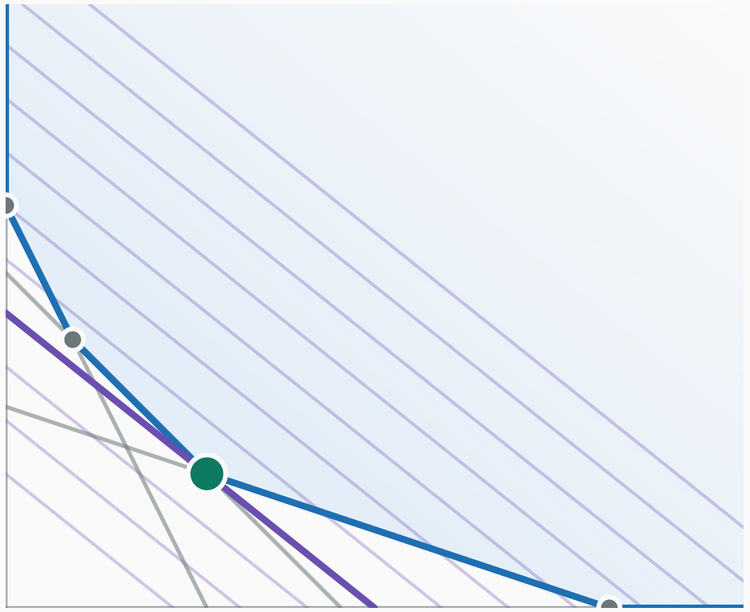
ALGORITHMS AND DATA STRUCTURES · LECTURE 13

Introduction to linear programming

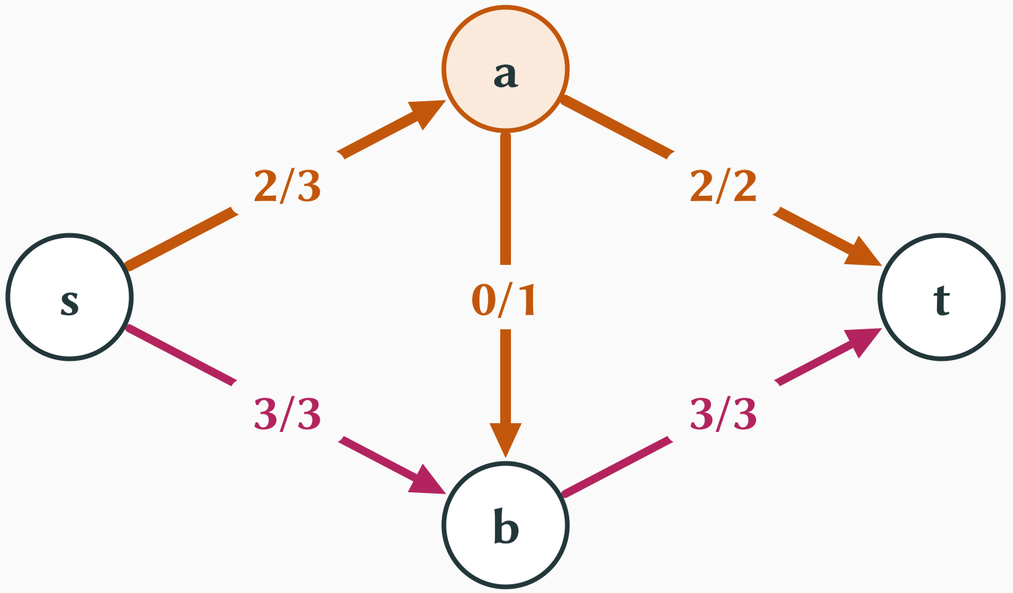
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Monday 2 November 2026 · 16:10–17:00



From flows to linear programs



Flow conservation at a: 2 in = 0 + 2 out.

Maximum flow is already a **linear program**: numbers to choose, linear constraints, a linear objective.

maximise $f_{sa} + f_{sb}$

subject to $0 \leqslant f_e \leqslant c_e$ for every edge e

$f_{sa} = f_{ab} + f_{at}$ (at a)

$f_{sb} + f_{ab} = f_{bt}$ (at b)

? *Which other problems have this form?*

Today **1** the model → **2** its geometry → **3** a first algorithm → **4** integer programs

A problem: the diet

Two energy drinks, **R** and **M**.

per can			**need per day**
calories	60	60	at least 300
vitamin A	12	6	at least 36
vitamin C	10	30	at least 90
cost	£4	£5	as small as possible

Each day we drink
x cans of **R**
and y cans of **M**.

? *How many of each, at the lowest cost?*

The diet as a linear program

	R	M	need
calories	60	60	≥ 300
vitamin A	12	6	≥ 36
vitamin C	10	30	≥ 90
cost	£4	£5	minimise

Unknowns: x cans of **R**,
y cans of **M** per day.

$$\textbf{minimise} \quad 4x + 5y$$

$$\textbf{subject to} \quad x + y \geqslant 5 \quad \div 60$$

$$2x + y \geqslant 6 \quad \div 6$$

$$x + 3y \geqslant 9 \quad \div 10$$

$$x, y \geqslant 0 \quad \text{no negative cans}$$

Linear programs

Linear program (LP), standard form.

$$\begin{aligned} \text{maximise} \quad & c_1 x_1 + \cdots + c_n x_n \\ \text{subject to} \quad & a_{i1} x_1 + \cdots + a_{in} x_n \leqslant b_i \\ & \text{for } i = 1, \ldots, m \\ & x_1, \ldots, x_n \geqslant 0 \end{aligned}$$

In matrix form:

maximise $c^T x$ subject to $Ax \leqslant b,\ x \geqslant 0$

Given: A $(m \times n)$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.
Unknown: $x \in \mathbb{R}^n$.

Linear: only sums of (number) × (unknown).

Example with $n = m = 3$:

$$\begin{aligned} \text{maximise} \quad & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leqslant 5 \\ & 4x_1 + x_2 + 2x_3 \leqslant 11 \\ & 3x_1 + 4x_2 + 2x_3 \leqslant 8 \\ & x_1, x_2, x_3 \geqslant 0 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} \quad c = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

Optimum: $x^* = (2, 0, 1)$, value 13

Every LP can be written in standard form

With the objective negated:

$$
\begin{aligned}
\textbf{maximise} \quad & -4x - 5y \\
\textbf{subject to} \quad & -x - y \leqslant -5 \\
& -2x - y \leqslant -6 \\
& -x - 3y \leqslant -9 \\
& x, y \geqslant 0
\end{aligned}
$$

minimise f	$\to$	maximise $-f$
$a^T x \geqslant \beta$	$\to$	$-a^T x \leqslant -\beta$
$a^T x = \beta$	$\to$	$a^T x \leqslant \beta,\ -a^T x \leqslant -\beta$
x_j any sign	$\to$	$x_j = x_j^+ - x_j^-,\ x_j^+, x_j^- \geqslant 0$

Standard form: maximise, $\leqslant$, $x \geqslant 0$.

Every LP can be written as maximise $c^T x$ subject to $Ax \leqslant b,\ x \geqslant 0$.

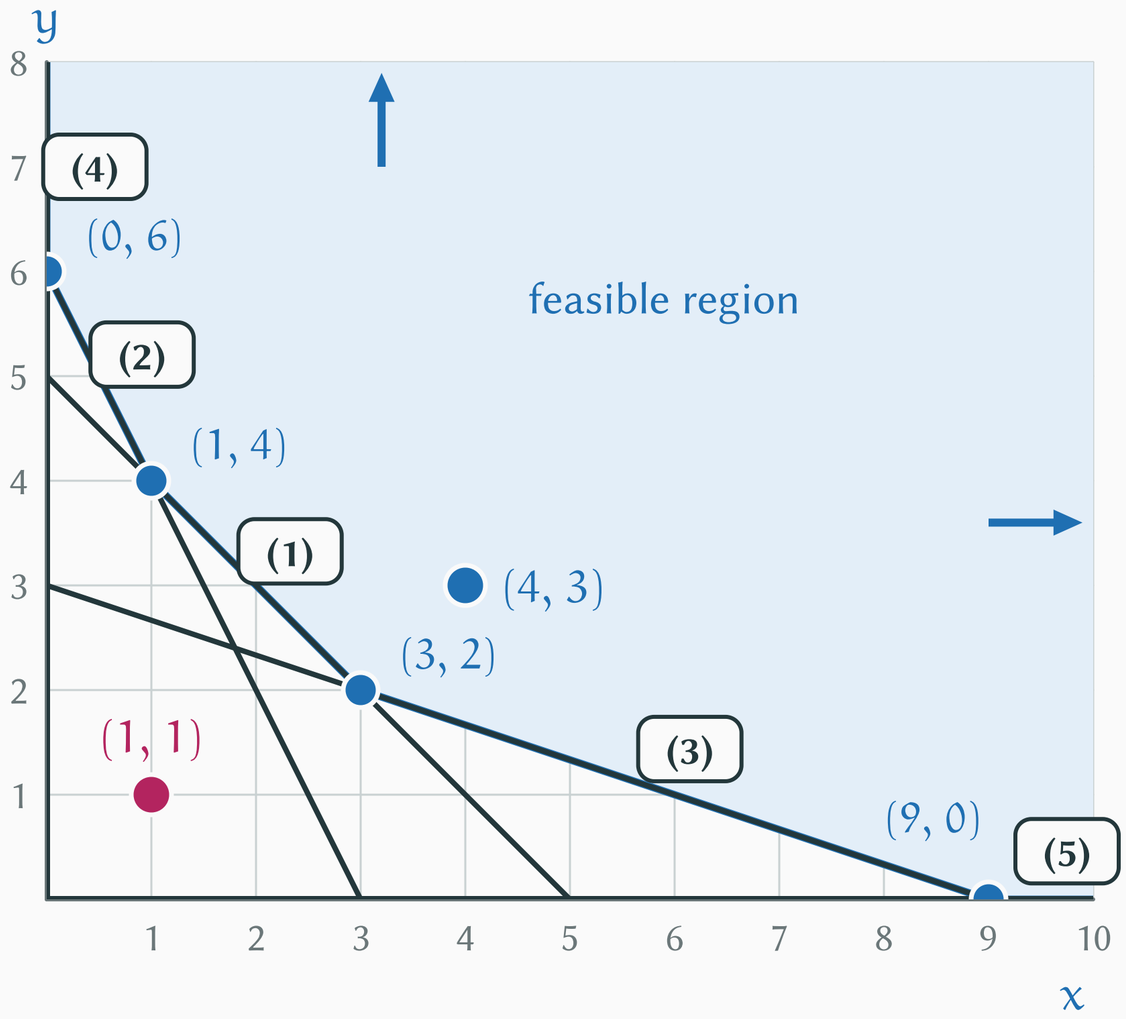
The feasible region

$$
\begin{aligned}
&(1) && x + y \geqslant 5 \\
&(2) && 2x + y \geqslant 6 \\
&(3) && x + 3y \geqslant 9 \\
&(4) && x \geqslant 0 \\
&(5) && y \geqslant 0
\end{aligned}
$$

Solution: a point (x, y).

Feasible solution: a solution that satisfies every constraint.

Feasible region: all of them: a **polyhedron** (a polytope if bounded).



The objective: level lines

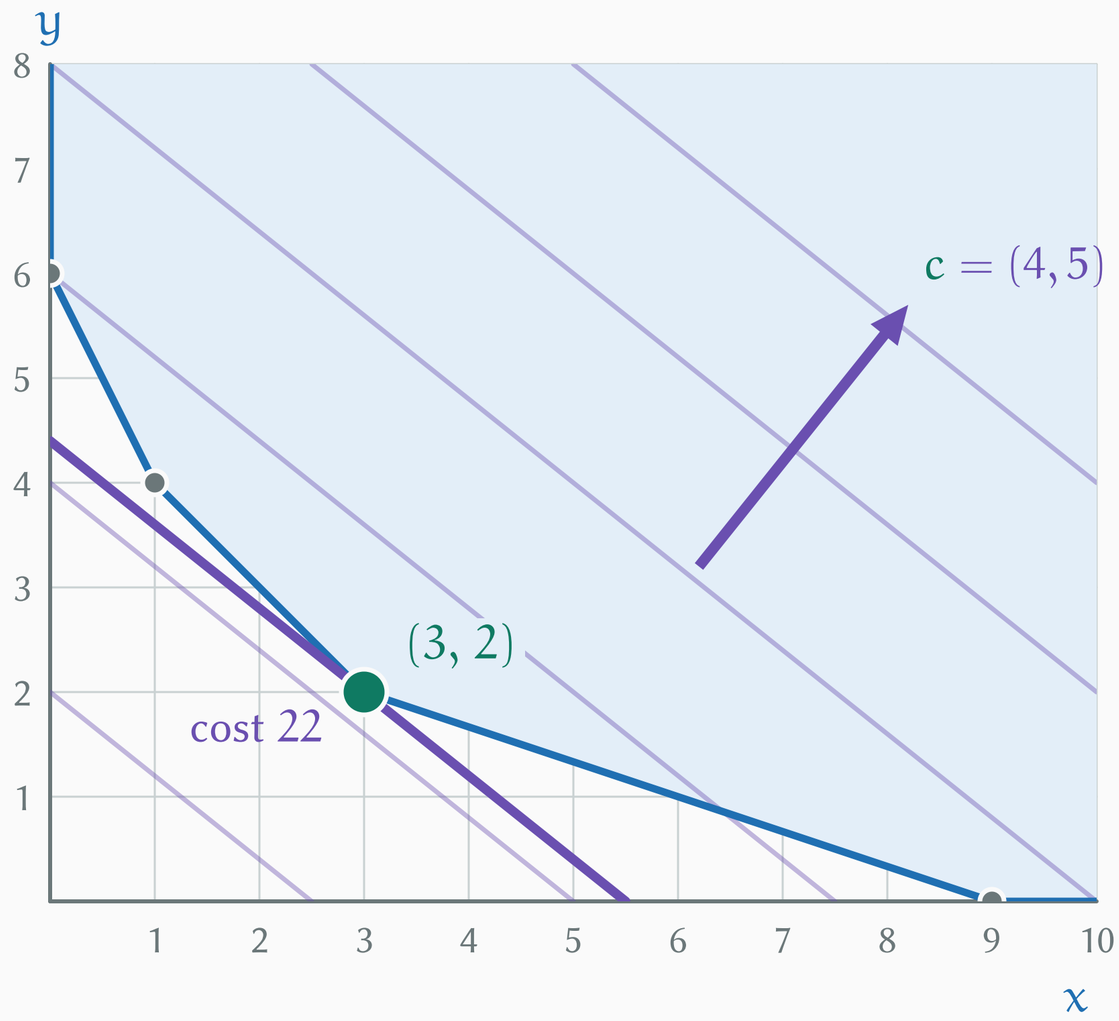
Cost of a diet: $4x + 5y$.

A **level line** joins the diets of equal cost: $4x + 5y = k$.

The cost grows along $c = (4, 5)$, perpendicular to the level lines.

Push the line towards smaller cost, as long as it still touches the region:
cost $k = 22$

Optimal solution: $x = 3, y = 2$.
Optimal value: $4 \cdot 3 + 5 \cdot 2 = 22$

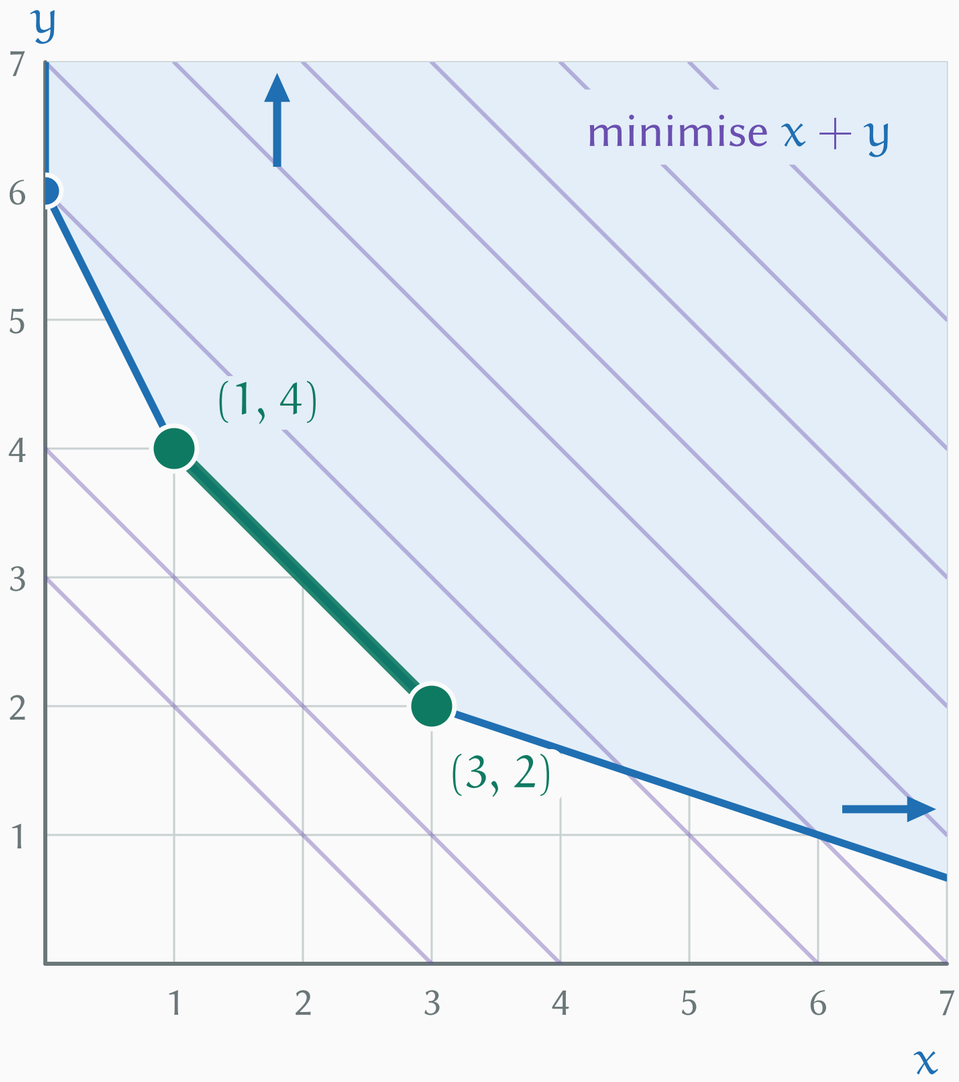


The optimum is at a vertex

Vertex. A feasible point where n independent constraints are tight (hold with equality).

Fact. If an LP has an optimal solution, then a vertex is optimal.

1. Let x^* be optimal, not a vertex: fewer than n independent constraints are tight. Then some vector $d \in \mathbb{R}^n$, $d \neq 0$, keeps all of them tight: tight at $x^* + td$ for every t.
2. $x^* + td$ is feasible for small $|t|$, so $c^\mathsf{T} d = 0$: the value does not change along d.
3. Walk along $\pm d$ until a new constraint is tight ($x \geqslant 0$ stops us): after at most n rounds, a vertex. □



Try all vertices

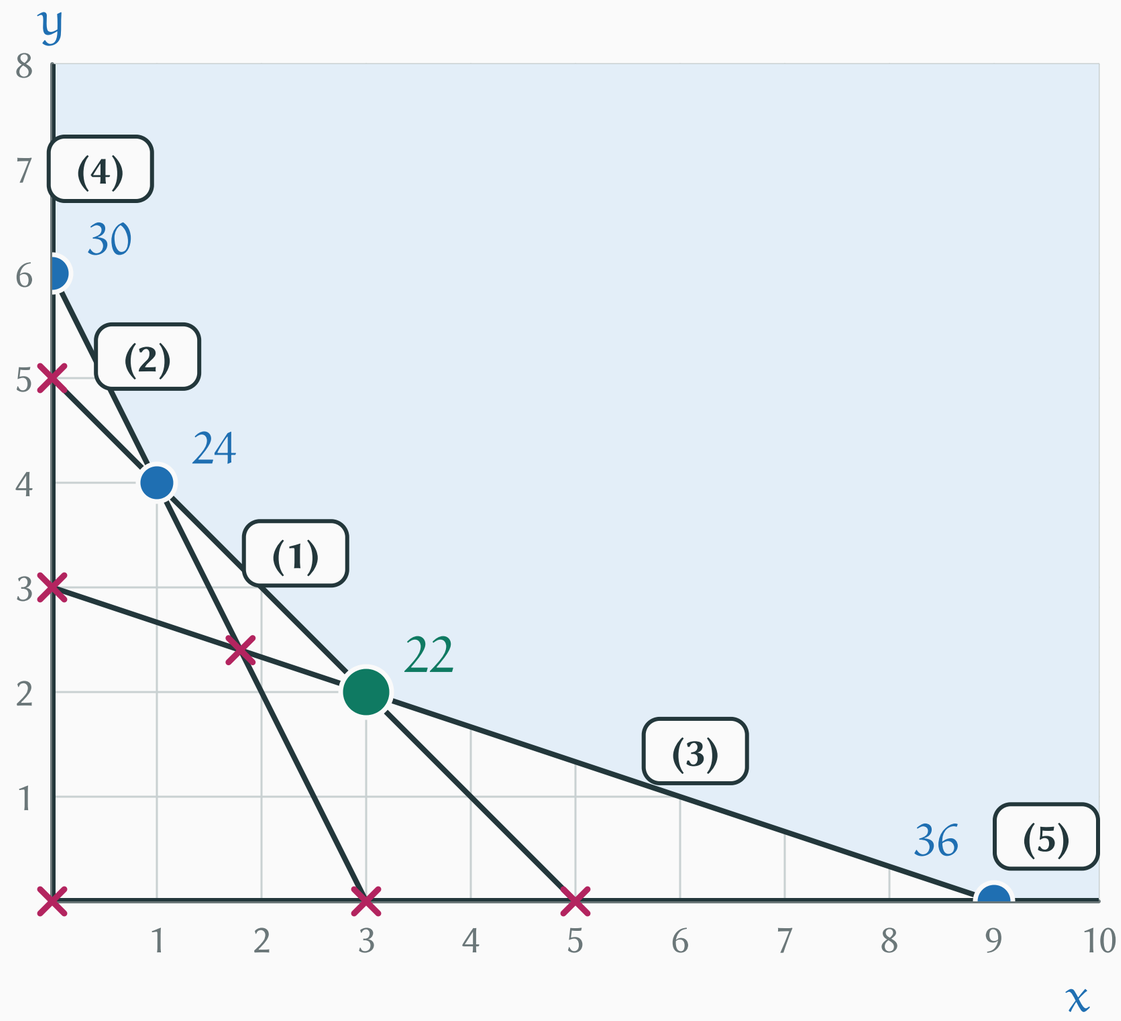
$m = 3$ constraints and $n = 2$ unknowns, $x, y \geqslant 0$.

A vertex: $n = 2$ of these $m + n = 5$ tight.

Candidates: $\binom{m+n}{n} = \binom{5}{2} = 10$ pairs.

lines	point	feasible?	cost
(1), (2)	(1, 4)	✓	24
(1), (3)	(3, 2)	✓	22
(1), (4)	(0, 5)	✗ (2)	
(1), (5)	(5, 0)	✗ (3)	
(2), (3)	(9/5, 12/5)	✗ (1)	
(2), (4)	(0, 6)	✓	30
(2), (5)	(3, 0)	✗ (1) (3)	
(3), (4)	(0, 3)	✗ (1) (2)	
(3), (5)	(9, 0)	✓	36
(4), (5)	(0, 0)	✗ (1) (2) (3)	

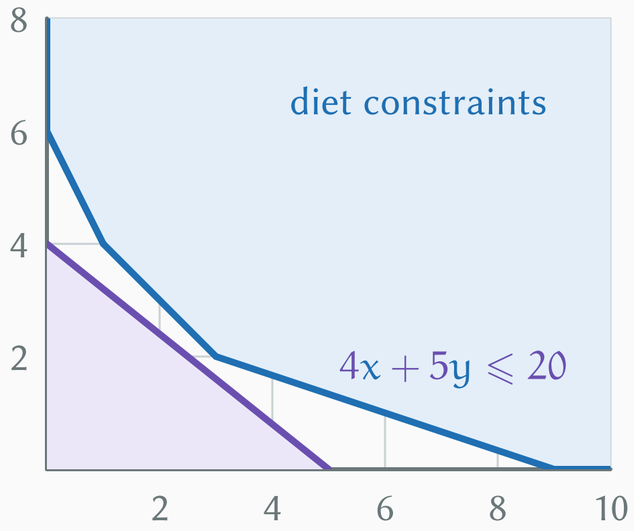
In general: $\binom{m+n}{n} \geqslant \left(\frac{m+n}{n}\right)^n$ candidates
$m = n$: at least 2^n, e.g. $\binom{100}{50} \approx 10^{29}$



Three outcomes

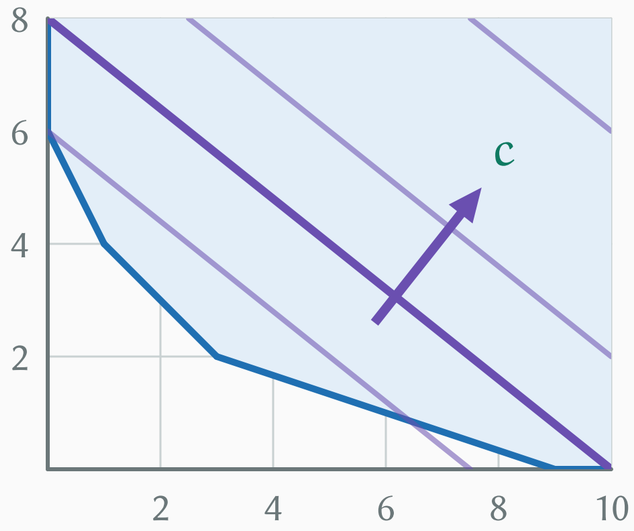
? *What can happen when we try to solve an LP?*

infeasible



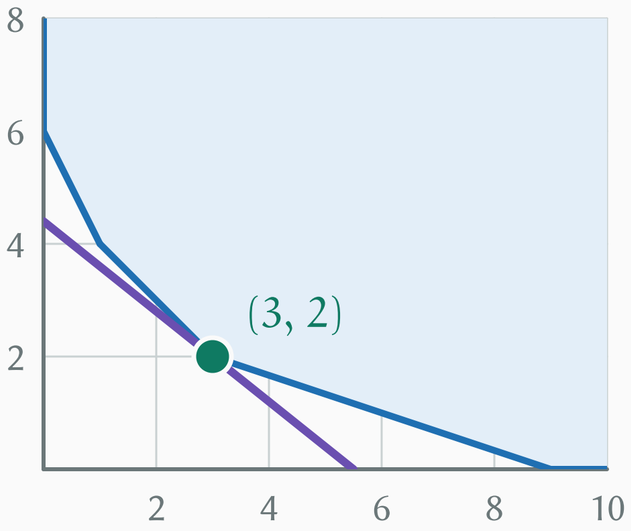
budget $4x + 5y \leqslant 20$:
no feasible solution

unbounded



maximise $4x + 5y$:
arbitrarily good values

optimal

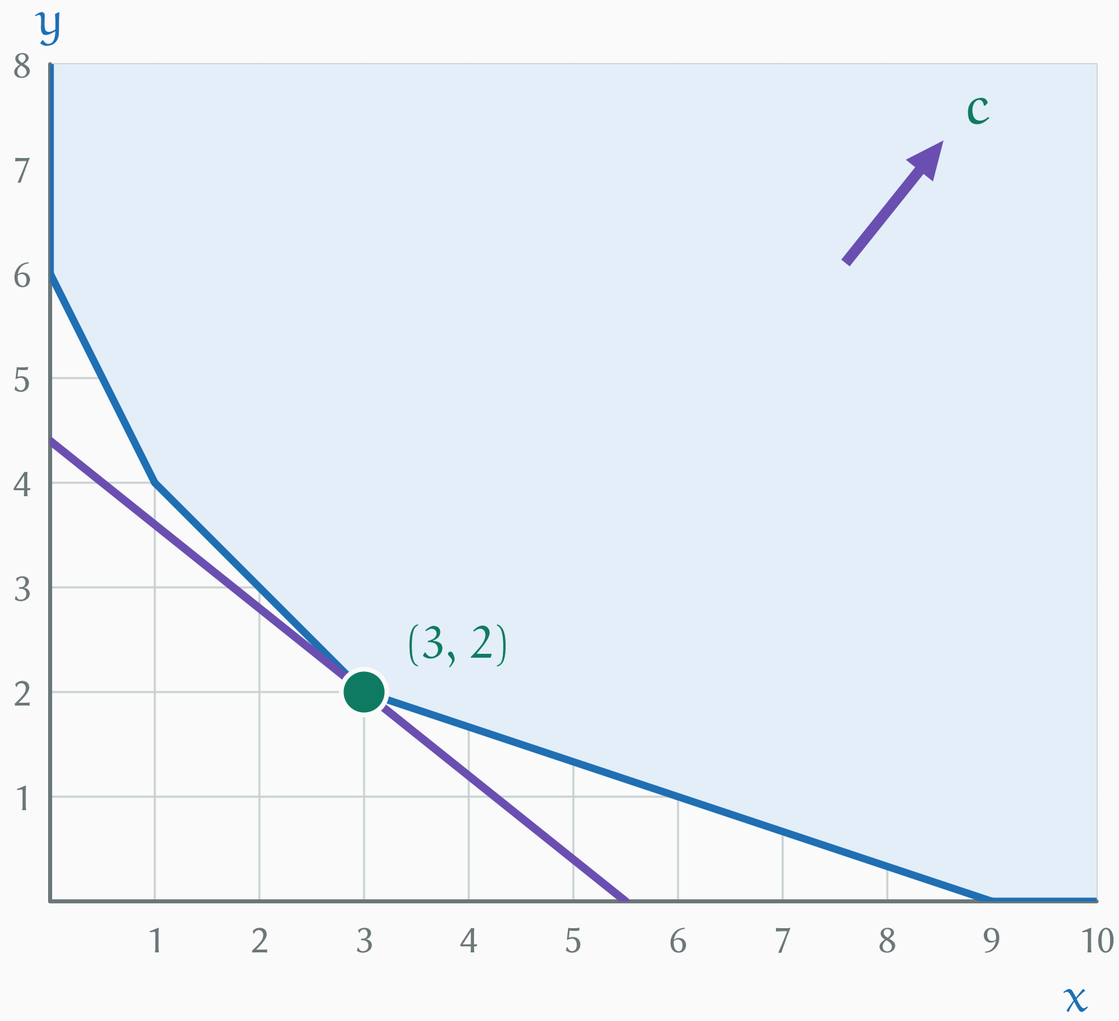


minimise $4x + 5y$:
best value 22, at $(3, 2)$

Every LP is infeasible, unbounded, or has an optimal solution.

Turning the objective

minimise $4x + 5y$ for cost vector c



> **optimal:** the vertex $(3, 2)$, value 22

Other choices of c:

$(3, 1)$: the vertex $(0, 6)$

$(2, 1)$: the whole edge from $(1, 4)$ to $(0, 6)$

$(1, 1)$: the whole edge from $(1, 4)$ to $(3, 2)$

$(1, 3)$: the whole edge from $(3, 2)$ to $(9, 0)$

$(-1, -1)$: unbounded

Eliminating a variable

Fourier–Motzkin elimination (Fourier 1826, Motzkin 1936)

(1) $x \geqslant 5 - y$

(2) $x \geqslant (6 - y)/2$

(3) $x \geqslant 9 - 3y$

(4) $x \geqslant 0$

(5) $y \geqslant 0$

(6) $x \leqslant (30 - 5y)/4$

(6) is new: a budget, the cost is at most £30.

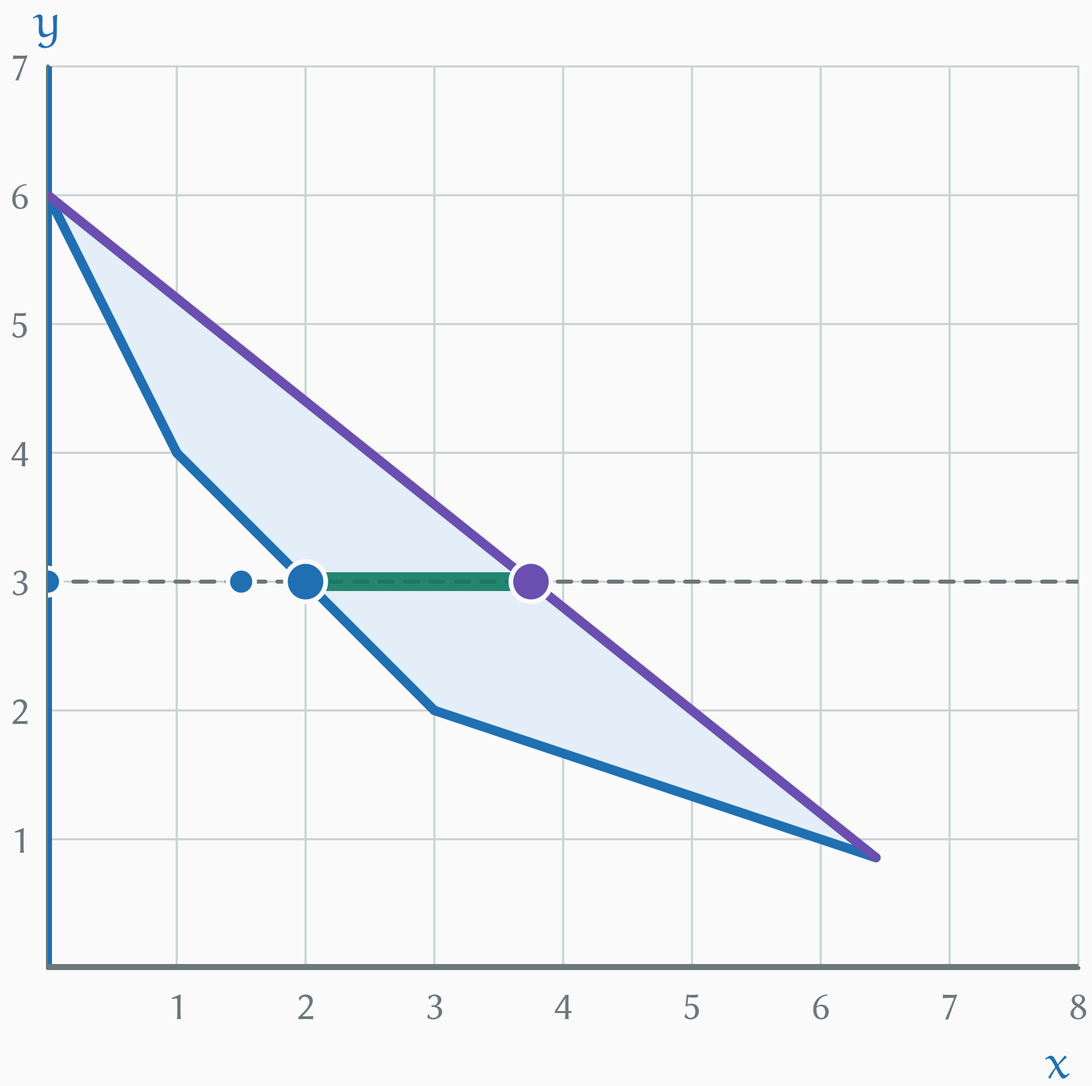
$$y = 3$$

largest lower bound 2

smallest upper bound 3.75

✓ an x exists, from 2 to 3.75

Lemma. For a given y, an x exists iff every lower bound $\leqslant$ every upper bound.



Pairs of bounds

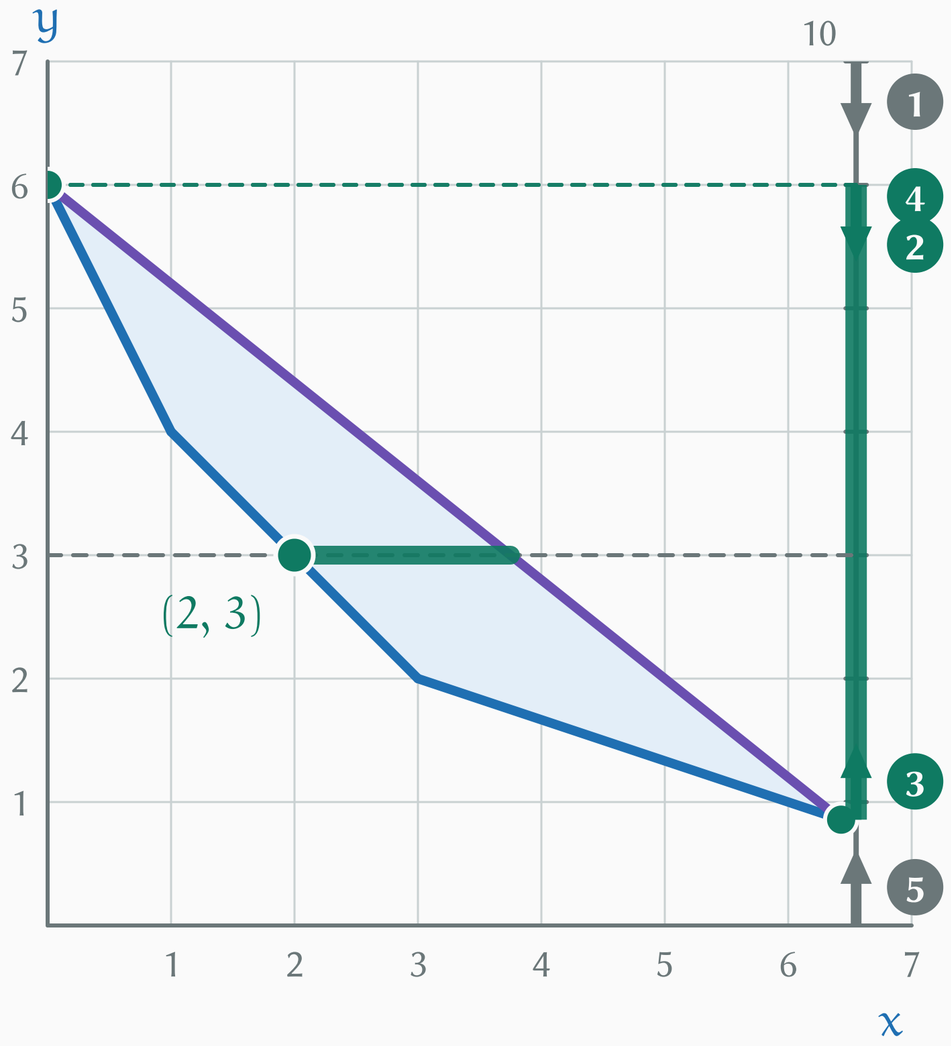
(1) $x \geqslant 5 - y$
(2) $x \geqslant (6 - y)/2$
(3) $x \geqslant 9 - 3y$
(4) $x \geqslant 0$
(5) $y \geqslant 0$
(6) $x \leqslant (30 - 5y)/4$

Each lower bound with each upper bound:

1. $5 - y \leqslant (30 - 5y)/4 \Rightarrow y \leqslant 10$
2. $(6 - y)/2 \leqslant (30 - 5y)/4 \Rightarrow y \leqslant 6$
3. $9 - 3y \leqslant (30 - 5y)/4 \Rightarrow y \geqslant 6/7$
4. $0 \leqslant (30 - 5y)/4 \Rightarrow y \leqslant 6$
5. $y \geqslant 0$ (no x in it)

Result. The system (1)–(6) has a solution iff the system $6/7 \leqslant y \leqslant 6$ has one.

Take $y = 3$: $2 \leqslant x \leqslant 15/4$, e.g. $(x, y) = (2, 3)$.



Optimising: the cost as a variable

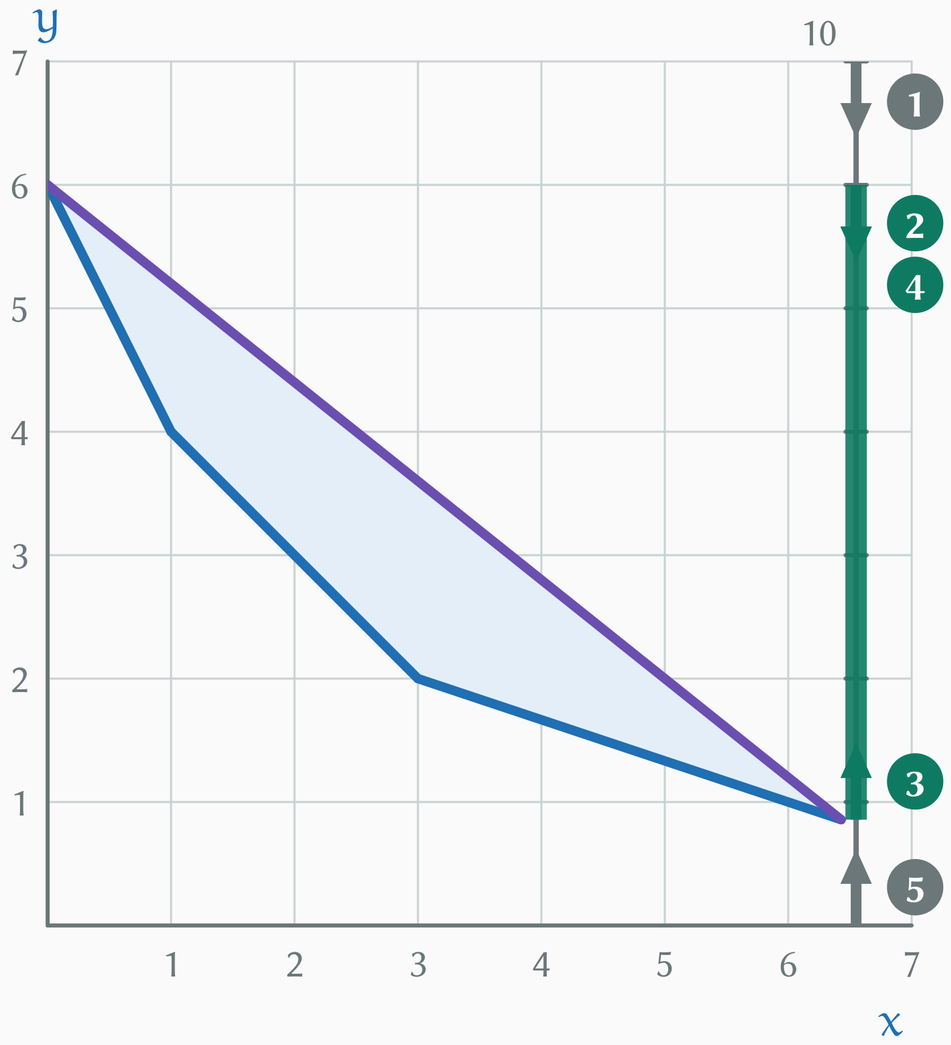
Let x_0 be the cost: **minimise** x_0
subject to (1)–(5) and (6′) $4x + 5y \leqslant x_0$

Eliminate x: each lower bound with the upper bound.

1. $5 - y \leqslant (x_0 - 5y)/4 \Rightarrow x_0 \geqslant 20 + y$
2. $(6 - y)/2 \leqslant (x_0 - 5y)/4 \Rightarrow x_0 \geqslant 12 + 3y$
3. $9 - 3y \leqslant (x_0 - 5y)/4 \Rightarrow x_0 \geqslant 36 - 7y$
4. $0 \leqslant (x_0 - 5y)/4 \Rightarrow x_0 \geqslant 5y$
5. $y \geqslant 0$ (no x in it)

Try other costs: $x_0 = 30$

✓ $6/7 \leqslant y \leqslant 6$



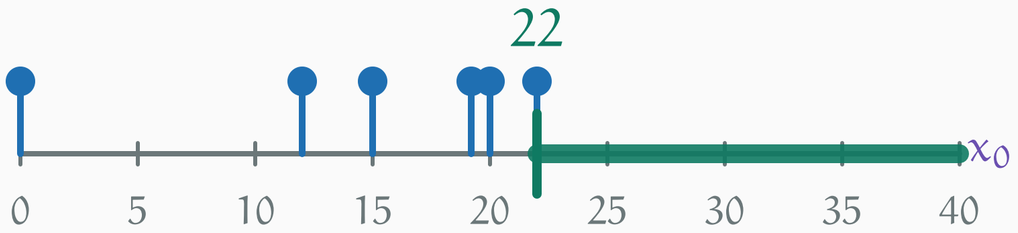
Optimising: eliminate y, read off the cost

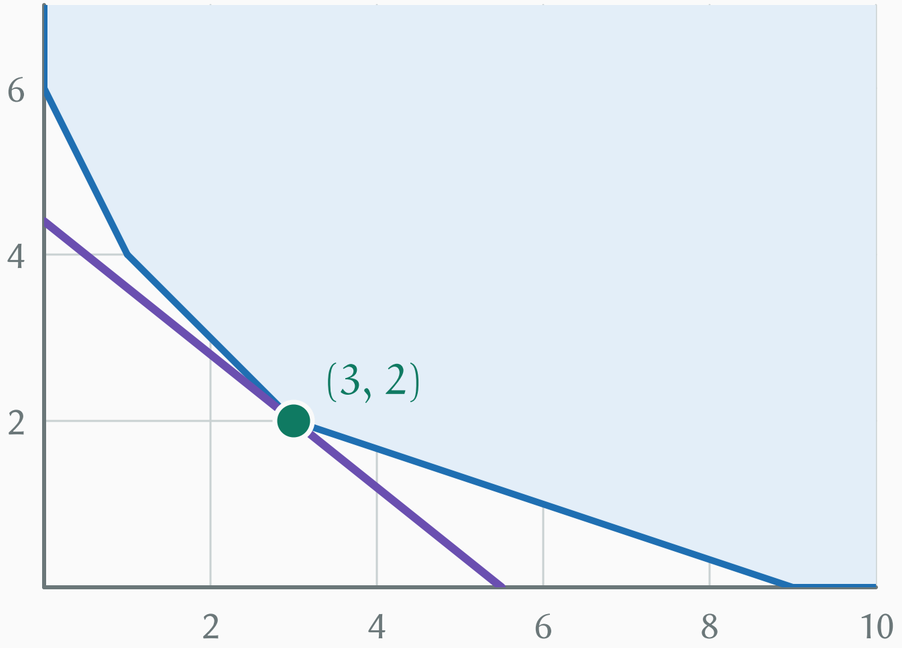
Eliminate y: pair each lower bound with each upper bound. The results bound x_0.

	$y \geqslant 0$	$y \geqslant (36 - x_0)/7$
$y \leqslant x_0 - 20$	$x_0 \geqslant 20$	$x_0 \geqslant 22$
$y \leqslant (x_0 - 12)/3$	$x_0 \geqslant 12$	$x_0 \geqslant 19.2$
$y \leqslant x_0/5$	$x_0 \geqslant 0$	$x_0 \geqslant 15$

> $x_0 \geqslant 22$: the least cost is 22

$x_0 = 22 \Rightarrow 2 \leqslant y \leqslant 2 \Rightarrow 3 \leqslant x \leqslant 3$
so the cheapest diet is $(x, y) = (3, 2)$.





Fourier–Motzkin blows up

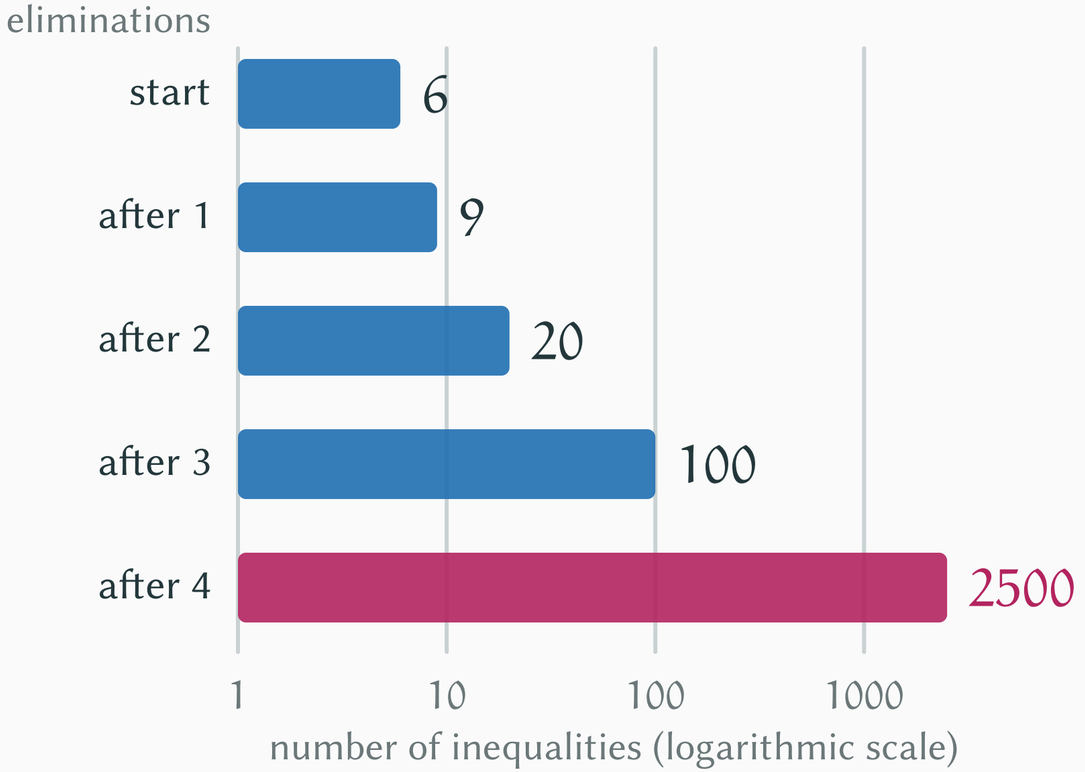
p lower and q upper bounds
give $p \cdot q$ new inequalities.

Example: 4 × 1 = 4 and 2 × 3 = 6 (the last two slides).

$p + q \leqslant m$, so after one step
at most $m^2/4$ inequalities ($m \geqslant 4$).

> After k steps: up to about $(m/4)^{2^k}$,
> **doubly exponential** in k.

Correct, but hopeless for LPs with
thousands of unknowns.



What elimination proves

Eliminate until only x_0 is left (diet region):

the LP	what is left on x_0	outcome
minimise $4x + 5y$	$x_0 \geqslant 22$ (the largest of 6)	**optimal**, 22
maximise $4x + 5y$	no inequality is left	**unbounded**
minimise $4x + 5y$, with $4x + 5y \leqslant 20$	$0 \leqslant -1$ (false)	**infeasible**

Theorem. If an LP with rational data has an optimal solution, it has a **rational** one, and its value is rational.

1. Every new inequality is a rational combination of old ones.
2. The optimal value is the best of finitely many such bounds on x_0.
3. Back-substitution picks each unknown in a rational interval.

Integer linear programs

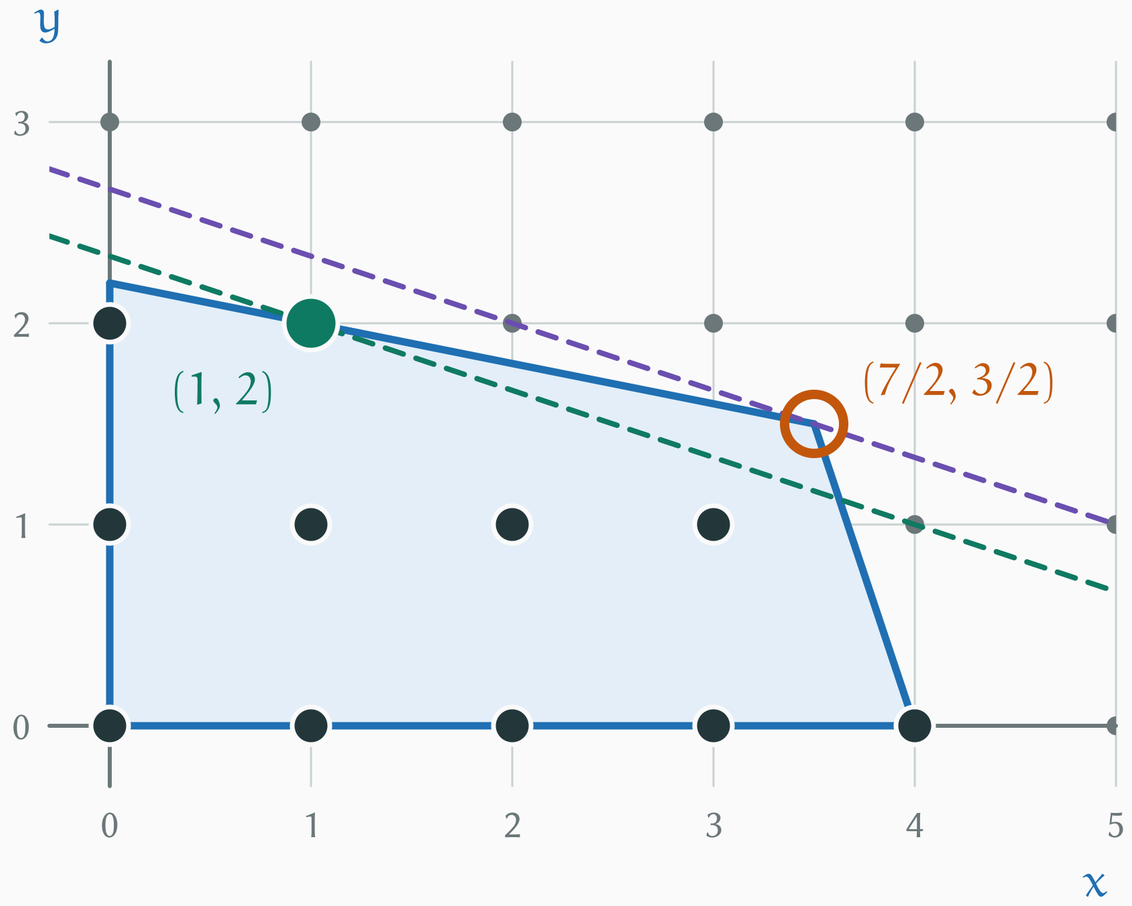
ILP: an LP whose unknowns must be **integers**.

$$\begin{aligned} &\textbf{maximise} && x + 3y \\ &\textbf{subject to} && 3x + y \leqslant 12 \\ & && x + 5y \leqslant 11 \\ & && x, y \geqslant 0 \qquad \text{integers} \end{aligned}$$

Feasible: the 11 **integer points** inside the polygon.

LP relaxation (drop "integers"):
optimum $(7/2, 3/2)$, value 8

ILP optimum $(1, 2)$, value $7 \leqslant 8$



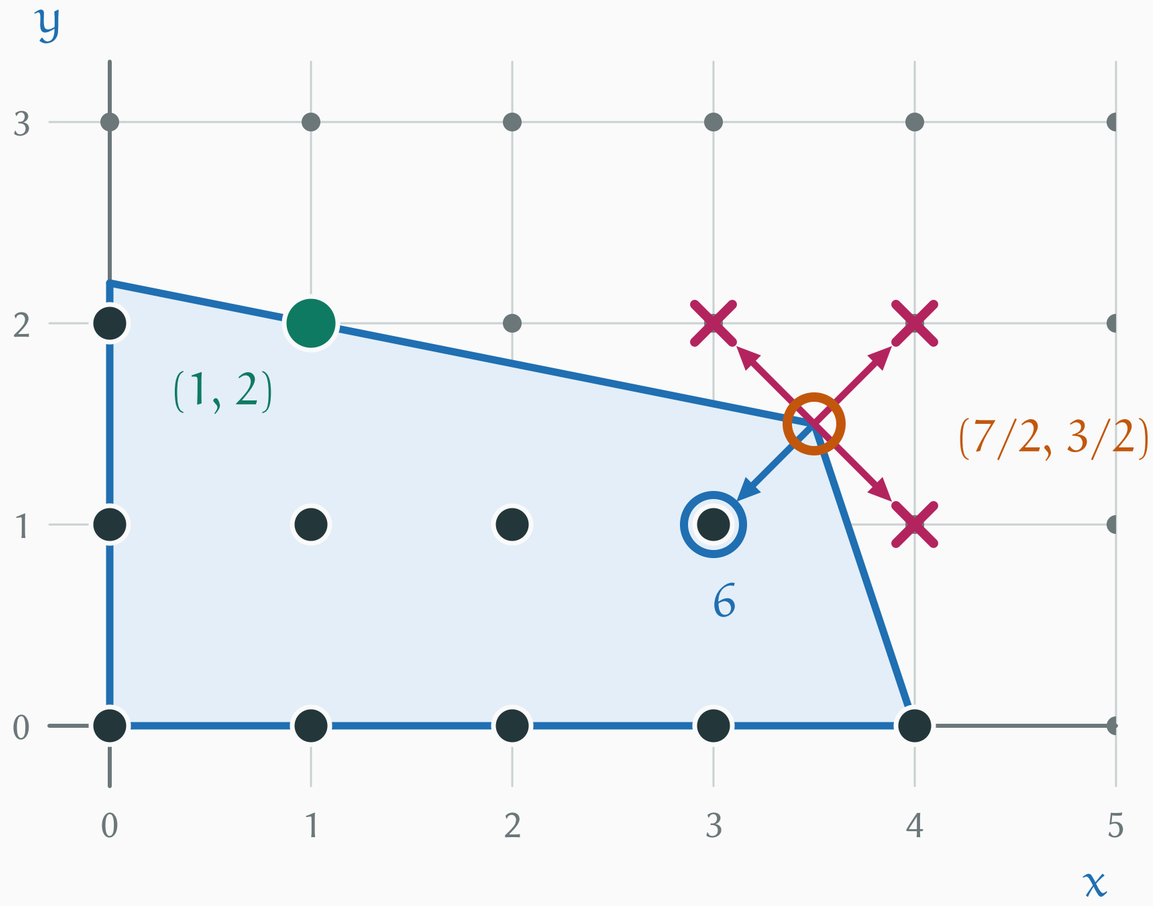
Rounding does not work

? *Solve the LP, then round?*

The four roundings of $(7/2, 3/2)$:

rounding	feasible?	value
$(3, 1)$	✓	6
$(3, 2)$	✗ $x + 5y \leqslant 11$ fails	
$(4, 1)$	✗ $3x + y \leqslant 12$ fails	
$(4, 2)$	✗ $3x + y \leqslant 12, x + 5y \leqslant 11$ fails	

Best rounding: $(3, 1)$, value 6.
The optimum $(1, 2)$, value 7, is not near.



Easy and hard

	Linear programs	Integer linear programs
feasible set	a polyhedron	the integer points in it
optimum	at a vertex	not necessarily at a vertex
methods	simplex (next lecture), ellipsoid, interior point	try all integer points, or branch and bound, cutting planes: still exponential
time	**weakly polynomial** polynomial in m, n and the bit length of the numbers	**NP-hard** no polynomial algorithm unless P = NP

Summary

1. **An LP:** linear objective, linear constraints.
   Infeasible, unbounded or optimal.
2. **Geometry:** an optimum can be moved to a vertex.
   There may be exponentially many vertices.
3. **Fourier–Motzkin:** eliminate unknowns by pairing bounds.
   Correct, but doubly exponential.
4. **Integer LPs:** NP-hard. Vertices and rounding fail.
   LPs can be solved in weakly polynomial time.

Next: the simplex method.

> If an LP has an optimal solution, a vertex is optimal.

Reference: Vanderbei, Chapter 1