AGTA Tutorial 7 Solutions

Exercise 1.

In a VCG-based auction, four *identical* items are being auctioned simultaneously. Suppose there are three buyers (bidders), A, B, and C who provide their claimed valuation functions v_A, v_B, and v_C as follows; $v_x(j)$ denotes the value, in pounds, that bidder x has for receiving j items:

	valuation				
bidder x	$v_x(0)$	$v_x(1)$	$v_x(2)$	$v_x(3)$	$v_x(4)$
$x := A$	0	3	6	9	12
$x := B$	0	2	7	9	14
$x := C$	0	3	8	11	13

An allocation outcome for this auction is specified by three numbers $j_A, j_B, j_C \in \{0, 1, 2, 3, 4\}$, such that for $x \in \{A, B, C\}$, j_x is the number of (identical) items allocated to bidder x, and such that $j_A + j_B + j_C \leq 4$. Each bidder will also be asked to pay a certain amount (in British pounds), p_A, p_B, and p_C, respectively, for their allocation.

What are VCG allocations, and VCG payments, for this auction? In other words, how many items will each bidder get, and what price will each pay for the items they get, if the VCG mechanism is used?

Solution 1. By inspection, we can see that the maximum achievable social welfare is 15. The allocation that achieves this social welfare is one in which bidder B is receiving two of the items and bidder C is receiving the remaining two items. To compute the payment of each bidder i, we have to compute the difference between the maximum social welfare achievable had i not participated in the auction, and the social welfare of the the allocation above, where i has participated in the action, but excluding the bidder's contribution. Using this we can calculate:

$$p_A = 15 - 15 = 0, \quad p_B = 14 - 8 = 6, \quad \text{and} \quad p_C = 14 - 7 = 7.$$

Exercise 2. Answer the following questions.

A. Explain why the second-price auction is a special case of the VCG mechanism.

B. Explain how to derive the second-price auction allocation and payments from Myerson's characterisation for single-parameter domains. In other words, explain why the allocation rule of the auction is monotone, and why the second price payment for the winner of the auction is the critical value.

C. We saw that the second price auction maximises the social welfare, so its approximation ratio with regard to this objective is 1. What about its Price of Anarchy (with regard to all pure Nash equilibria, not just dominant strategy ones)? Is it also 1? Justify your answer.

Solution 2. **A.** The second price auction allocates the item to the highest bidder, therefore it maximises the social welfare. Additionally, we can see that the payments of the second-price auction are consistent with those of the VCG mechanism: all bidders besides the winner pay 0, and the highest bidder (the winner) pays the value of the second highest bid (value). This latter quantity is exactly the maximum social welfare if we remove the highest bidder from the auction. Noting that the social welfare of all bidders without the highest bidder (when the highest bidder participates in the auction) is 0, this is precisely the VCG payment.

B. It is straightforward to see that the only possible monotone allocation f assigns the item to the highest bidder. To compute the payments, we can then use the definition of the critical value:

$$c_i(\mathbf{v}_{-i}) = \sup_{v_i : f(v_i, \mathbf{v}_{-i}) \notin W_i} v_i$$

We can then observe that the critical value for the winning bidder is exactly the second highest value. Since the winner pays the critical value, the second condition of Myerson's characterisation is satisfied.

C. Consider an instance of the second-price auction with two bidders with values v_1, v_2, such that $v_1 > v_2$. Consider the strategy profile in which bidder 1 bids $b_1 = 0$ and bidder 2 bids $b_2 > v_1$, and observe that this is a pure Nash equilibrium. Indeed, bidder 2 wins the item and pays 0, so the bidder does not wish to change her bid. Bidder 1 loses the item and has utility 0; to change her utility, she would have to bid $b_1' > b_2$, but then she would have to pay $b_2 > v_1$, and her utility would be negative.

In this equilibrium, the social welfare is v_2 and the ratio of the maximum social welfare to the equilibrium social welfare (and hence a lower bound on the Price of Anarchy) is v_1/v_2. This can be made arbitrarily bad.

Remark: There is something "unsatisfying" about this Nash equilibrium: namely that the pure strategy of bidder 2 is *weakly dominated* by the strategy in which the bidder bids truthfully. In other words, the bidder does not really have any reason to bid like this, but, within the definition of Nash equilibria, she might. Sometimes in the literature of the Price of Anarchy, such equilibria are however deemed "unnatural" and are ruled out from consideration.

Exercise 3. Consider an auction of m goods with n *unit-demand bidders*: a bidder i is unit-demand if there exist $v_i^1, v_i^2, \ldots, v_i^m \in \mathbb{R}_{\geq 0}$ such that for any subset of goods S, we have $v_i(S) = \max_{j \in S} v_i^j$. We also assume $v_i(\emptyset) = 0$. Prove that for auctions with unit-demand bidders, the VCG mechanism can be implemented to run in polynomial time in n and m.

Solution 3. To show that the VCG mechanism can be implemented to run in polynomial time, we have to show that the allocation that maxises the social welfare can be computed in polynomial time; the payments can then also be computed in polynomial time, as these involve maximising the social welfare on a subset of bidders.

Consider a complete bipartite graph $G = (A, B, E)$ with the bidders on one side A and the items on the other side B (and one edge $(i, j) \in E$ for every $i \in A$ and $j \in B$). For each edge $(i, j) \in E$, let $w(i, j) = v_i^j$ be the weight of the edge. Consider any arbitrary allocation X and let $y_i \in \arg\max_{j \in X_i} v_i^j$ be the most valuable item that bidder i receives in this allocation. Now consider the matching μ on G such that $\mu(i) = y_i$; this is a matching because in any allocation X, each item can only be assigned to at most one bidder. By definition, the total weight of μ is equal to the social welfare of X, i.e.,

$$\sum_{i \in A} w(i, \mu(i)) = \sum_{i \in A} w(i, y_i) = \sum_{i \in A} \max_{j \in X_i} v_i^j = \sum_{i \in A} v_i(X_i),$$

where the last equation follows from the definition of the unit-demand bidders. Conversely, for any matching μ in G with total weight $\sum_{i \in A} w(i, \mu(i))$, we can construct an allocation X with the same social welfare as this weight, by assigning to every bidder only the item that they are matched with in μ (or no item, if they are not matched with any item in μ).

From the above, it follows that a matching of maximum weight corresponds to an allocation of maximum welfare and vice versa. A maximum weight matching can be computed in polynomial time (e.g., via using a polynomial-time algorithm for max-flow), and hence the social welfare-maximising allocation can be computed in polynomial time as well.

Exercise 4 (Knapsack Auctions)**.** In a *knapsack auction*, each bidder i has a publicly known weight w_i and a privately owned value v_i. A mechanism elicits bids b_i from the bidders (i.e., reported values) and outputs a set of winners $\mathcal{W}$, under the constraint that $\sum_{i \in \mathcal{W}} w_i \leq W$, for a given weight *capacity* W. The goal of the mechanism design is to design a *truthful* mechanism that maximises the sum of bids (or, *reported values*), i.e., $\sum_{i \in \mathcal{W}} b_i$.

Unfortunately, the problem of maximising the social welfare in a knapsack auction is a known NP-hard problem. For that reason, the mechanism designer decides to apply the following greedy algorithm for approximating the maximum social welfare:

- Initialise $\mathcal{X} = \emptyset$, and let $i^* \in \arg\max_i b_i$ be a bidder with the highest bid.
- Sort the bidders in term of non-increasing *bang-per-back*, i.e., b_i/w_i.
- Add bidders one by one to $\mathcal{X}$ until adding the next bidder would violate the capacity constraint.
- If the social welfare of the bidders in $\mathcal{X}$ is at least b_{i^*}, let $\mathcal{W} = \mathcal{X}$, otherwise let $\mathcal{W} = \{i^*\}$. You may assume that $w_i \leq W$ for any bidder i, as otherwise this bidder can not be a winner in any feasible solutions, and we can disregard the bidder from the auction.

It can be shown that this algorithm achieves a 2-approximation to the maximum social welfare (*bonus: think about how to prove that*), as long as the elicited bids are the truthful bids.

A. Show that the algorithm used above induces a monotone rule.

B. Compute the appropriate payments so that the resulting mechanism is truthful.

Solution 4.

A. To show that the algorithm induces a monotone allocation rule, we need to show the following: Assume that a bidder i is a winner when she bids b_i; then she will still be a winner when she bids $b'_i > b_i$. We consider two cases:

(a) Bidder i is a winner outside the set $\mathcal{X}$ (which means that bidder i is bidder i^* and the sum of bids of the bidders in $\mathcal{X}$ is smaller than b_{i^*}). If the bidder bids $b'_i > b_i = b_{i^*}$ instead, she will still be the winner with the highest bid, and the value of that bid will be higher than the sum of bids of bidders in $\mathcal{X}$. Hence, $\mathcal{W} = \{i^*\}$ in this case.

(b) Bidder i is a winner in the set $\mathcal{X}$ (which means that the sum of bids of bidders in $\mathcal{X}$ is at least b_{i^*}). When bidding b'_i, the bidder's bang-per-buck increases, i.e., $b'_i/w_i > b_i/w_i$, which implies that, since the bidder was part of $\mathcal{X}$ before, she will still be part of $\mathcal{X}$, as $\mathcal{X}$ is constructed by adding the bidders with the highest bang-per-buck greedily. Hence, even if the relative order in which bidders are included in $\mathcal{X}$ has changed, exactly the same set of bidders will be included in the set, but the sum of bids of $\mathcal{X}$ has increased. Since bidders in $\mathcal{X}$ were winning before, they are still winning after the update.

B. Since this is a single-parameter domain, every winner is going to pay the critical value (bid) and every loser is going to pay 0. Again, we need to consider two cases, depending on whether the winner is in the set $\mathcal{X}$, or a singleton. In the former case, the critical value is the minimum bid that would ensure that the bidder is included in the set, and that the social welfare of $\mathcal{X}$ is higher than any singleton bid.

Let ℓ be the last bidder in the set $\mathcal{X}$ in terms of non-increasing order of bang-per-buck; then, the minimum bid that bidder i has to make to be in the set $\mathcal{X}$ is $b^{(1)} = \frac{b_\ell}{w_\ell} \cdot w_i$. Furthermore, let $b_{i^*} = \max_j b_j$; in order for the set $\mathcal{X}$ to be winning, it has to be the case that

$$\sum_{k \in \mathcal{X} \setminus \{i\}} b_k + b_i \geq b_{i^*} \Rightarrow b^{(2)} = b_i^* - \sum_{k \in \mathcal{X} \setminus \{i\}} b_k$$

Overall, the critical bid (and hence the payment) should be $b^c = \max\{b^{(1)}, b^{(2)}\}$.

In the latter case, the critical value is the minimum bid that would ensure that the bidder has the highest bid, and this bid is higher that the social welfare of the bidders in set $\mathcal{X}$. This can be written as

$$b^c = \max\{\max_{j \neq i} b_k, \quad \sum_{k \in \mathcal{X}} b_k\}.$$