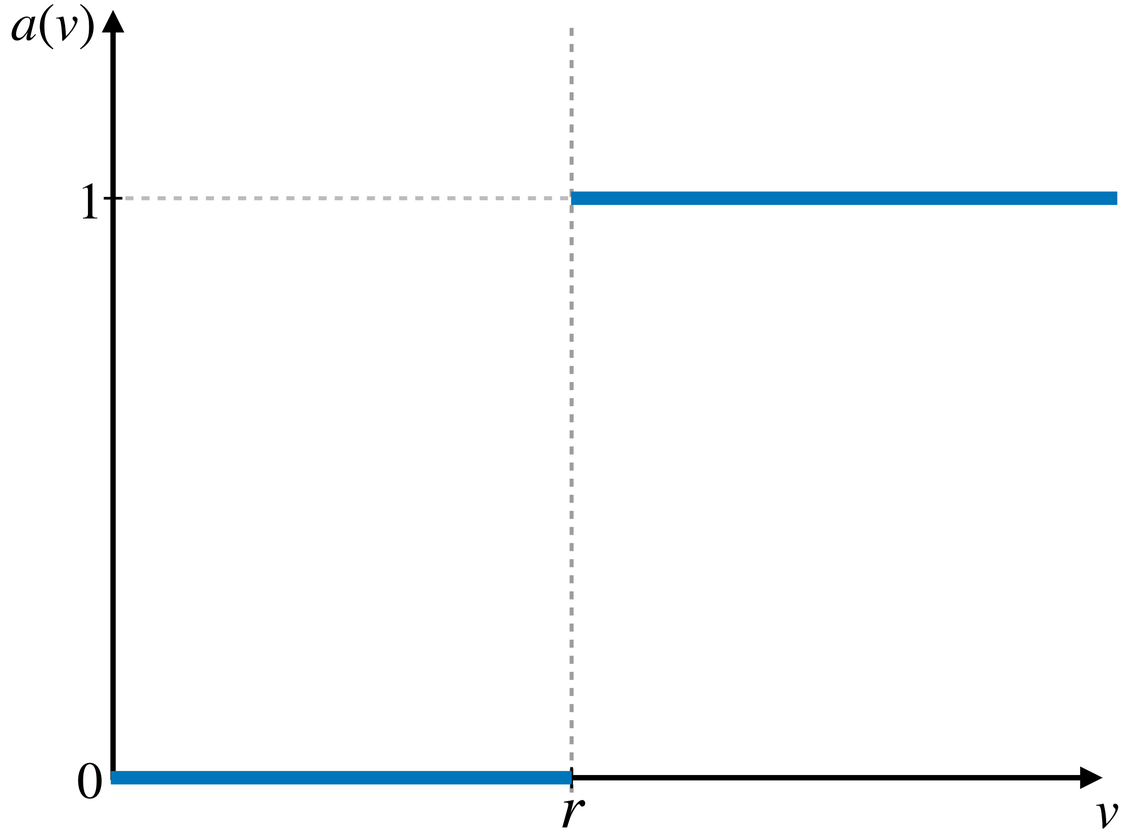

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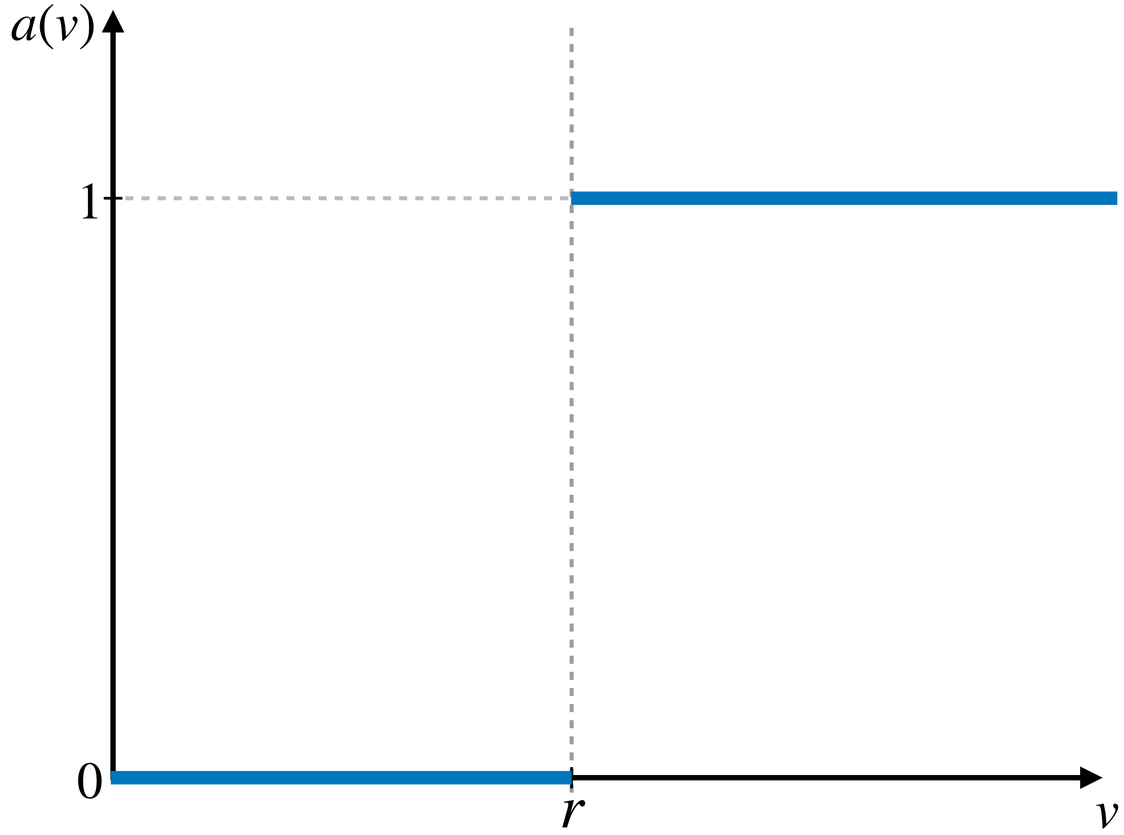

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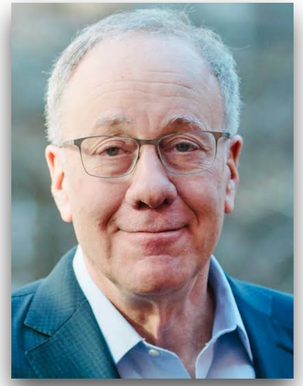

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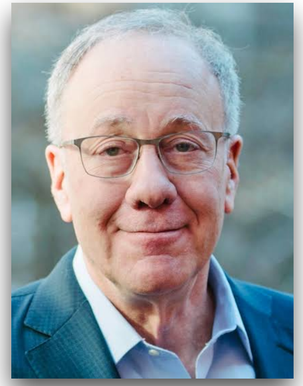

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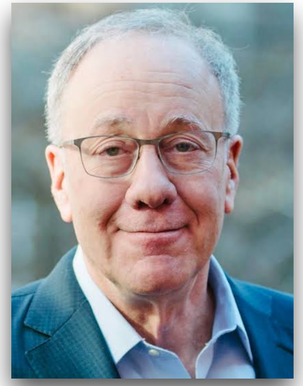

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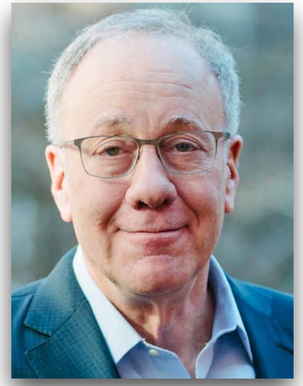

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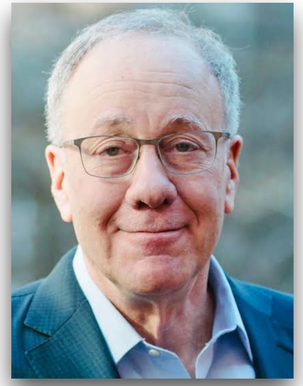

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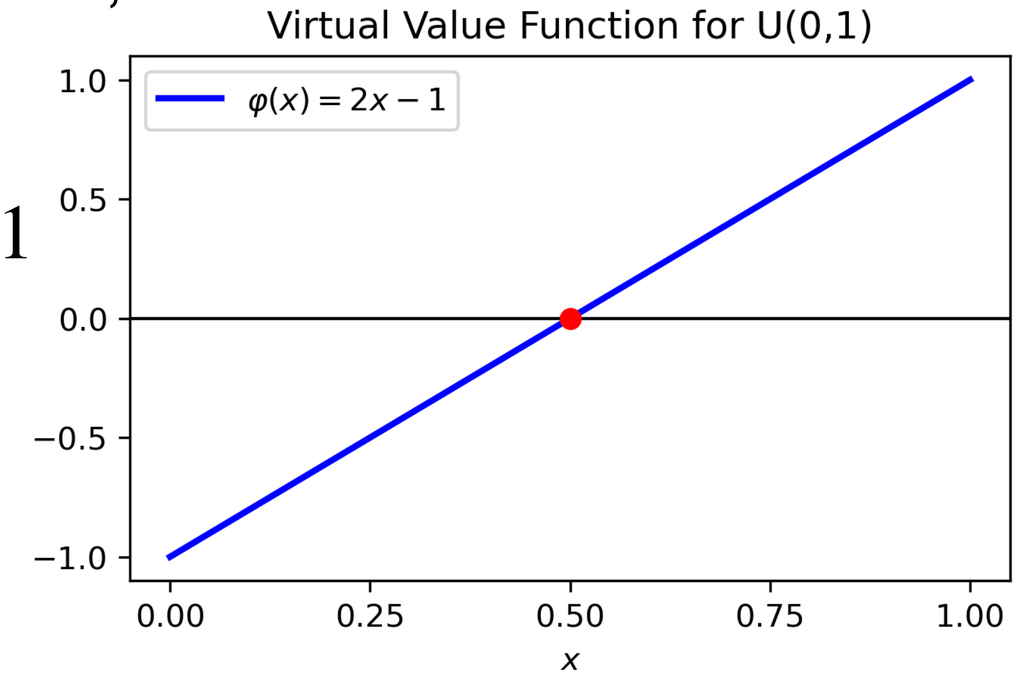

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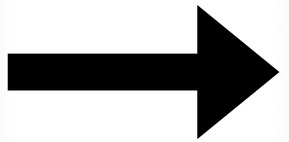

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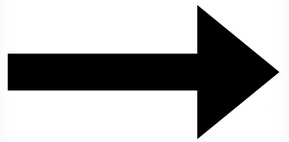

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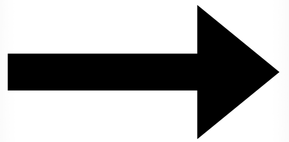

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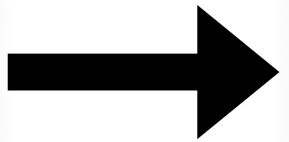

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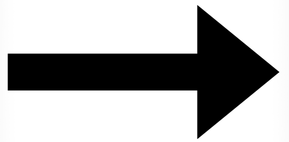

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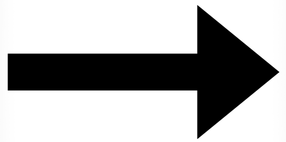

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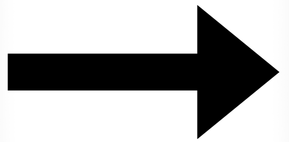

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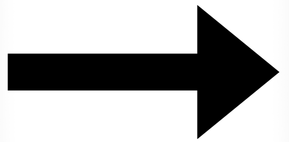

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Assuming *strictly regular* priors:

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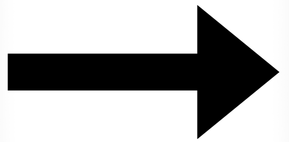

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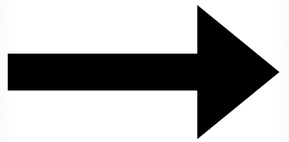

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- Simplify notation: $v := v_i$, $a(v) := a_i(v, \boldsymbol{v}_{-i})$, $p(v) := p_i(v, \boldsymbol{v}_{-i})$, $F(v) := F_i(v)$, ...

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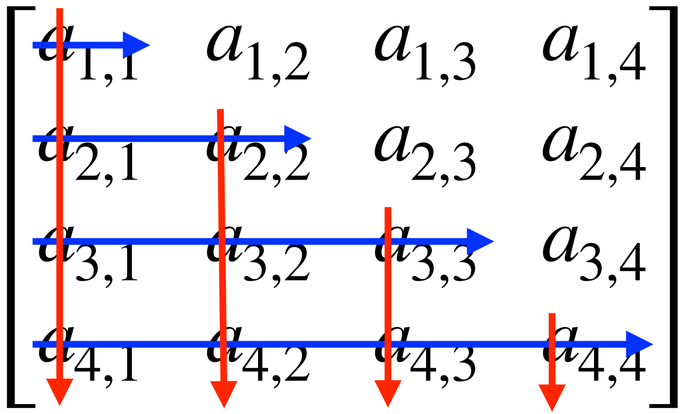

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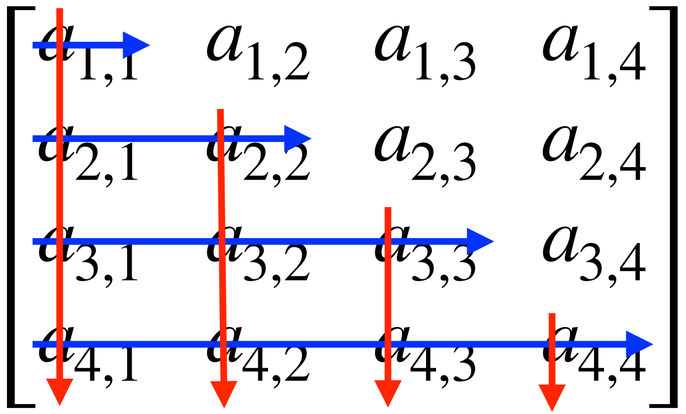

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“Simplicity” vs Optimality

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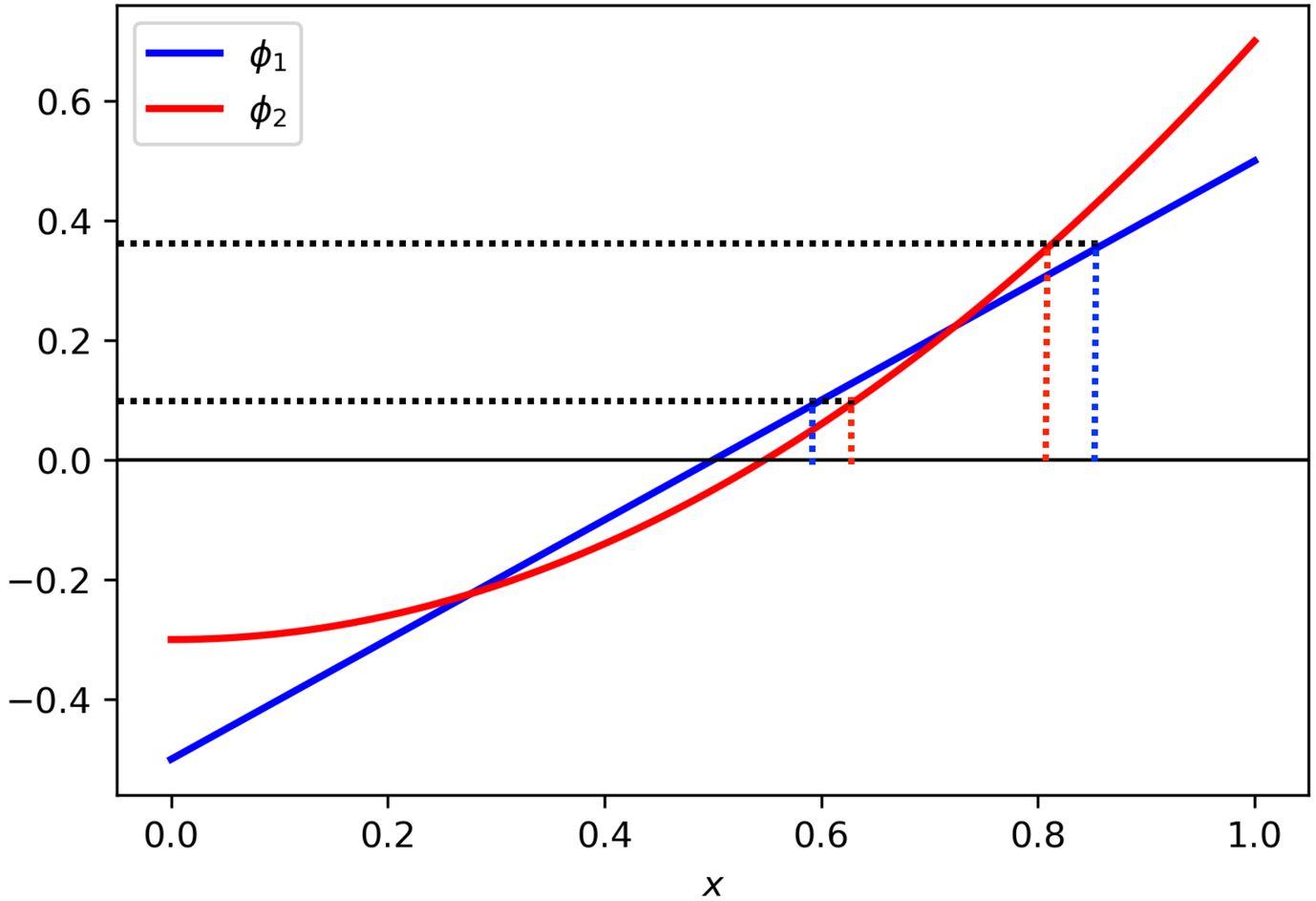

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The Bullow-Klemperer Approximation

Identical Bidders

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THEOREM (J. Bulow & P. Klemperer [1996])

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The Bullow-Klemperer Approximation

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COROLLARY

For n bidders with regular iid priors, the second-price auction achieves at least a $\frac{n-1}{n}$-fraction of the optimal expected revenue.

Second-Price with Reserves & Pricing

Optimal Revenue Approximation for Non-Identical Regular Bidders

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A Small Glimpse Beyond: Multi-item Auctions

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