Conjunctive Queries: Evaluation and Static Analysis

(Chapter 14 and 15 of DBT)

[DBT] Database Theory, https://github.com/pdm-book/community

A Core Relational Query Language

Conjunctive Queries (CQ)

= {σ,π,⋈}-fragment of relational algebra

= relational calculus using only ∃ and ∧

= simple SELECT-FROM-WHERE SQL queries
(only AND and equality in the WHERE clause)

Syntax of Conjunctive Queries

$$Q(\mathbf{x}) \; := \; \exists \mathbf{y} \, (R_1(\mathbf{v_1}) \; \wedge \; \cdots \; \wedge \; R_m(\mathbf{v_m}))$$

- $R_1,\ldots,R_m$ are relation names
- $\mathbf{x}$, $\mathbf{y}$, $\mathbf{v_1},\ldots,\mathbf{v_m}$ are tuples of variables
- each variable mentioned in $\mathbf{v_i}$ appears either in $\mathbf{x}$ or $\mathbf{y}$
- the variables in $\mathbf{x}$ are free called distinguished or output variables

It is very convenient to see conjunctive queries as rule-based queries of the form

$$Q(\mathbf{x}) \; \text{:-} \; \underbrace{R_1(\mathbf{v_1}),\ldots,R_m(\mathbf{v_m})}_{}$$

this is called the body of Q that can be seen as a set of atoms

Pattern Matching Problem

List the airlines that fly directly from London to Glasgow

	Airport(VIE,Vienna),
Flight(VIE,LHR,BA),	Airport(LHR,London),
Flight(LHR,EDI,BA),	Airport(LGW,London),
Flight(LGW,GLA,U2),	Airport(LCA,Larnaca),
Flight(LCA,VIE,OS),	Airport(GLA,Glasgow),
	Airport(EDI,Edinburgh)

Q(z) :- Airport(x,London), Airport(y,Glasgow), Flight(x,y,z)

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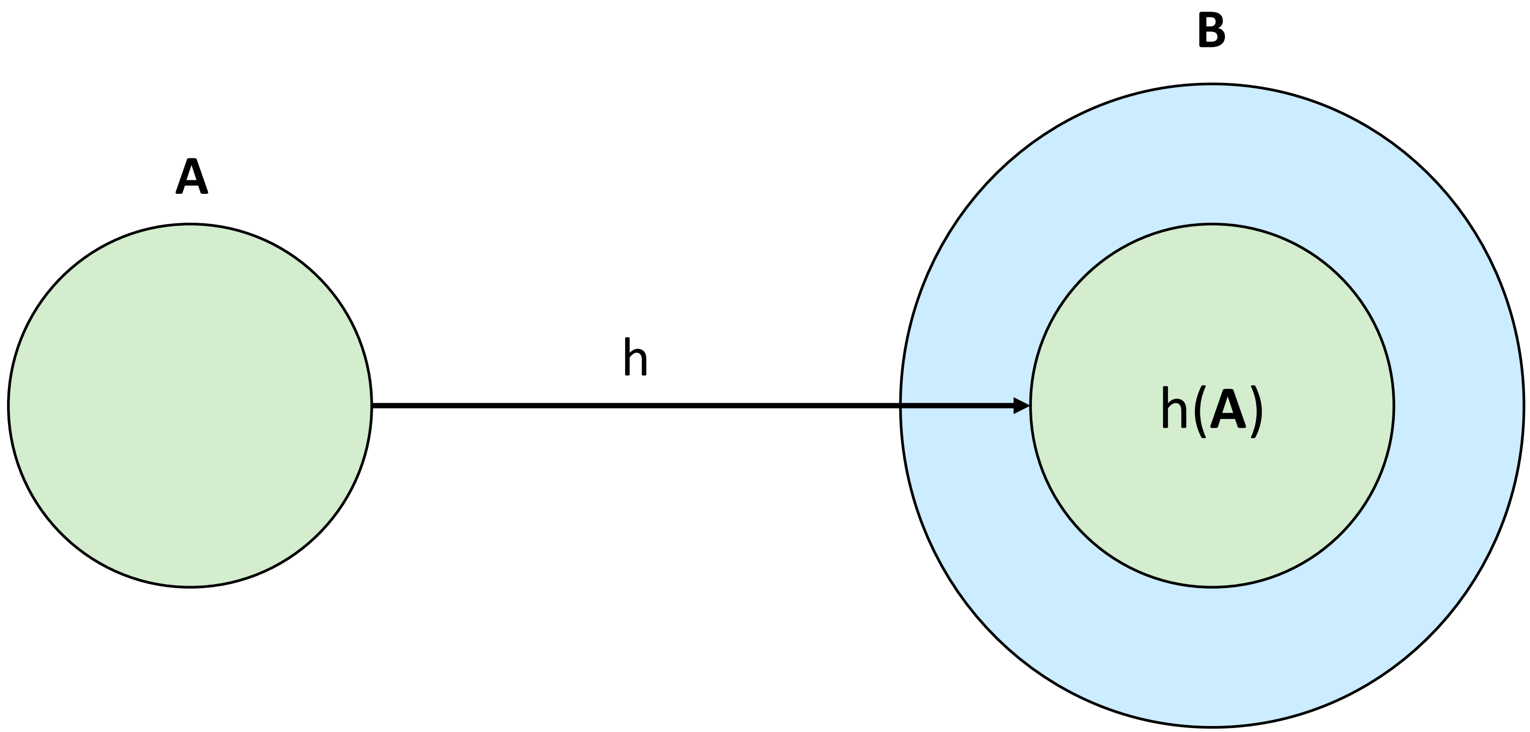
Homomorphism



h : terms(**A**) → terms(**B**) that is the identity on constants

Semantics of Conjunctive Queries

- A match of a conjunctive query $Q(x_1,\ldots,x_k)$:- body in a database D is a homomorphism h from the set of atoms body to the set of atoms D

- The answer to $Q(x_1,\ldots,x_k)$:- body over D is the set of k-tuples

$$Q(D) \ := \ \{(h(x_1),\ldots,h(x_k)) \mid h \text{ is a match of } Q \text{ in } D\}$$

- The answer consists of the witnesses for the distinguished variables of Q

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Flight(LCA,VIE,OS),	**Airport(GLA,Glasgow)**,
	Airport(EDI,Edinburgh)

$\{x \mapsto \text{LGW},\ y \mapsto \text{GLA},\ z \mapsto \text{U2},$
$\text{London} \mapsto \text{London},\ \text{Glasgow} \mapsto \text{Glasgow}\}$

Q(z) :- Airport(x,London), Airport(y,Glasgow), Flight(x,y,z)

Query Evaluation

- Understand the complexity of evaluating a conjunctive query over a database
- What to measure? Queries may have a large output and it would be misleading to count the output as "complexity"
- We therefore consider the following decision problem for **CQ**

CQ-Evaluation

Input: a database D, a CQ $Q(x_1,\ldots,x_k)$:- body, and a tuple $(a_1,\ldots,a_k)$ of values

Question: $(a_1,\ldots,a_k) \in Q(D)$?

combined complexity

Data Complexity of Query Evaluation

- Measures the complexity in terms of the size of the database - the query is fixed
- Meaningful in practice since the database is usually much bigger than the query
- We consider the following decision problem for a fixed CQ $Q(x_1,\ldots,x_k)$:- body

Q-Evaluation

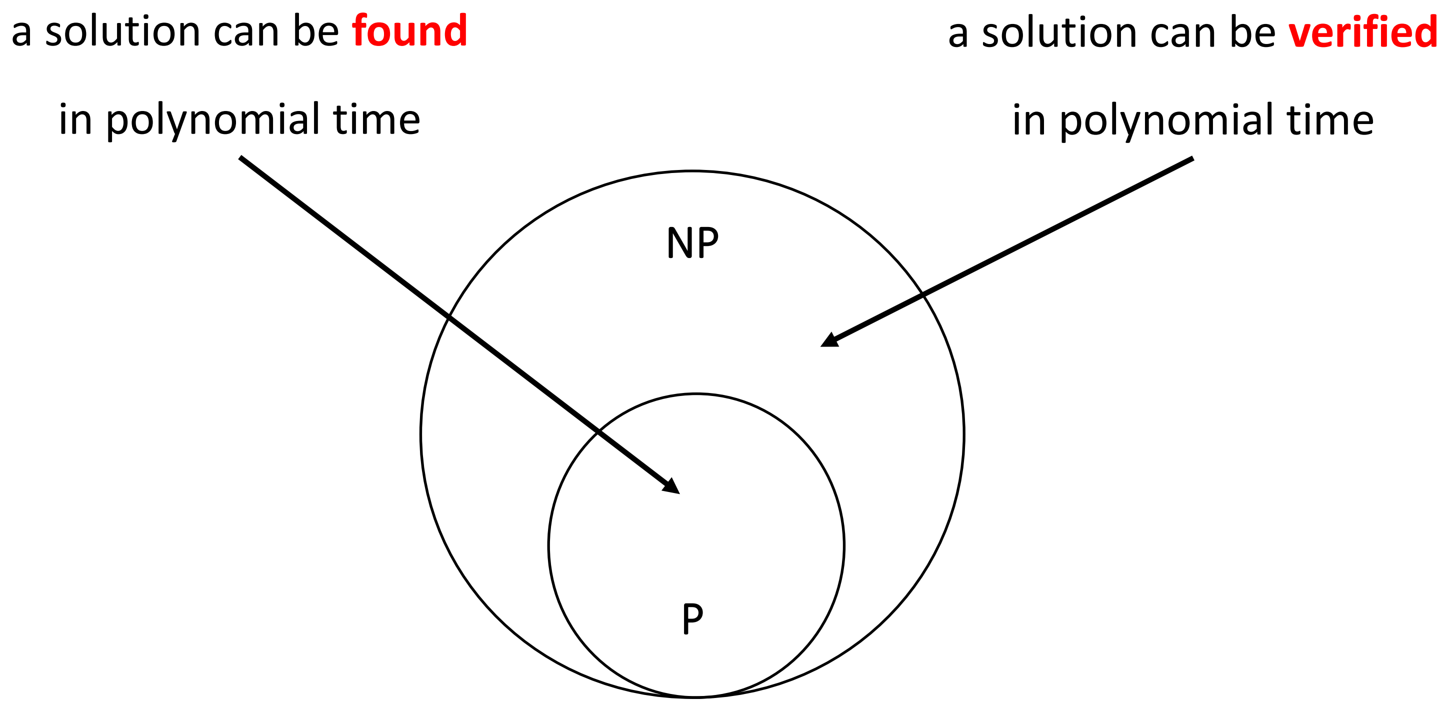
Input: a database D, and a tuple $(a_1,\ldots,a_k)$ of values

Question: $(a_1,\ldots,a_k) \in Q(D)$?

Complexity of Query Evaluation

Theorem: CQ-Evaluation is NP-complete and in PTIME in data complexity

Few Words about NP



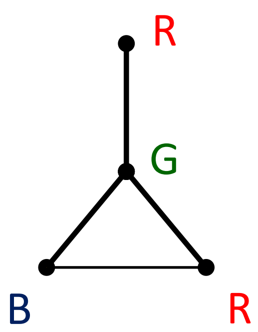
- $P \subseteq NP$, but it is open whether $P \subset NP$ or $P = NP$ - one of the most important questions in mathematics and theoretical computer science
- One of the Millennium Prize Problems selected by the Clay Mathematics Institute in 2000 (1 million USD prize for the first correct solution)

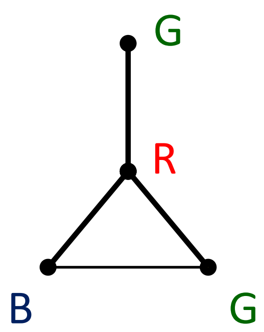
3-Colorability Problem

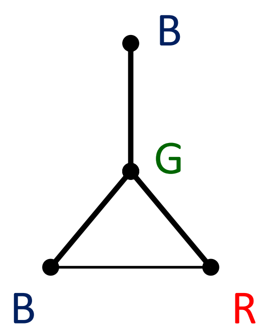
3COL

Input: an undirected graph **G** = (V,E)

Question: is there a function c : V → {R,G,B} such that (v,u) ∈ E ⇒ c(v) ≠ c(u)?







3-Colorability Problem

3COL

Input: an undirected graph **G** = (V,E)

Question: is there a function c : V → {R,G,B} such that (v,u) ∈ E ⇒ c(v) ≠ c(u)?

Theorem: 3COL is NP-complete

Proof:

- **Guess** a function c : V → {R,G,B} in polynomial time, and **verify** that (v,u) ∈ E ⇒ c(v) ≠ c(u) in polynomial time
- 3COL is one of the most difficult problems in the complexity class NP (it is unlikely to be solvable in polynomial time unless P = NP)

Complexity of Query Evaluation

Theorem: CQ-Evaluation is NP-complete and in PTIME in data complexity

Proof:

(NP-membership) Guess-and-verify:

- Consider a database D, a CQ $Q(x_1,\ldots,x_k)$:- body, and a tuple $(a_1,\ldots,a_k)$ of values
- Guess a substitution h : terms(body) $\rightarrow$ terms(D) that is the identity on constants
- Verify that h is a match of Q in D, i.e., h(body) $\subseteq$ D and $(h(x_1),\ldots,h(x_k)) = (a_1,\ldots,a_k)$

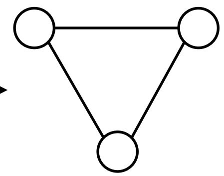
(NP-hardness) Reduction from 3COL

NP-hardness

(NP-hardness) Reduction from 3COL

> 3COL
>
> **Input:** an undirected graph **G** = (V,E)
>
> **Question:** is there a function c : V → {R,G,B} such that $(v,u) \in E \Rightarrow c(v) \neq c(u)$?

Lemma: G is 3-colorable iff **G** can be mapped to $\mathbf{K_3}$, i.e., **G** $\xrightarrow{\text{hom}}$



therefore, **G** is 3-colorable iff there is a match of Q_G in D = {E(x,y),E(y,z),E(z,x)}

the Boolean CQ that represents **G**

Complexity of Query Evaluation

Theorem: CQ-Evaluation is NP-complete and in PTIME in data complexity

Proof:

(NP-membership) Guess-and-verify:

- Consider a database D, a CQ $Q(x_1,\ldots,x_k)$:- body, and a tuple $(a_1,\ldots,a_k)$ of values
- Guess a substitution h : terms(body) → terms(D) that is the identity on constants
- Verify that h is a match of Q in D, i.e., $h(\text{body}) \subseteq D$ and $(h(x_1),\ldots,h(x_k)) = (a_1,\ldots,a_k)$

(NP-hardness) Reduction from 3-colorability

(in PTIME) For every substitution h : terms(body) → terms(D) that is the identity on constants, check if $h(\text{body}) \subseteq D$ and $(h(x_1),\ldots,h(x_k)) = (a_1,\ldots,a_k)$

Static Analysis

CQ-Satisfiability

Input: a conjunctive query Q

Question: is there a database D such that Q(D) is non-empty?

- If the answer is no, then the input query Q makes no sense
- **CQ**-Evaluation becomes trivial - the answer is always NO!

Static Analysis

CQ-Equivalence

Input: two conjunctive queries Q_1 and Q_2

Question: $Q_1 \equiv Q_2$? or $Q_1(D) = Q_2(D)$ for every database D?

- Replace a query Q_1 with a query Q_2 that is easier to evaluate
- But, we have to be sure that $Q_1(D) = Q_2(D)$ for every database D

Static Analysis

CQ-Containment

Input: two conjunctive queries Q_1 and Q_2

Question: $Q_1 \subseteq Q_2$? or $Q_1(D) \subseteq Q_2(D)$ for every database D?

- Equivalence boils down to two containment checks
- Clearly, $Q_1(D) = Q_2(D)$ iff $Q_1(D) \subseteq Q_2(D)$ and $Q_2(D) \subseteq Q_1(D)$

Complexity of Static Analysis

CQ-Satisfiability

Input: a conjunctive query Q

Question: is there a database D such that $Q(D)$ is non-empty?

CQ-Equivalence

Input: two conjunctive queries Q_1 and Q_2

Question: $Q_1 \equiv Q_2$? or $Q_1(D) = Q_2(D)$ for every database D?

CQ-Containment

Input: two conjunctive queries Q_1 and Q_2

Question: $Q_1 \subseteq Q_2$? or $Q_1(D) \subseteq Q_2(D)$ for every database D?

Canonical Database

- Convert a conjunctive query Q into a database D[Q] - the canonical database of Q

- Given a conjunctive query of the form $Q(\mathbf{x})$:- body, D[Q] is obtained from body by replacing each variable x with a new value $c(x) = \underline{x}$

- E.g., given Q(x,y) :- R(x,y), P(y,z,w), R(z,x), then $D[Q] = \{R(\underline{x},\underline{y}), P(\underline{y},\underline{z},\underline{w}), R(\underline{z},\underline{x})\}$

- **Note:** The mapping c : {variables in body} → {new values} is a bijection, where c(body) = D[Q] and c^{-1}(D[Q]) = body

Satisfiability of CQs

CQ-Satisfiability

Input: a conjunctive query Q

Question: is there a database D such that Q(D) is non-empty?

Theorem: A conjunctive query Q is always satisfiable

Proof: Due to its canonical database - Q(D[Q]) is trivially non-empty

Equivalence and Containment of CQs

CQ-Equivalence

Input: two conjunctive queries Q_1 and Q_2

Question: $Q_1 \equiv Q_2$? or $Q_1(D) = Q_2(D)$ for every database D?

CQ-Containment

Input: two conjunctive queries Q_1 and Q_2

Question: $Q_1 \subseteq Q_2$? or $Q_1(D) \subseteq Q_2(D)$ for every database D?

$$Q_1 \equiv Q_2 \quad \text{iff} \quad Q_1 \subseteq Q_2 \text{ and } Q_2 \subseteq Q_1$$

$$Q_1 \subseteq Q_2 \quad \text{iff} \quad Q_1 \equiv (Q_1 \wedge Q_2)$$

…thus, we can safely focus on **CQ**-Containment

Homomorphism Theorem

A query homomorphism from $Q_1(x_1,\ldots,x_k)$:- $body_1$ to $Q_2(y_1,\ldots,y_k)$:- $body_2$

is a substitution $h : terms(body_1) \rightarrow terms(body_2)$ such that:

1. h is a homomorphism from $body_1$ to $body_2$
2. $(h(x_1),\ldots,h(x_k)) = (y_1,\ldots,y_k)$

Homomorphism Theorem: Let Q_1 and Q_2 be conjunctive queries. It holds that:

$Q_1 \subseteq Q_2$ iff there exists a query homomorphism from Q_2 to Q_1

Homomorphism Theorem: Example

$Q_1(x,y)$:- $R(x,z), S(z,z), R(z,y)$

$h = \{a \mapsto x,\ b \mapsto y,\ c \mapsto z,\ d \mapsto z\}$

$Q_2(a,b)$:- $R(a,c), S(c,d), R(d,b)$

- h is a query homomorphism from Q_2 to Q_1 $\Rightarrow$ $Q_1 \subseteq Q_2$
- But, there is no homomorphism from Q_1 to Q_2 $\Rightarrow$ $Q_1 \subset Q_2$

Homomorphism Theorem: Proof

Assume that $Q_1(x_1,\dots,x_k)$:- $body_1$ and $Q_2(y_1,\dots,y_k)$:- $body_2$

($\Rightarrow$) $Q_1 \subseteq Q_2 \;\Rightarrow\;$ there exists a query homomorphism from Q_2 to Q_1

- Clearly, $(c(x_1),\dots,c(x_k)) \in Q_1(D[Q_1])$ - recall that $D[Q_1] = c(body_1)$
- Since $Q_1 \subseteq Q_2$, we conclude that $(c(x_1),\dots,c(x_k)) \in Q_2(D[Q_1])$
- Therefore, there exists a homomorphism h such that $h(body_2) \subseteq D[Q_1] = c(body_1)$

  and $h((y_1,\dots,y_k)) = (c(x_1),\dots,c(x_k))$
- By construction, $c^{-1}(c(body_1)) = body_1$

  and $c^{-1}((c(x_1),\dots,c(x_k))) = (x_1,\dots,x_k)$
- Therefore, $c^{-1} \circ h$ is a

  query homomorphism from Q_2 to Q_1

$$
\begin{tikzcd}
Q_2(y_1,\dots,y_k) \arrow[d] & :- & body_2 \arrow[d, "h"] \arrow[dd, bend left=60, "c^{-1}\circ h"] \\
Q_1(c(x_1),\dots,c(x_k)) \arrow[d] & :- & c(body_1) \arrow[d, "c^{-1}"] \\
Q_1(x_1,\dots,x_k) & :- & body_1
\end{tikzcd}
$$

Homomorphism Theorem: Proof

Assume that $Q_1(x_1,\ldots,x_k)$:- $body_1$ and $Q_2(y_1,\ldots,y_k)$:- $body_2$

($\Leftarrow$) $Q_1 \subseteq Q_2$ $\Leftarrow$ there exists a query homomorphism from Q_2 to Q_1

- Consider a database D, and a tuple $\mathbf{t}$ such that $\mathbf{t} \in Q_1(D)$
- We need to show that $\mathbf{t} \in Q_2(D)$
- Clearly, there exists a homomorphism g such that $g(body_1) \subseteq D$ and $g((x_1,\ldots,x_k)) = \mathbf{t}$
- By hypothesis, there exists a query homomorphism h from Q_2 to Q_1
- Therefore, $g(h(body_2)) \subseteq D$ and

  $g(h((y_1,\ldots,y_k))) = \mathbf{t}$, which implies that $\mathbf{t} \in Q_2(D)$

$$
\begin{tikzcd}
Q_2(y_1,\ldots,y_k) \arrow[d] & :- & body_2 \arrow[d, "h"] \arrow[dd, bend left=60, "g \circ h"] \\
Q_1(x_1,\ldots,x_k) \arrow[d] & :- & body_1 \arrow[d, "g"] \\
\mathbf{t} & & D
\end{tikzcd}
$$

Existence of a Query Homomorphism

Theorem: Let Q_1 and Q_2 be conjunctive queries. The problem of deciding whether there exists a query homomorphism from Q_2 to Q_1 is NP-complete

Proof:

(NP-membership) Guess a substitution and verify that is a query homomorphism

(NP-hardness) Easy reduction from **CQ**-Evaluation

By applying the homomorphism theorem we get that:

Corollary: **CQ**-Equivalence and **CQ**-Containment are NP-complete