AGTA Tutorial 7

Exercise 1.

In a VCG-based auction, four *identical* items are being auctioned simultaneously. Suppose there are three buyers (bidders), A, B, and C who provide their claimed valuation functions v_A, v_B, and v_C as follows; $v_x(j)$ denotes the value, in pounds, that bidder x has for receiving j items:

	valuation				
bidder x	$v_x(0)$	$v_x(1)$	$v_x(2)$	$v_x(3)$	$v_x(4)$
$x := A$	0	3	6	9	12
$x := B$	0	2	7	9	14
$x := C$	0	3	8	11	13

An allocation outcome for this auction is specified by three numbers $j_A, j_B, j_C \in \{0, 1, 2, 3, 4\}$, such that for $x \in \{A, B, C\}$, j_x is the number of (identical) items allocated to bidder x, and such that $j_A + j_B + j_C \leq 4$. Each bidder will also be asked to pay a certain amount (in British pounds), p_A, p_B, and p_C, respectively, for their allocation.

What are VCG allocations, and VCG payments, for this auction? In other words, how many items will each bidder get, and what price will each pay for the items they get, if the VCG mechanism is used?

Exercise 2. Answer the following questions.

A. Explain why the second-price auction is a special case of the VCG mechanism.

B. Explain how to derive the second-price auction allocation and payments from Myerson's characterisation for single-parameter domains. In other words, explain why the allocation rule of the auction is monotone, and why the second price payment for the winner of the auction is the critical value.

C. We saw that the second price auction maximises the social welfare, so its approximation ratio with regard to this objective is 1. What about its Price of Anarchy (with regard to all pure Nash equilibria, not just dominant strategy ones)? Is it also 1? Justify your answer.

Exercise 3. Consider an auction of m goods with n *unit-demand bidders*: a bidder i is unit-demand if there exist $v_i^1, v_i^2, \ldots, v_i^m \in \mathbb{R}_{\geq 0}$ such that for any subset of goods S, we have $v_i(S) = \max_{j \in S} v_i^j$. We also assume $v_i(\emptyset) = 0$. Prove that for auctions with unit-demand bidders, the VCG mechanism can be implemented to run in polynomial time in n and m.

Exercise 4 (Knapsack Auctions)**.** In a *knapsack auction*, each bidder i has a publicly known weight w_i and a privately owned value v_i. A mechanism elicits bids b_i from the bidders (i.e., reported values) and outputs a set of winners $\mathcal{W}$, under the constraint that $\sum_{i \in \mathcal{W}} w_i \leq W$, for a given weight *capacity* W. The goal of the mechanism design is to design a *truthful* mechanism that maximises the sum of bids (or, *reported values*), i.e., $\sum_{i \in \mathcal{W}} b_i$.

Unfortunately, the problem of maximising the social welfare in a knapsack auction is a known NP-hard problem. For that reason, the mechanism designer decides to apply the following greedy algorithm for approximating the maximum social welfare:

- Initialise $\mathcal{X} = \emptyset$, and let $i^* \in \arg\max_i b_i$ be a bidder with the highest bid.
- Sort the bidders in term of non-increasing *bang-per-back*, i.e., b_i/w_i.
- Add bidders one by one to $\mathcal{X}$ until adding the next bidder would violate the capacity constraint.
- If the social welfare of the bidders in $\mathcal{X}$ is at least b_{i^*}, let $\mathcal{W} = \mathcal{X}$, otherwise let $\mathcal{W} = \{i^*\}$. You may assume that $w_i \leq W$ for any bidder i, as otherwise this bidder can not be a winner in any feasible solutions, and we can disregard the bidder from the auction.

It can be shown that this algorithm achieves a 2-approximation to the maximum social welfare (*bonus: think about how to prove that*), as long as the elicited bids are the truthful bids.

A. Show that the algorithm used above induces a monotone rule.

B. Compute the appropriate payments so that the resulting mechanism is truthful.