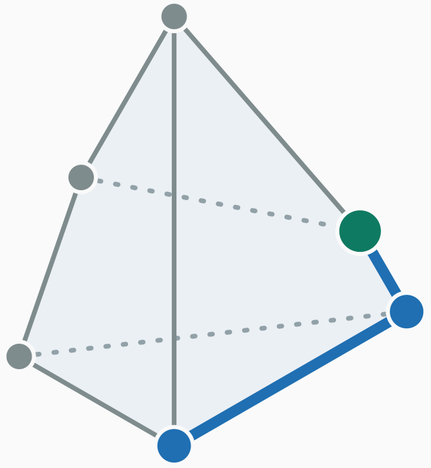
ALGORITHMS AND DATA STRUCTURES · LECTURE 14

The simplex method



Heng Guo

School of Informatics, University of Edinburgh

Thursday 5 November 2026 · 16:10–17:00

From vertices to a walk

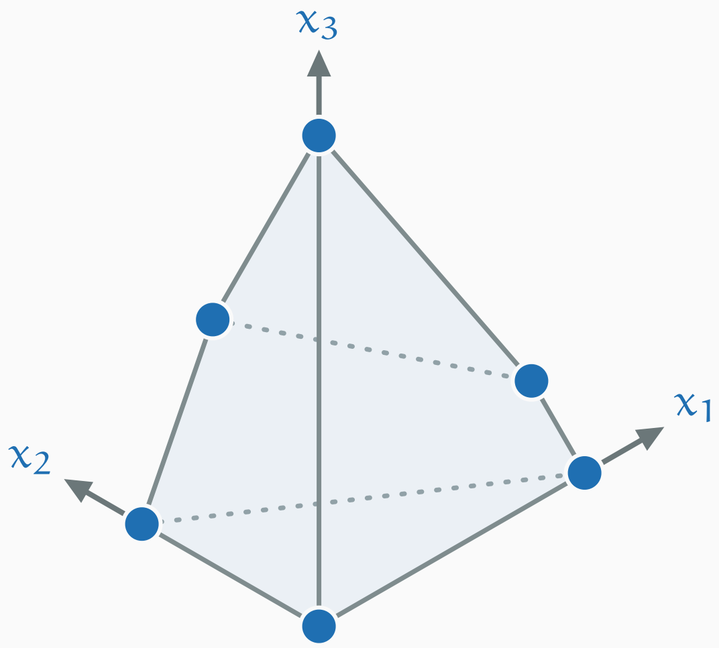
> **Lecture 13.** If an LP has an optimal solution, then a vertex is optimal.

Trying all vertices: $\binom{m+n}{n}$ candidates.
Far too many: $\binom{100}{50} \approx 10^{29}$.

? *Find the best vertex without visiting them all?*

Idea: walk from vertex to vertex, uphill.

This is the **simplex method** (Dantzig, 1947).



Why the simplex method?

Not polynomial: for the classical pivot rules, some LPs need exponentially many pivots (**next lecture**).

Klee and Minty, 1972.

Other algorithms: the ellipsoid method and interior-point methods are weakly polynomial.

Khachiyan, 1979 (ellipsoid); Karmarkar, 1984 (interior point).

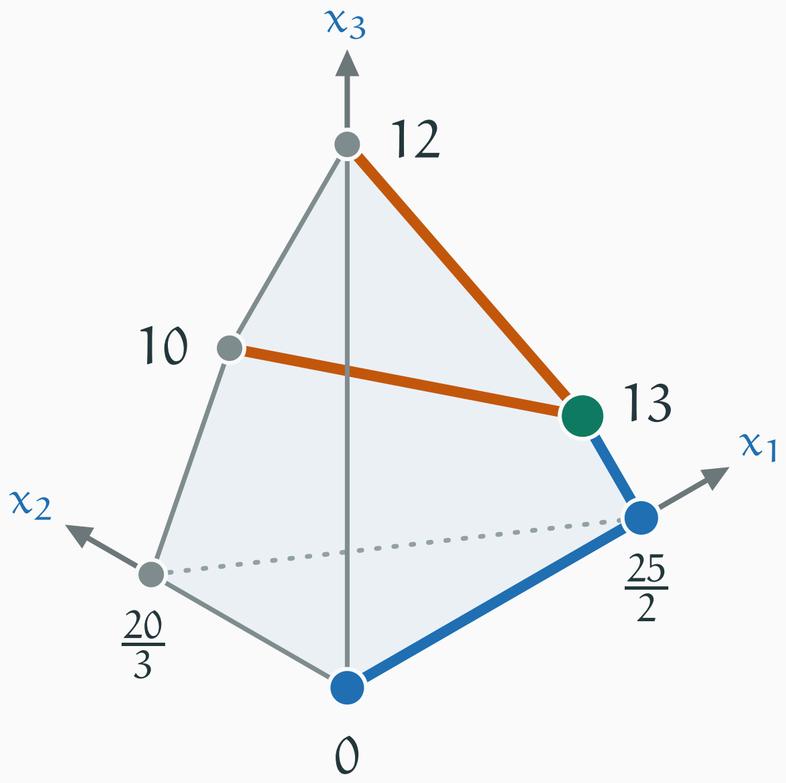
? *Then why study the simplex method?*

1. **Insight** into LPs: vertices, proofs of optimality, duality (**next lecture**).
2. **Fast in practice.**
   Smoothed analysis: after a small random perturbation of the data, the expected number of pivots is polynomial.
   Spielman and Teng, 2001.
3. **Open problem:** a pivot rule that makes it polynomial may still exist.

Walking from vertex to vertex

Our example.

$$\begin{aligned} \text{maximise} \quad & \zeta = 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leqslant 5 \\ & 4x_1 + x_2 + 2x_3 \leqslant 11 \\ & 3x_1 + 4x_2 + 2x_3 \leqslant 8 \\ & x_1, x_2, x_3 \geqslant 0 \end{aligned}$$



ζ at every vertex.

Walk: move to a better neighbouring vertex.
Stop when no neighbour is better.

At $(2, 0, 1)$, $\zeta = 13$.

No neighbour is better $(25/2, 12, 10)$: **stop**.

? *How do we find the neighbours without a picture?*

Vertices and dictionaries

How do we describe a vertex without a picture?

Slack variables

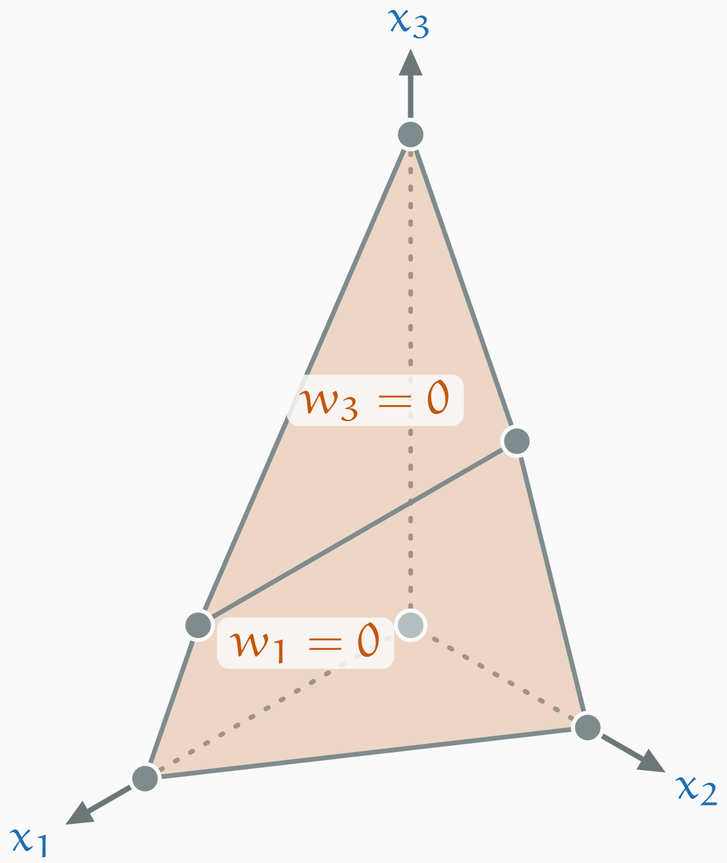
$$
\begin{aligned}
\text{maximise} \quad & \zeta = 0 \;\; + 5x_1 + 4x_2 + 3x_3 \\
\text{subject to} \quad & w_1 = 5 \;\; - 2x_1 - 3x_2 - \;\; x_3 \\
& w_2 = 11 - 4x_1 - \;\; x_2 - 2x_3 \\
& w_3 = 8 \;\; - 3x_1 - 4x_2 - 2x_3 \\
& x_1, x_2, x_3, w_1, w_2, w_3 \geqslant 0
\end{aligned}
$$

Slack variable w_i: the room that is left in constraint i.

Constraint i holds $\iff$ $w_i \geqslant 0$: **the same LP**.

$x_j = 0$: the point is on a coordinate plane.

$w_i = 0$: constraint i is tight, the point is on its face.
A vertex: **three of the six variables are** 0.



Slack variables in general

The LP, with three variables and three constraints:

$$\begin{aligned}
\text{maximise} \quad & c_1x_1 + c_2x_2 + c_3x_3 \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leqslant b_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leqslant b_2 \\
& a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \leqslant b_3 \\
& x_1, x_2, x_3 \geqslant 0
\end{aligned}$$

In matrix form:

maximise $c^T x$ subject to $Ax \leqslant b$, $x \geqslant 0$.

With slack variables w_1, w_2, w_3:

$$\begin{aligned}
\text{maximise} \quad & \zeta = 0 + c_1x_1 + c_2x_2 + c_3x_3 \\
\text{subject to} \quad & w_1 = b_1 - a_{11}x_1 - a_{12}x_2 - a_{13}x_3 \\
& w_2 = b_2 - a_{21}x_1 - a_{22}x_2 - a_{23}x_3 \\
& w_3 = b_3 - a_{31}x_1 - a_{32}x_2 - a_{33}x_3 \\
& x_1, x_2, x_3, w_1, w_2, w_3 \geqslant 0
\end{aligned}$$

In matrix form:

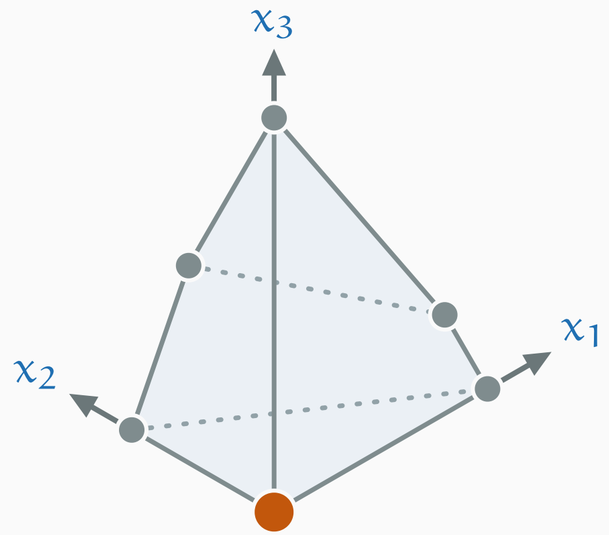
$\zeta = c^T x$, $w = b - Ax$, $x, w \geqslant 0$.

Dictionaries

Dictionary. The basic variables and ζ, written with the non-basic variables.

basic | non-basic

$$\begin{aligned} \zeta &= 0 &&+ 5x_1 + 4x_2 + 3x_3 \\ \hline w_1 &= 5 &&- 2x_1 - 3x_2 - x_3 \\ w_2 &= 11 &&- 4x_1 - x_2 - 2x_3 \\ w_3 &= 8 &&- 3x_1 - 4x_2 - 2x_3 \end{aligned}$$



Basic solution: set the non-basic variables to 0.

$x = (0, 0, 0)$ $w = (5, 11, 8)$ $\zeta = 0$

Feasible: all basic variables $\geqslant 0$.

Vertex: three variables are 0.

The pivot

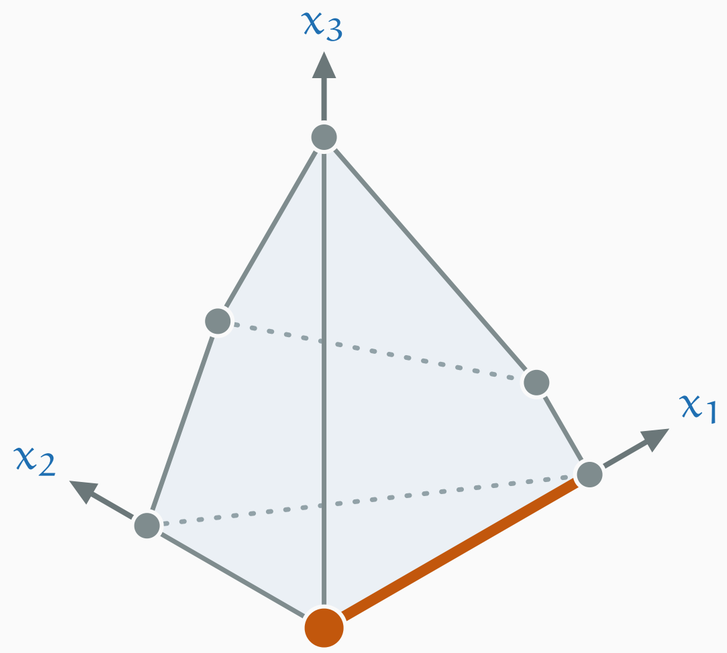
How do we move to a better neighbouring vertex?

Step 1: the entering variable

$$
\begin{aligned}
\zeta &= 0 \quad &&+ 5x_1 + 4x_2 + 3x_3 \\
\hline
w_1 &= 5 \quad &&- 2x_1 - 3x_2 - x_3 \\
w_2 &= 11 \quad &&- 4x_1 - x_2 - 2x_3 \\
w_3 &= 8 \quad &&- 3x_1 - 4x_2 - 2x_3
\end{aligned}
$$

Entering variable: x_1, the largest coefficient (5).

Any positive coefficient would do.

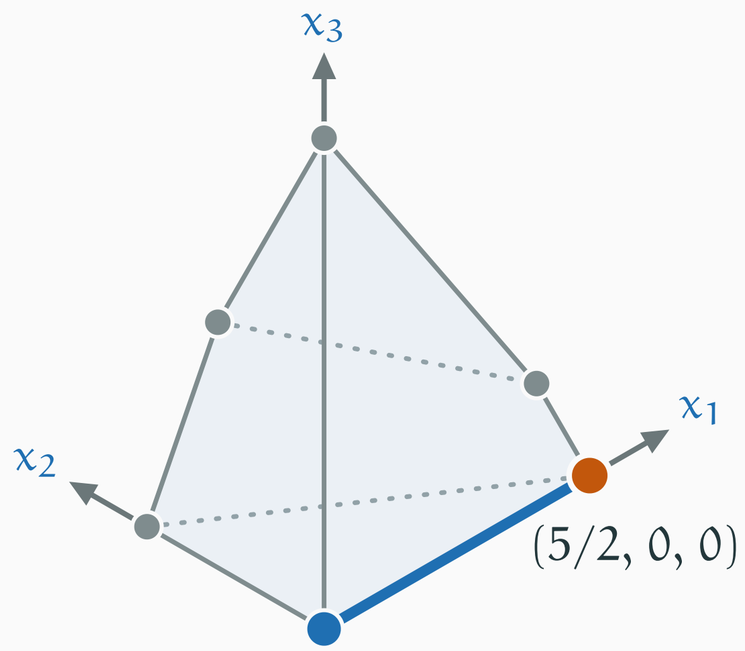


Step 2: how far? The ratio test

$$\begin{array}{rcrrrrr} \zeta & = & 0 & +5x_1 & +4x_2 & +3x_3 & \\ \hline w_1 & = & 5 & \boxed{-2x_1} & -3x_2 & -x_3 & \frac{5}{2} \\ w_2 & = & 11 & -4x_1 & -x_2 & -2x_3 & \frac{11}{4} \\ w_3 & = & 8 & -3x_1 & -4x_2 & -2x_3 & \frac{8}{3} \end{array}$$

limit value

New vertex: $x = (5/2, 0, 0)$ $w = (0, 1, 1/2)$ $\zeta = 25/2$



Step 3: the new dictionary

$$
\begin{aligned}
\zeta &= 0 + 5x_1 + 4x_2 + 3x_3 \\
\hline
w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\
w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\
w_3 &= 8 - 3x_1 - 4x_2 - 2x_3
\end{aligned}
\quad\longrightarrow\quad
\begin{aligned}
\zeta &= \tfrac{25}{2} - \tfrac{5}{2}w_1 - \tfrac{7}{2}x_2 + \tfrac{1}{2}x_3 \\
\hline
x_1 &= \tfrac{5}{2} - \tfrac{1}{2}w_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
w_3 &= \tfrac{1}{2} + \tfrac{3}{2}w_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3
\end{aligned}
$$

Same system, new point of view: a new vertex, and ζ went up.

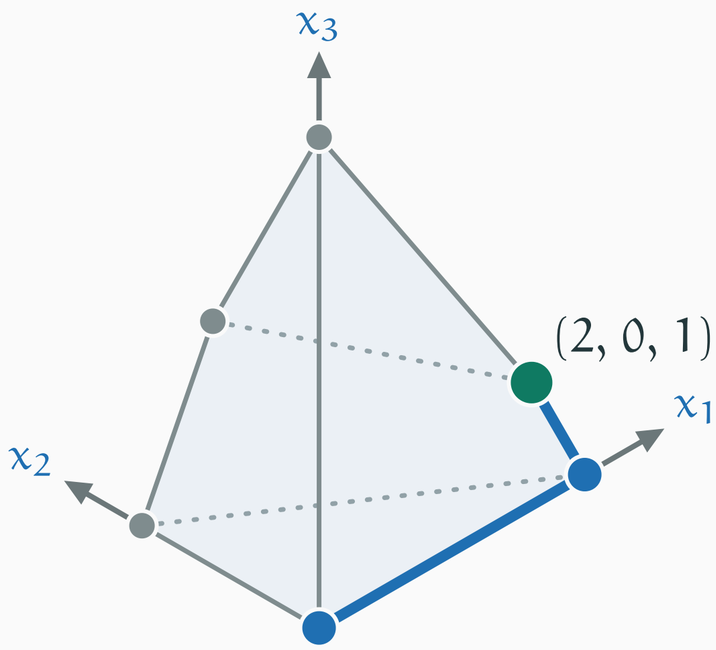
before	$x = (0, 0, 0)$	$w = (5, 11, 8)$	$\zeta = 0$
after	$x = (5/2, 0, 0)$	$w = (0, 1, 1/2)$	$\zeta = 25/2$

New basic variables: $x_1 = 5/2$, $w_2 = 1$, $w_3 = 1/2$.
New non-basic variables: $w_1 = x_2 = x_3 = 0$.

Again: a second pivot

$$\begin{aligned}
\zeta &= 13 - w_1 - 3x_2 - w_3 \\
\hline
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}$$

value



No positive coefficient: **stop**.

$x = (2, 0, 1)$ $w = (0, 1, 0)$ $\zeta = 13$

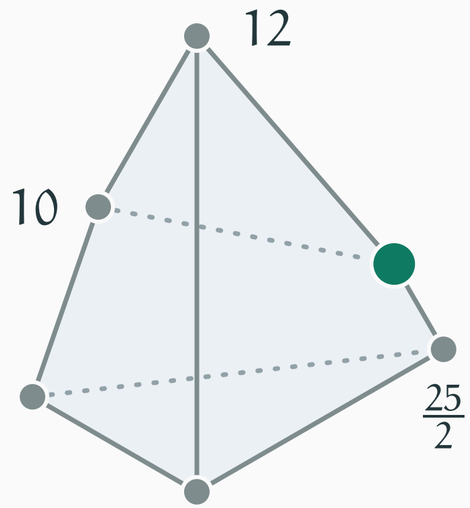
The algorithm

When can we stop, and does the method always stop?

Why we can stop

Optimality. If a feasible dictionary has no positive coefficient in the ζ row, its basic solution is optimal.

1. The last dictionary: $\zeta = 13 - w_1 - 3x_2 - w_3$
2. A rewriting of the objective: true for **every** point.
3. Feasible: $w_1, x_2, w_3 \geqslant 0$, so $\zeta \leqslant 13$.
4. The basic solution has $\zeta = 13$: **optimal**.



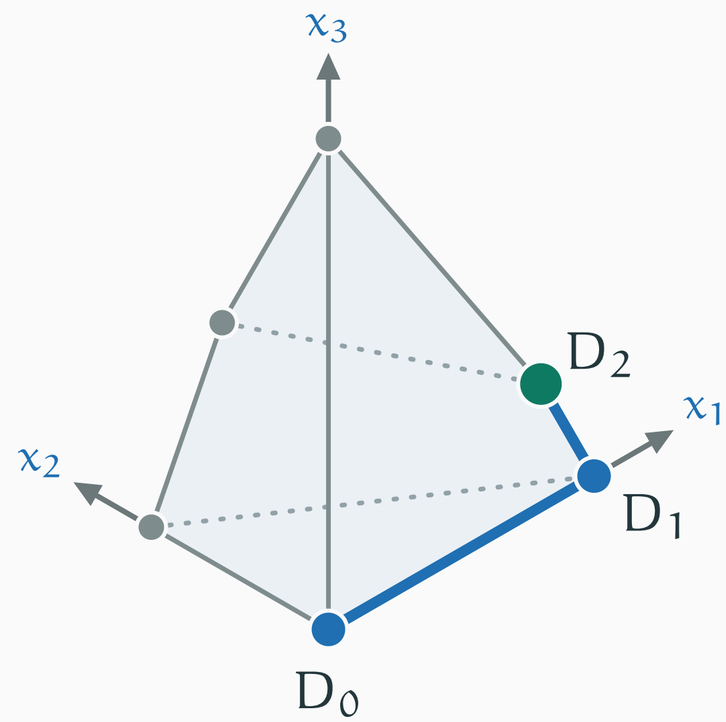
$$\begin{aligned}\zeta &= 13 - (5 - 2x_1 - 3x_2 - x_3) - 3x_2 - (8 - 3x_1 - 4x_2 - 2x_3) \\ &= 5x_1 + 4x_2 + 3x_3\end{aligned}$$

Dictionaries are vertices, pivots are edges

	non-basic variables	vertex	ζ
D_0	x_1, x_2, x_3	$(0, 0, 0)$	0
D_1	x_2, x_3, w_1	$(5/2, 0, 0)$	$25/2$
D_2	x_2, w_1, w_3	$(2, 0, 1)$	13

A **pivot** swaps a non-basic and a basic variable:
one constraint is released, another becomes tight.
The other two stay tight: we slide along an **edge**.

> **Fact.** A feasible dictionary is a vertex.
> No positive coefficient: no edge goes up.



1 $D \leftarrow$ initial dictionary ($b \geqslant 0$)
2 **while** some $\widehat{c}_j > 0$ **do**
3 choose k with $\widehat{c}_k > 0$
4 choose r minimising $\widehat{b}_r / \widehat{a}_{rk}$
5 $D \leftarrow \text{pivot}(D, k, r)$
6 **return** basic solution of D, value $\widehat{\zeta}$

Notation. The current dictionary, its entries with hats:

$\zeta = \widehat{\zeta} + \sum_j \widehat{c}_j x_j, \quad x_i = \widehat{b}_i - \sum_j \widehat{a}_{ij} x_j$ (i basic, j non-basic)

For example, in matrix form:

$$\zeta = \widehat{\zeta} + \begin{pmatrix} \widehat{c}_1 & \widehat{c}_2 & \widehat{c}_3 \end{pmatrix} x_N$$

$$x_B = \begin{pmatrix} \widehat{b}_1 \\ \widehat{b}_2 \\ \widehat{b}_3 \end{pmatrix} - \begin{pmatrix} \widehat{a}_{11} & \widehat{a}_{12} & \widehat{a}_{13} \\ \widehat{a}_{21} & \widehat{a}_{22} & \widehat{a}_{23} \\ \widehat{a}_{31} & \widehat{a}_{32} & \widehat{a}_{33} \end{pmatrix} x_N$$ (x_B basic, x_N non-basic)

$$\begin{aligned} \zeta &= 13 - w_1 - 3x_2 - w_3 \\ \hline x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3 \end{aligned}$$

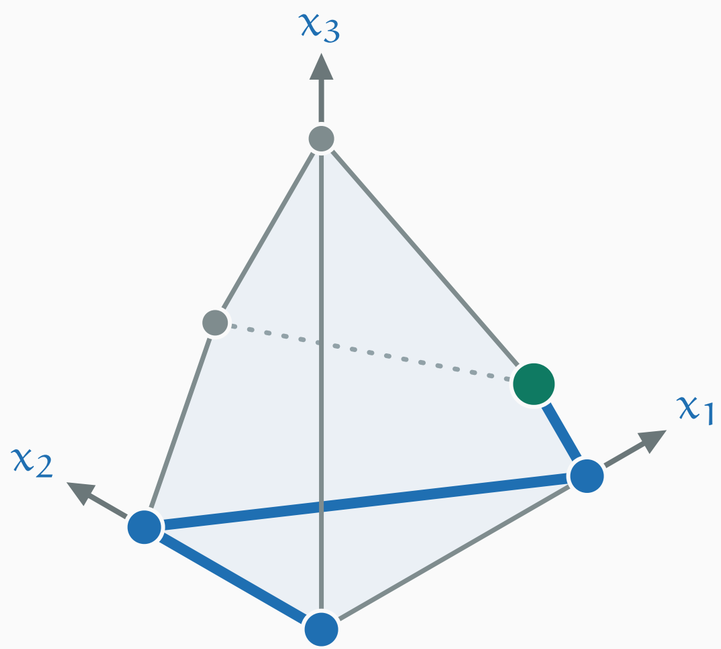
Optimal

$x = (2, 0, 1) \quad w = (0, 1, 0) \quad \zeta = 13$

The entering variable is a free choice

Any variable with a positive coefficient may enter.

first	entering variables	ζ after each pivot	pivots
x_1	x_1, x_3	25/2, 13	2
x_2	x_2, x_1, x_3	20/3, 25/2, 13	3
x_3	x_3, x_1	12, 13	2



Different path, different length, same optimum.
Common rule: the largest coefficient (x_1).

Pivot rules: the largest coefficient is Dantzig's rule, the classical one.
Others: largest increase, steepest edge, Bland's rule (**next lecture**), random edge, random facet.

Does the simplex method stop?

1. A pivot with step $t > 0$ raises ζ by $\widehat{c}_k\, t > 0$.
2. Thus ζ increases strictly: $0 < 25/2 < 13$.
   No dictionary returns.
3. A dictionary is fixed by its basic variables:
   at most $\binom{m+n}{m}$ dictionaries.
4. **Hence the method stops.**

Catch. If the step is $t = 0$, then ζ stays:
the same vertex, a new dictionary.
It can cycle.
Next lecture: degeneracy.

$$\begin{aligned}
\zeta &= 3 - \tfrac{1}{2}x_1 + 2x_2 - \tfrac{3}{2}w_1 && \text{limit}\\
\hline
x_3 &= 1 - \tfrac{1}{2}x_1 - \tfrac{1}{2}w_1 && -\\
w_2 &= 0 + x_1 - x_2 + w_1 && 0
\end{aligned}$$

Ratio 0: step $t = 0$.
After the pivot ζ is still 3.

Potential problems

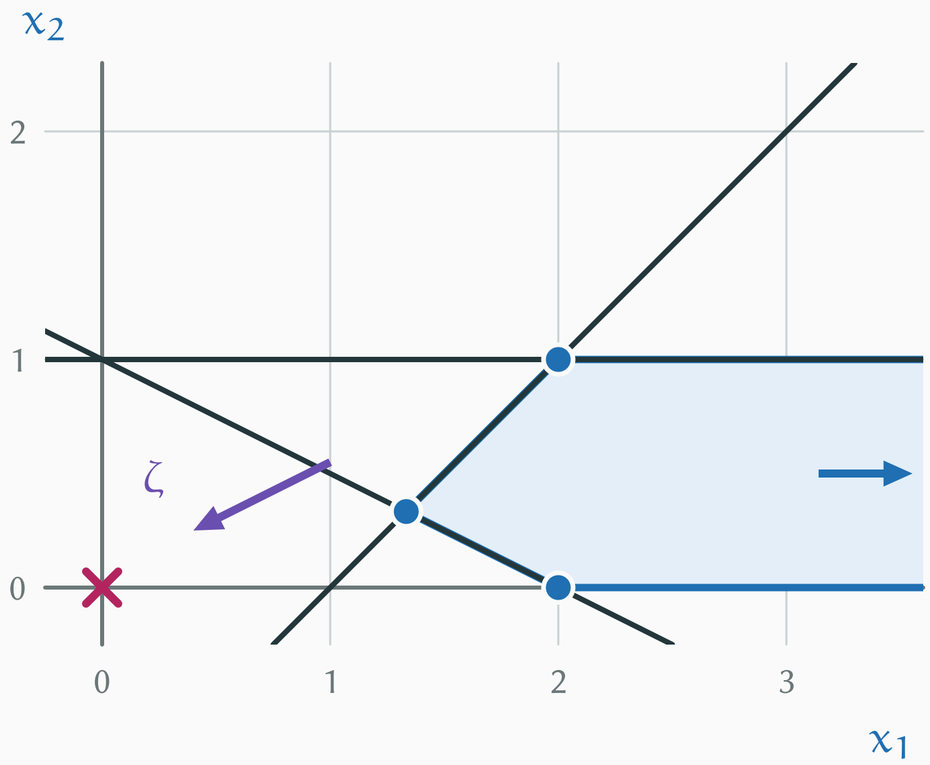
What if the origin is not feasible, or the LP is unbounded?

Potential issue 1: an infeasible start

$$
\begin{aligned}
\text{maximise} \quad & -2x_1 - x_2 \\
\text{subject to} \quad & -x_1 + x_2 \leqslant -1 \\
& -x_1 - 2x_2 \leqslant -2 \\
& x_2 \leqslant 1 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

$$
\begin{aligned}
\zeta &= 0 - 2x_1 - x_2 \\
\hline
w_1 &= -1 + x_1 - x_2 \\
w_2 &= -2 + x_1 + 2x_2 \\
w_3 &= 1 - x_2
\end{aligned}
$$

Basic solution: $x = (0, 0)$ $w = (-1, -2, 1)$: **infeasible**.



? *Where do we start?*

We need a feasible vertex.

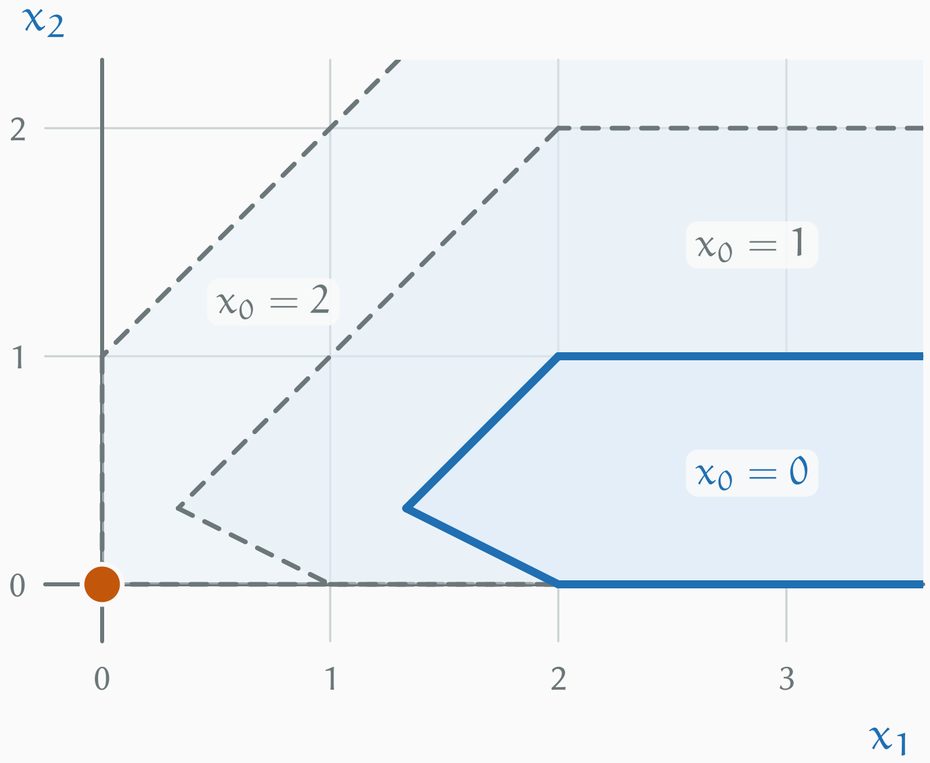
The auxiliary problem

Auxiliary LP. Relax every constraint by x_0, and make x_0 small.

$$\begin{aligned} \text{maximise} \quad & -x_0 \\ \text{subject to} \quad & -x_1 + x_2 - x_0 \leqslant -1 \\ & -x_1 - 2x_2 - x_0 \leqslant -2 \\ & x_2 - x_0 \leqslant 1 \\ & x_0, x_1, x_2 \geqslant 0 \end{aligned}$$

Lemma. The LP is feasible
$\iff$ the auxiliary LP has optimum 0.

$\Leftarrow$: with $x_0 = 0$ the original constraints hold.
$\Rightarrow$: a feasible x gives $(x, 0)$ with value 0, and $-x_0 \leqslant 0$.



Easy start: $x = 0$ with x_0 large enough is feasible.

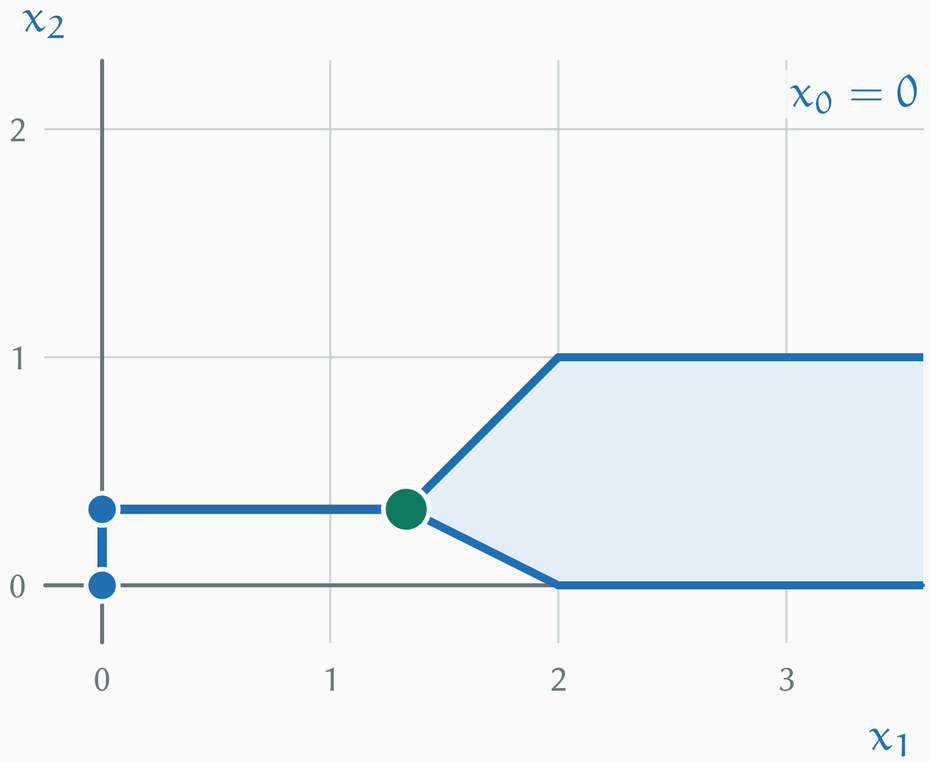
Phase 1: solve the auxiliary LP

$$\zeta = 0 - x_0$$

$$\begin{aligned} x_2 &= \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2 \\ x_1 &= \frac{4}{3} - x_0 + \frac{2}{3}w_1 + \frac{1}{3}w_2 \\ w_3 &= \frac{2}{3} + x_0 + \frac{1}{3}w_1 - \frac{1}{3}w_2 \end{aligned}$$

Optimum $x_0 = 0$: the LP is **feasible**.

A vertex of it: $(4/3, 1/3)$.



Phase 2: back to the original LP

$$
\begin{aligned}
\zeta &= 0 - x_0 \\
\hline
x_2 &= \tfrac{1}{3} - \tfrac{1}{3}w_1 + \tfrac{1}{3}w_2 \\
x_1 &= \tfrac{4}{3} - x_0 + \tfrac{2}{3}w_1 + \tfrac{1}{3}w_2 \\
w_3 &= \tfrac{2}{3} + x_0 + \tfrac{1}{3}w_1 - \tfrac{1}{3}w_2
\end{aligned}
\qquad \longrightarrow \qquad
\begin{aligned}
\zeta &= -3 - w_1 - w_2 \\
\hline
x_2 &= \tfrac{1}{3} - \tfrac{1}{3}w_1 + \tfrac{1}{3}w_2 \\
x_1 &= \tfrac{4}{3} + \tfrac{2}{3}w_1 + \tfrac{1}{3}w_2 \\
w_3 &= \tfrac{2}{3} + \tfrac{1}{3}w_1 - \tfrac{1}{3}w_2
\end{aligned}
$$

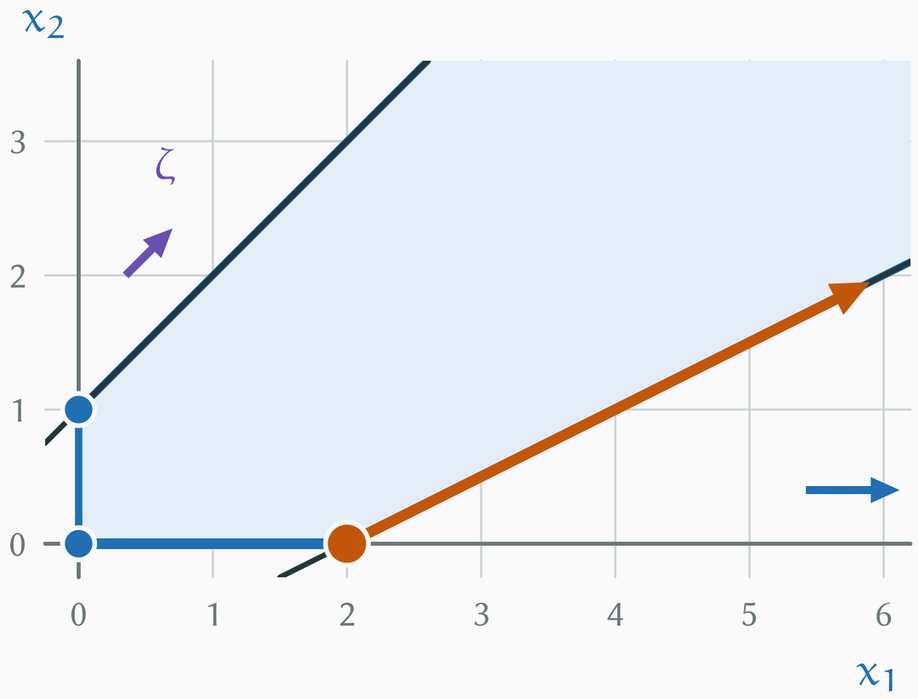
No positive coefficient: **optimal**, $x = (4/3, 1/3)$ $\quad \zeta = -3$

$$
\begin{aligned}
\zeta &= -2x_1 - x_2 \\
&= -2\left(\tfrac{4}{3} + \tfrac{2}{3}w_1 + \tfrac{1}{3}w_2\right) - \left(\tfrac{1}{3} - \tfrac{1}{3}w_1 + \tfrac{1}{3}w_2\right) \\
&= -3 - w_1 - w_2
\end{aligned}
$$

Potential issue 2: an unbounded LP

$$\begin{aligned} \text{maximise} \quad & x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leqslant 1 \\ & x_1 - 2x_2 \leqslant 2 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

$$\begin{aligned} \zeta &= 2 - w_2 + 3x_2 && \text{limit} \\ \hline w_1 &= 3 - w_2 + x_2 && - \\ x_1 &= 2 - w_2 + 2x_2 && - \end{aligned}$$



x_2 can grow forever: ζ grows without bound.

Unbounded.

The whole method: three outcomes

```
1  if some b_i < 0 then phase 1
2      solve the auxiliary LP (maximise −x_0)
3      if its optimum is < 0 then return infeasible
4      D ← feasible dictionary without x_0
5  else D ← initial dictionary
6  while some ĉ_j > 0 do phase 2
7      choose k with ĉ_k > 0
8      if all â_ik ⩽ 0 then return unbounded
9      r ← row minimising b̂_r/â_rk, â_rk > 0
10     D ← pivot(D, k, r)
11 return basic solution of D (optimal)
```

Infeasible. Phase 1 ends with $x_0 > 0$.
$x_1 + x_2 \leqslant 1$ and $x_1 + x_2 \geqslant 2$:
optimum $-1/2$.

Unbounded. No row limits
the entering variable.

Optimal. No positive coefficient.

How fast is it? A little history

HOW FAST

One pivot: $O(mn)$ operations.

Number of pivots: **no polynomial bound** is known.

$n = m$	vertices at most $\binom{2n}{n}$	pivots, average	largest
5	252	**2.1**	6
10	184 756	**3.1**	6
15	155 117 520	**3.9**	8
20	137 846 528 820	**5.2**	12

Random LPs, 100 for each n.

WHY STUDY IT

- insight into LPs: geometry, duality (**next lecture**)
- fast in practice and used in most LP solvers

HISTORY

Invented in 1947
by **George Dantzig**.

Dantzig, *Origins of the Simplex Method.*
In: *A History of Scientific Computing*, 1990.

Summary

1. **Dictionaries.** Slack variables make $Ax \leqslant b$ equations.
   A feasible dictionary is a vertex.
2. **A pivot.** Entering variable: $\widehat{c}_k > 0$.
   Leaving variable: smallest ratio.
   Solve and substitute: a better neighbouring vertex.
3. **Stopping.** No $\widehat{c}_j > 0$: optimal.
   Finitely many dictionaries.
   Zero ratios (degeneracy): **next lecture**.
4. **Potential problems.** Infeasible start: auxiliary LP (phase 1).
   No limiting row: unbounded.

> The simplex method walks from vertex to vertex, always uphill, until no edge goes up.

Reference: Vanderbei, Chapter 2