AGTA Tutorial 8 Solutions

Exercise 1. Consider a single-item auction with n bidders with values drawn independently from the uniform distribution over $[0, 1]$.

A. Prove that, when $n = 2$, the expected revenue of the second price auction in this case is $1/3$.

B. Prove that, when $n = 2$, the expected revenue of the second price auction with reserve price at $1/2$ in this case is $5/12$.

C. What is the maximum possible expected revenue that can be extracted for this auction when $n = 2$?

D. Prove that the expected revenue of the second-price auction is maximum among all truthful auctions that *always* allocate the item to some bidder.

E. Using D above, prove that the expected revenue of the second-price auction with $n + 1$ bidders is at least as high as the expected revenue of the optimal (revenue-maximising) auction with n bidders (the Bulow-Klemperer Theorem).

Solution 1.

A. Let X_1 and X_2 be the random variables corresponding toe the bidders' valuations drawn from $U[0, 1]$, i.e., we have $X_1, X_2 \sim F_X$, where $F_X(x) = x$. Let $Z = \min\{X_1, X_2\}$ be (a random variable) equal to the second highest valuation. We have

$$\begin{aligned} F_Z(z) &= \Pr(Z \le z) = 1 - \Pr(Z > z) = 1 - \Pr(\min\{X_1, X_2\} > z) \\ &= 1 - \Pr(X_1 > z) \cdot \Pr(X_2 > z) = 1 - (1 - F_X(z))^2 = 1 - (1 - z)^2. \end{aligned}$$

Let $v^{(2)}$ denote the second highest value (bid), and note that this is a random variable as well. The expected revenue of the auction is then

$$\mathbb{E}[v^{(2)}] = \int_0^1 \Pr(Z \ge z) dz = \int_0^1 [1 - F_Z(z)] d_z = \int_0^1 (1 - z)^2 dz = \frac{1}{3}.$$

B. We will calculate the expected revenue of the auction conditioning on three events: (i) both bidders bid below $1/2$, (ii) only one bidder bids at least $1/2$, and (iii) both bidders bid at least $1/2$. Since the value of the bidders are drawn i.i.d. from $U[0, 1]$, the probability of these events is $1/4$, $1/2$, and $1/4$ respectively. Conditioned on (i), the expected revenue is 0, since the item is not sold. Conditioned on (ii), the expected revenue is $1/2$, as the payment to the winner is the reserve price $1/2$. It remains to calculate the expected revenue conditioned on (iii). By the definition of the random variable Z in part (A) above, this is equal to:

$$\frac{1}{4} \cdot \mathbb{E}\left[Z | Z \ge \frac{1}{2}\right] = \int_{1/2}^1 z \cdot 2(1 - z) dz = 1/6,$$

where above we used that $\mathbb{E}[Z|Z \ge \frac{1}{2}] = \frac{\int_{1/2}^1 z \cdot f_Z(z) d_z}{\Pr Z \ge \frac{1}{2}}$.

Overall, the expected revenue is $1/4 + 1/6 = 5/12$.

C. Since the values are drawn from $U[0,1]$, the virtual valuation function is $\phi(v) = 2v - 1$. We know that the optimal (revenue maximising) auction sells the item to the bidder with the highest non-negative virtual valuation, $\phi(v_i)$; if all bidders have negative virtual values, then the item remains unsold. A bidder with value v has negative virtual value if and only if $2v - 1 < 0$, i.e., when $v < 1/2$. In other words, the optimal auction sells the item to the bidder with the highest value, assuming that value is at least $1/2$. The payment is the second highest virtual value if that value is at least $1/2$, otherwise the payment is $1/2$. Therefore, the optimal auction is precisely the second price auction with reserve price at $1/2$, which by (B) above, has expected revenue $5/12$.

By the payment formula for optimal auctions, we know that the expected revenue is the expected maximum non-negative virtual value $\mathbb{E}[\max \phi(v_i)^+]$ where $\phi(v_i)^+ = \max\{\phi(v_i), 0\}$.

D. By Myerson's theorem, we know that the expected revenue of any auction is equal to the expected virtual social welfare, i.e.,

$$\mathbb{E}_{\mathbf{v}\sim F}\left[\sum_{i=1}^{n} p_i(\mathbf{v})\right] = \mathbb{E}_{\mathbf{v}\sim F}\left[\sum_{i=1}^{n} a_i(\mathbf{v}) \cdot \phi_i(v_i)\right]$$

Consider the optimal (revenue-maximising) auction among those that always sells the item to some bidder. Since this auction maximises the expected revenue, by the above, it maximizes the expected virtual social welfare. This means that it allocates the item to the bidder with the highest virtual value (even if this virtual value is negative, as the item is always sold). For the uniform distribution, the virtual valuation function is $\phi(x) = 2x - 1$, and hence the bidder with the highest value is also the bidder with the highest virtual value. Therefore the second-price auction achieves the maximum virtual social welfare, and hence the maximum revenue.

E. Consider the following auction $\mathcal{A}$ on $n+1$ bidders: Pick the first n bidders and run Myerson's optimal auction for those bidders. If the auction does not sell the item to some bidder, give the item to bidder $n+1$ for free (i.e., $p_n = 0$). Notice that, since the item is always allocated, the expected revenue of $\mathcal{A}$ is at most the expected revenue of the second-price auction. In turn, the expected revenue of $\mathcal{A}$ for $n+1$ bidders is by design the same as the expected revenue of the optimal (revenue-maximising) auction for n bidders. This proves the claim.

Exercise 2. Consider a single-item auction with n bidders, with values drawn independently from distributions $F_1, \ldots, F_n$. Show by example that, in an optimal auction, the bidder with the highest bid need not win, even if the bidder has positive virtual valuation. Give an intuitive explanation why this property might be beneficial to the expected revenue of the auction.

Solution 2. Consider the case when $n = 2$, and distributions $F_1 = U[0,1]$ and $F_2 = U[0,3/2]$. The virtual value functions are then $\phi_1(b) = 2b - 1$ and $\phi_2(b) = 2b - \frac{3}{2}$. Let $b_1 = 0.7$ and $b_2 = 0.8$ be the bids of the agents; in that case we have $\phi_1(b_1) = 0.4$ and $\phi_2(b_2) = 0.1$, and observe that $\phi_1(v_1), \phi_2(v_2) > 0$. The winner of the auction is the bidder with the highest (non-negative) virtual value, namely bidder 1. However, bidder 2 is the bidder with the highest bid.

In effect, the optimal auction discriminates against bidders for whom the values are generally expected to be higher, as this discourages such bidders from underbidding when their actual values are high. In other words, bidders with high maximum valuations are "forced" to bid more aggressively, and the auction extracts more revenue in expectation.

Exercise 3. Consider an all *All-Pay Auction* in which the values of the n bidders are drawn independently from the uniform distribution over $[0,1]$. In this auction format, the winner is the bidder with the highest bid, and the payment of each bidder is her bid, regardless of whether the bidder is a winner or not.

A. Show that, for the case of $n = 2$, the stategy profile in which every bidder $i \in \{1,2\}$ bids $b_i = v_i^2/2$ is a Bayes-Nash equilibrium of the auction.

B. For general n, the (unique) symmetric equilibrium of the auction is given by $b_i = \frac{n-1}{n} v_i^n$, for each bidder i. Show that the expected revenue of the auction in this case is $\frac{n-1}{n+1}$. How does that compare to the expected revenue of the First-Price auction with n bidders whose values are drawn independently from the uniform distribution over $[0,1]$?

Solution 3.

A. Assume that $b_2 = \frac{v_2^2}{2}$; we will show that bidding $b_1 = \frac{v_1^2}{2}$ is a strategy that maximises the expected utility of bidder 1 (a best response). Observe that bidder 1 wins the item if $b_1 > b_2 \Rightarrow b_1 > \frac{v_2^2}{2} \Rightarrow v_2 < \sqrt{2b_1}$. In that case, the utility of bidder 1 is $(v_1 - b_1)$, otherwise, the utility of bidder 1 (when she does not win) is $-b_1$. Overall, the expected utility of bidder 1 is:

$$U(b_1, b_2; v_1) = \int_0^{\sqrt{2b_1}} (v_1 - b_1)\ dv_2 - \int_{\sqrt{2b_i}}^{1} b_1\ dv_2 = (v_1 - b_1)\sqrt{2b_1} - b_1(1 - \sqrt{2b_1}) = v_1\sqrt{2b_1} - b_1$$

Setting the derivative of the expected utility (with respect to b_1) equal to 0, we obtain

$$\frac{v_1}{\sqrt{2b_1}} = 1 \Rightarrow b_1 = \frac{v_1^2}{2},$$

as desired.

The expected payment of any bidder is

$$\mathbb{E}_{v \in U[0,1]}\left[\frac{(n-1)v^n}{n}\right] = \frac{n-1}{n}\mathbb{E}[v^n] = \frac{n-1}{n}\int_0^1 v^n d_v = \frac{n-1}{n}\left[\frac{v^{n+1}}{n+1}\right]_0^1 = \frac{n-1}{n} \cdot \frac{1}{n+1} = \frac{n-1}{n(n+1)}$$

Since the values are drawn iid, the expected revenue for all bidders is n times the expected revenue of one bidder, hence $\frac{n-1}{n+1}$.

By the Revenue Equivalence theorem, since in both auction mechanisms (i) the winner is the bidder with the highest bid and (ii) bidders that bid 0 pay 0, the expected revenue of the two auction mechanisms is the same.