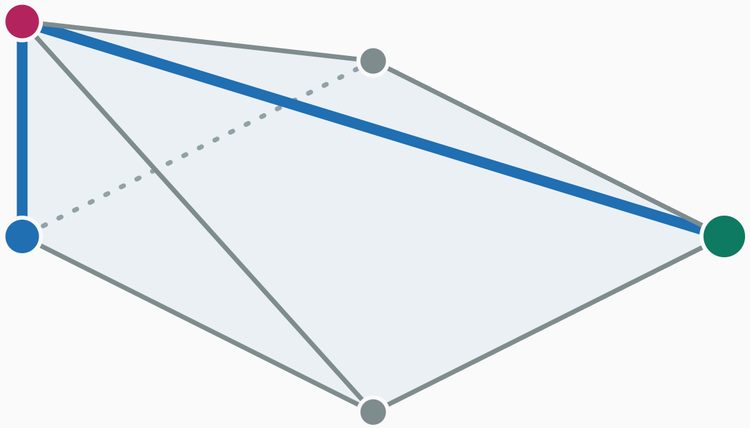
ALGORITHMS AND DATA STRUCTURES · LECTURE 15

Degeneracy, geometry and duality



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A loose end

Last lecture. A step $t > 0$ raises ζ:
no dictionary repeats,
so the method stops.

Loose end. A step $t = 0$ leaves ζ where it is:
same vertex, new dictionary.

? *Can the method go round in circles?*

$$\begin{aligned} \zeta &= 3 - \tfrac{1}{2}x_1 + 2x_2 - \tfrac{3}{2}w_1 && \text{limit} \\ \hline x_3 &= 1 - \tfrac{1}{2}x_1 - \tfrac{1}{2}w_1 && - \\ w_2 &= 0 + x_1 - x_2 + w_1 && 0 \end{aligned}$$

Degeneracy

What if the ratio test has a tie, and a pivot does not move?

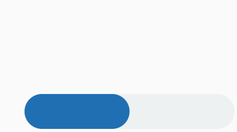
A tie in the ratio test

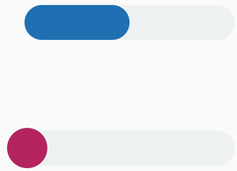
$$\zeta = 3 - \tfrac{1}{2}x_1 + 2x_2 - \tfrac{3}{2}w_1$$

$$x_3 = 1 - \tfrac{1}{2}x_1 - \tfrac{1}{2}w_1$$

$$w_2 = 0 + x_1 - x_2 + w_1$$

value

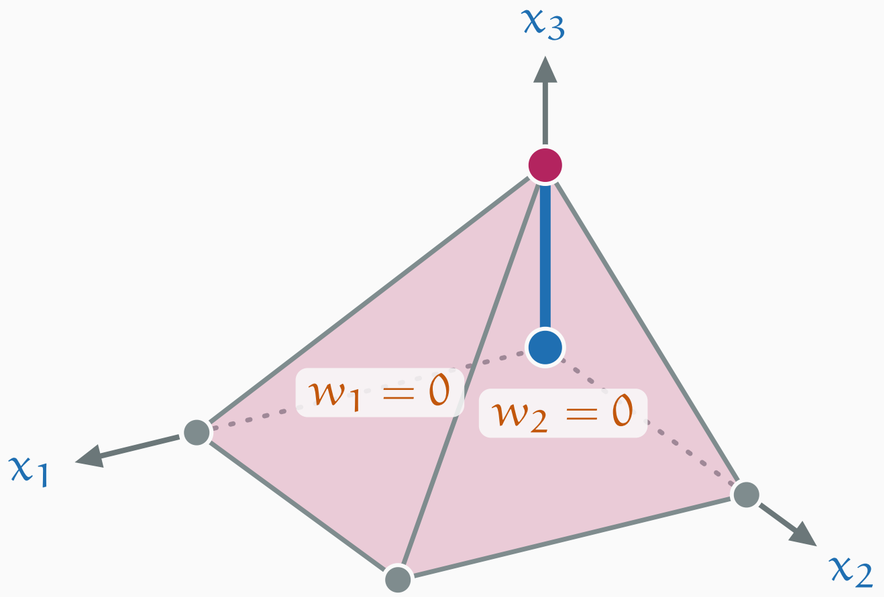




w_1 leaves (first row). Now w_2 is basic with value 0.

$x = (0, 0, 1)$ $w = (0, 0)$ $\zeta = 3$

Degenerate. A dictionary is **degenerate** if a basic variable is 0: some $\widehat{b}_i = 0$.



A pivot that does not move

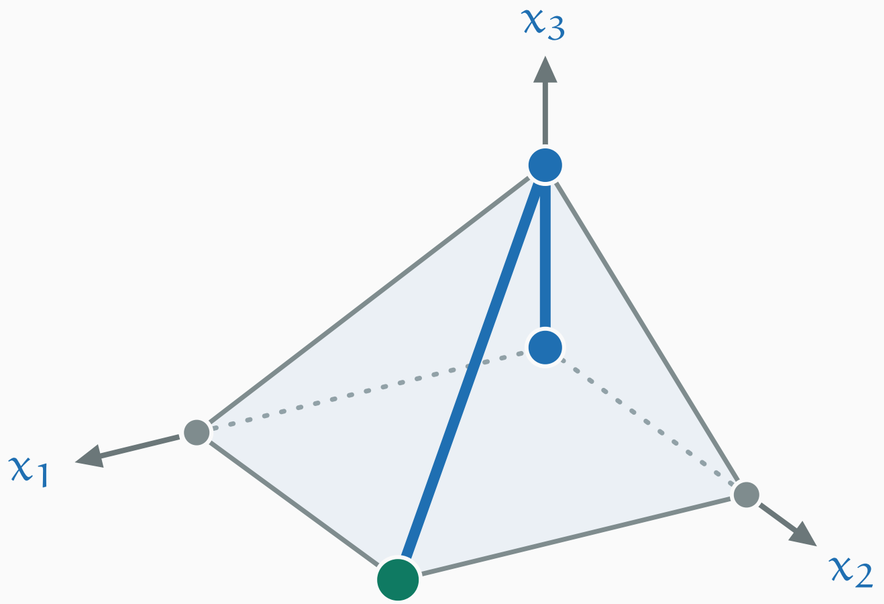
$$\begin{aligned} \zeta &= 3 + \tfrac{3}{2}x_1 - 2w_2 + \tfrac{1}{2}w_1 \\ \hline x_3 &= 1 \boxed{-\tfrac{1}{2}x_1} - \tfrac{1}{2}w_1 \\ x_2 &= 0 + x_1 - w_2 + w_1 \end{aligned}$$

	limit	value
	2	
	–	

A real step: ζ rises to 6.

In D_3, $\zeta = 6 - 3x_3 - 2w_2 - w_1$: **optimal.**

	non-basic	vertex	ζ
D_1	x_1, x_2, w_1	$(0, 0, 1)$	3
D_2	x_1, w_1, w_2	$(0, 0, 1)$	3
D_3	x_3, w_1, w_2	$(2, 2, 0)$	6



Facets, edges, vertices

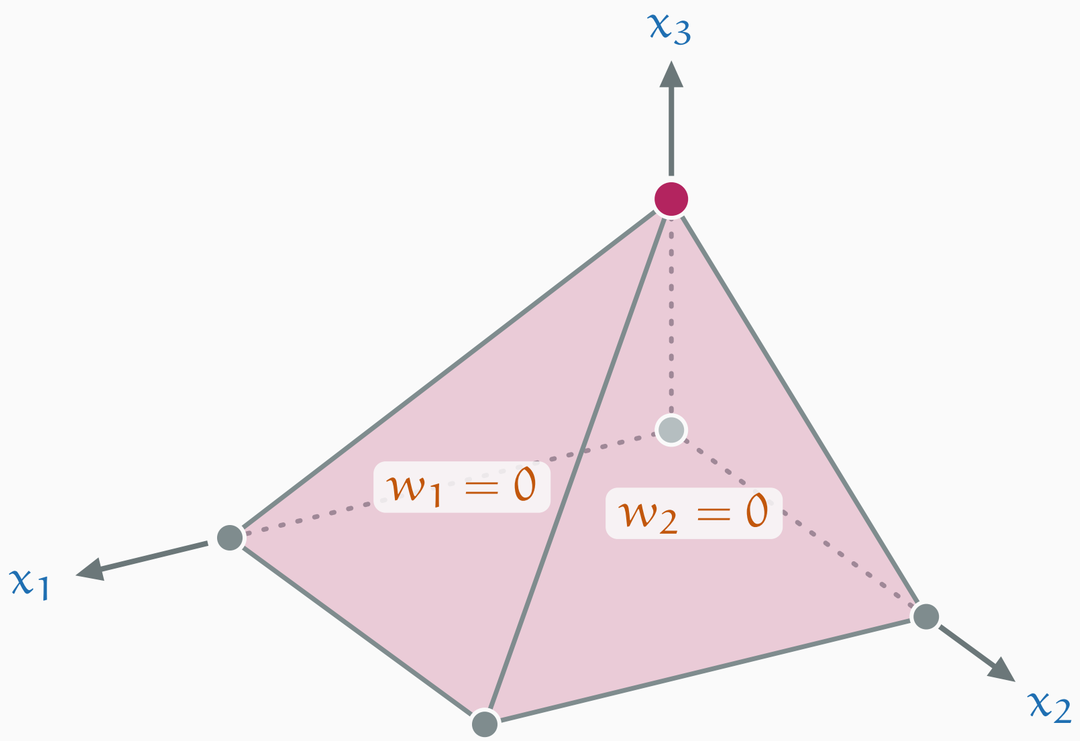
Half-spaces intersect in a **polyhedron**.
A **facet** lies in one boundary plane.

1. **facet:** one variable (x_j or w_i) is 0
2. **edge:** two variables are 0
3. **vertex:** $n = 3$ variables are 0

> **The apex:** four variables are 0:
> more than $n = 3$, so **degenerate**.

Any three of the four: **four dictionaries**.

x_1, x_2, w_1 D_1	x_1, x_2, w_2
x_1, w_1, w_2 D_2	x_2, w_1, w_2



Cycling and speed

Can the simplex method run forever, and how many pivots does it need?

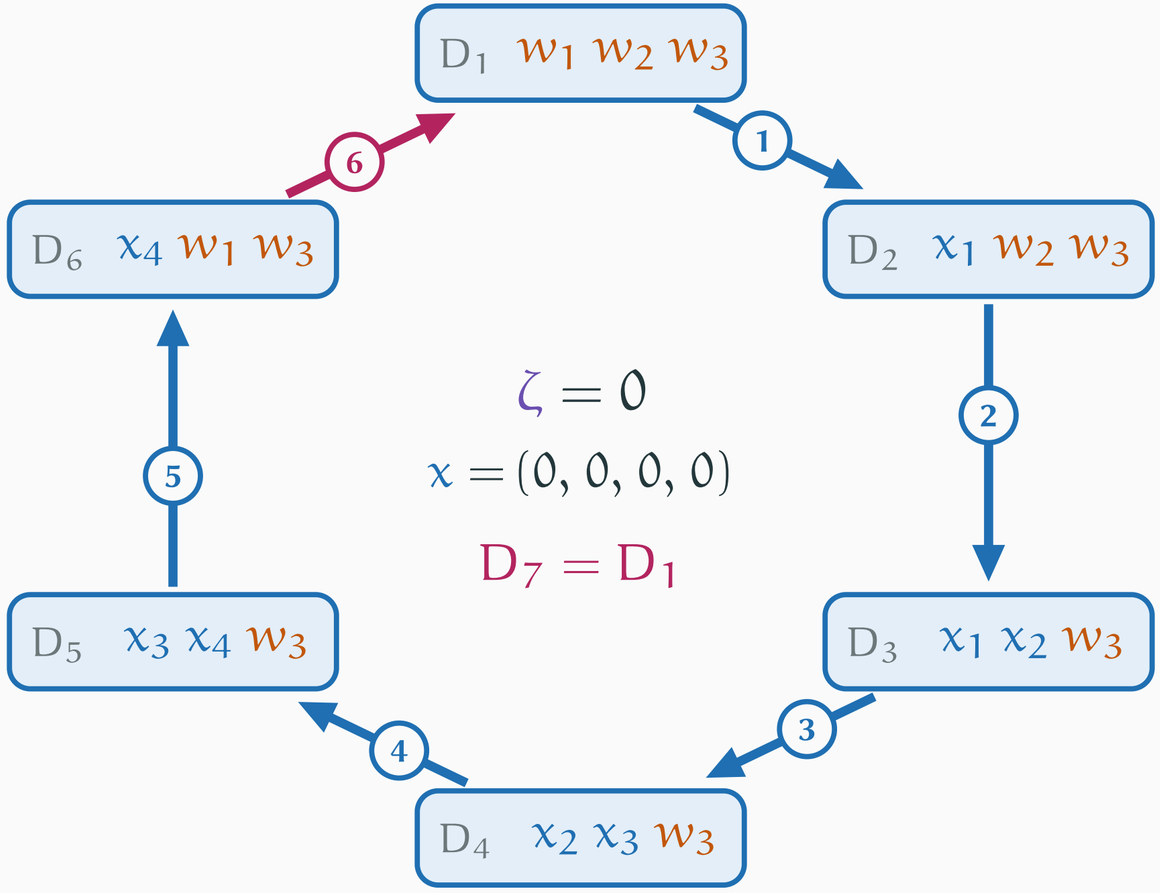
Degenerate pivots can go round in circles

$$\begin{aligned}
\zeta &= 0 + 10\,x_1 - 57\,x_2 - 9\,x_3 - 24\,x_4 \\
\hline
w_1 &= 0 - \tfrac{1}{2}x_1 + \tfrac{11}{2}x_2 + \tfrac{5}{2}x_3 - 9\,x_4 \\
w_2 &= 0 - \tfrac{1}{2}x_1 + \tfrac{3}{2}x_2 + \tfrac{1}{2}x_3 - x_4 \\
w_3 &= 1 - x_1
\end{aligned}$$

Largest coefficient enters. Ties: first row.

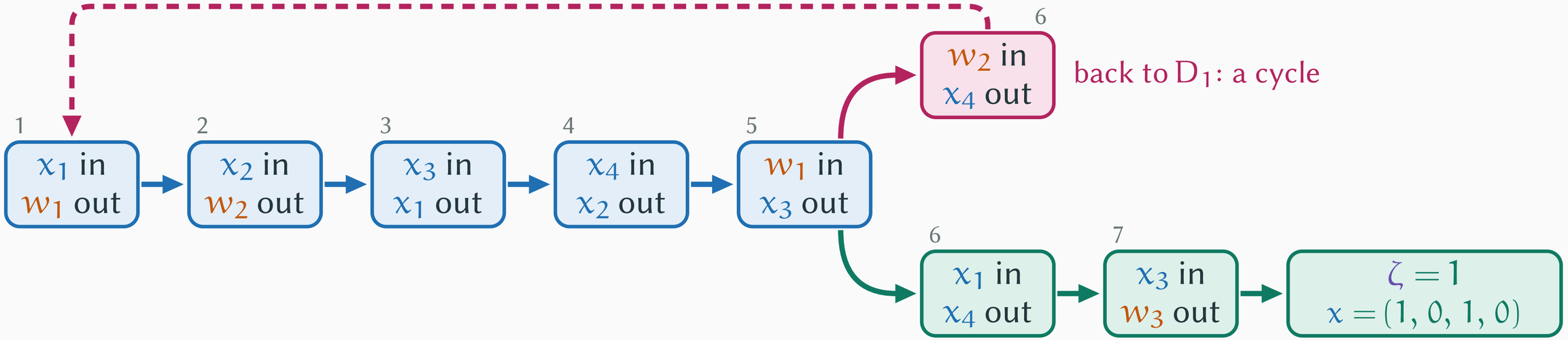
1	x_1 in, w_1 out	4	x_4 in, x_2 out
2	x_2 in, w_2 out	5	w_1 in, x_3 out
3	x_3 in, x_1 out	6	w_2 in, x_4 out

The dictionary after pivot 6 is D_1 again:
the method cycles.



Bland's rule

Bland's rule. Number the variables $x_1, \dots, x_n, w_1, \dots, w_m$.
Always take the **smallest index**:
entering: among the positive $\widehat{c}_j$;
leaving: among the rows tied in the ratio test.



Pivot 7 has step $t = 1$: ζ jumps to 1. Optimal.

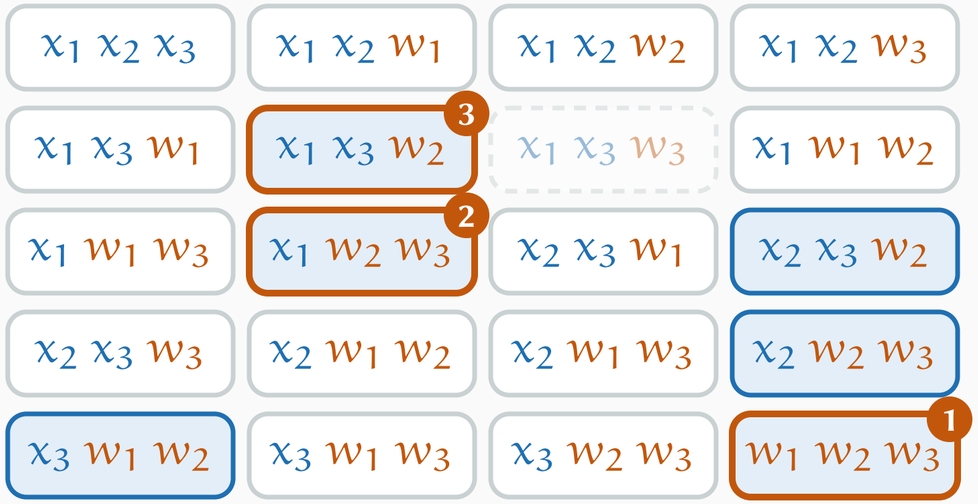
Theorem (Bland, 1977).
With Bland's rule the simplex method never cycles.

The simplex method terminates

Theorem. If the simplex method does not cycle, it terminates.

1. A dictionary is determined by its basic variables.
2. At most $\binom{n+m}{m}$ choices: here $\binom{6}{3} = 20$.
3. No dictionary repeats: at most $\binom{n+m}{m}$ pivots.

With Bland's rule **the method always stops.**

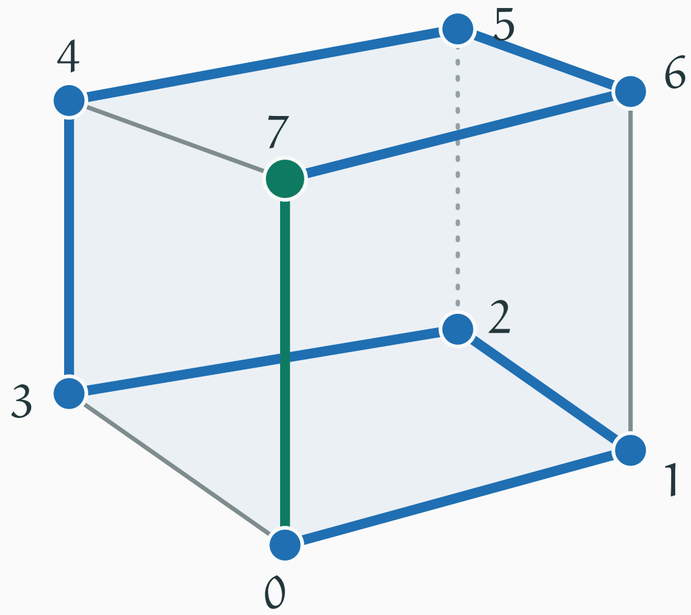


Blue: vertices.
Orange: Thursday's run.

Fundamental theorem. Every LP is infeasible, unbounded, or has an optimal solution.
A feasible LP has a feasible vertex.
If an optimum exists, a vertex is optimal.

How many pivots? The Klee–Minty cube

Klee–Minty: n variables,
a deformed cube with 2^n vertices.
The largest-coefficient rule
visits **all** of them: $2^n - 1$ pivots.



The optimum is a neighbour of the start: **one pivot** would do.

n	2	3	4	5	6	7	8
vertices	4	8	16	32	64	128	256
pivots	3	7	15	31	63	127	255

Bad news: exponential in the worst case for the classical rules.

Good news: fast in practice.
Smoothed analysis: polynomial after a small random perturbation of the data.
Spielman and Teng, 2004.
Ellipsoid and interior-point methods are weakly polynomial.

Duality

How can we prove that a solution is optimal?

How can we prove an upper bound?

$$
\begin{aligned}
\text{maximise} \quad & 5x_1 + 4x_2 + 3x_3 \\
\text{subject to} \quad (1)\ & 2x_1 + 3x_2 + x_3 \leqslant 5 \\
(2)\ & 4x_1 + x_2 + 2x_3 \leqslant 11 \\
(3)\ & 3x_1 + 4x_2 + 2x_3 \leqslant 8 \\
& x_1, x_2, x_3 \geqslant 0
\end{aligned}
$$

The simplex method found $x^* = (2, 0, 1)$ with value 13.

? *How can we prove that nothing is better?*

Idea: multiply constraint i by $y_i \geqslant 0$ and add.
If the new coefficients still dominate $5, 4, 3$, we get an upper bound.

y	$A^T y$	$\geqslant c$?	$b^T y$
$(1, 0, 0)$	$(2, 3, 1)$	**no**	5
$(1, 1, 1)$	$(9, 8, 5)$	**yes**	24
$(0, 0, 5/3)$	$(5, 20/3, 10/3)$	**yes**	$40/3$
$(1, 0, 1)$	$(5, 7, 3)$	**yes**	13

$$
\begin{aligned}
& 5x_1 + 4x_2 + 3x_3 \\
\leqslant\ & 5x_1 + 7x_2 + 3x_3 \\
=\ & (2x_1 + 3x_2 + x_3) + (3x_1 + 4x_2 + 2x_3) \\
\leqslant\ & 5 + 8 = 13
\end{aligned}
$$

The dual

Primal

maximise $c^T x$

subject to $Ax \leqslant b, \quad x \geqslant 0$

n variables, m constraints

	x_1	x_2	x_3		
(1)	2	3	1	$\leqslant$	5
(2)	4	1	2	$\leqslant$	11
(3)	3	4	2	$\leqslant$	8
c	5	4	3		max

Dual

minimise $b^T y$

subject to $A^T y \geqslant c, \quad y \geqslant 0$

m variables, n constraints

	y_1	y_2	y_3		
(1)	2	4	3	$\geqslant$	5
(2)	3	1	4	$\geqslant$	4
(3)	1	2	2	$\geqslant$	3
b	5	11	8		min

A **constraint** of the primal is a **variable** of the dual (row 1 and y_1).

A **variable** x_1 of the primal is a **constraint** of the dual (column 1 and row 1).

Weak duality

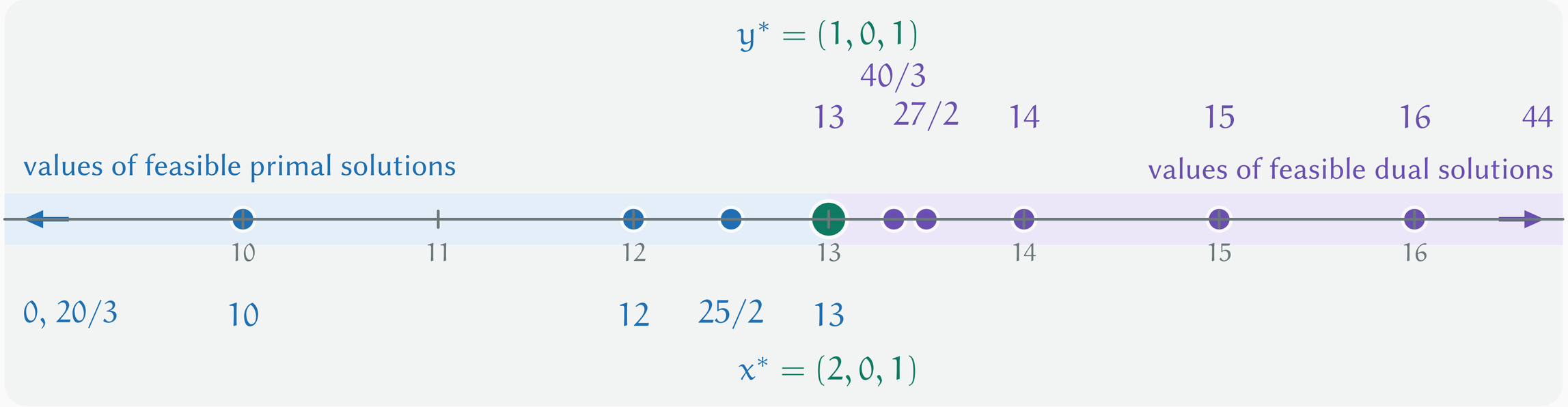
Weak duality. If x is feasible for the primal and y for the dual, then $c^\top x \leqslant b^\top y$.

$$c^\top x \leqslant (A^\top y)^\top x = y^\top (Ax) \leqslant y^\top b = b^\top y$$

$A^\top y \geqslant c$ and $x \geqslant 0$ — only transposing — $Ax \leqslant b$ and $y \geqslant 0$

Consequence. If $c^\top x = b^\top y$ for feasible x, y, then both are optimal: **a certificate**.

Squeezing the optimum



The two ranges touch: no gap. The best values are equal, $13 = 13$.

Strong duality (von Neumann, 1947).
If the primal has an optimal solution x^*, then the dual has an optimal solution y^*, and $c^T x^* = b^T y^*$.

Four cases

Duality, complete statement. Exactly one of these four cases occurs.

Primal	Dual	Example	Its dual
optimal, value 13	**optimal**, value 13	Vanderbei's LP, as before: $13 = 13$	its dual, as before
infeasible	**unbounded**	maximise $x_1 + x_2$ $x_1 + x_2 \leqslant 1$ $-x_1 - x_2 \leqslant -2$	minimise $y_1 - 2y_2$ $y_1 - y_2 \geqslant 1$ $y_1 - y_2 \geqslant 1$
unbounded	**infeasible**	maximise $x_1 + x_2$ $-x_1 + x_2 \leqslant 1$ $x_1 - 2x_2 \leqslant 2$	minimise $y_1 + 2y_2$ $-y_1 + y_2 \geqslant 1$ $y_1 - 2y_2 \geqslant 1$
infeasible	**infeasible**	maximise $2x_1 - x_2$ $x_1 - x_2 \leqslant 1$ $-x_1 + x_2 \leqslant -2$	minimise $y_1 - 2y_2$ $y_1 - y_2 \geqslant 2$ $-y_1 + y_2 \geqslant -1$

By weak duality, a feasible solution on one side bounds the other side:

unbounded forces **infeasible**.

Reading the dual solution off the final dictionary

The last dictionary of Thursday's run:

$$\begin{aligned}
\zeta &= 13 - w_1 - 3x_2 - w_3 \\
\hline
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3
\end{aligned}$$

Let $y_i = -(\text{coefficient of } w_i \text{ in the } \zeta \text{ row})$, and 0 if w_i is basic:

$y = (1, 0, 1)$

Is y feasible for the dual? $A^T y \geqslant c$:

$(A^T y)_j$	5	7	3
c_j	5	4	3
slack	0	3	0
coefficient of x_j	0	−3	0

$b^T y = 5 \cdot 1 + 11 \cdot 0 + 8 \cdot 1 = 13 = \zeta^*$

y is an optimal solution of the dual.
The dual solution was in the dictionary all along.

Why the dual solution is optimal

Strong duality. If the primal has an optimal solution x^*, the dual has one, y, with $b^\mathsf{T} y = c^\mathsf{T} x^*$.

1. In the optimal dictionary no coefficient is positive.
Let $s_j = -\widehat{c}_j$, $y_i = -\widehat{c}_{n+i}$ ($\geqslant 0$): $\zeta = \zeta^* - s^\mathsf{T} x - y^\mathsf{T} w$.

in our example $s = (0, 3, 0)$, $y = (1, 0, 1)$

2. A rewriting, so true for **every** x, with $w = b - Ax$:
$c^\mathsf{T} x = \zeta^* - s^\mathsf{T} x - y^\mathsf{T}(b - Ax)$.

3. Coefficients of x: $c = A^\mathsf{T} y - s$, so $A^\mathsf{T} y \geqslant c$: y is dual feasible.

$(5, 4, 3) = (5, 7, 3) - (0, 3, 0)$

4. Constants: $0 = \zeta^* - y^\mathsf{T} b$, so $b^\mathsf{T} y = \zeta^*$.

$0 = 13 - 13$

By weak duality y is optimal for the dual. □

Complementary slackness

Complementary slackness. Feasible x and y are both optimal

if and only if $y_i w_i = 0$ and $x_j s_j = 0$ for all i, j.

Slacks: $w_i = b_i - (Ax)_i$ and $s_j = (A^T y)_j - c_j$, both $\geqslant 0$.

1. $b^T y - c^T x = \sum_i y_i w_i + \sum_j x_j s_j$
   expand, and $y^T A x = x^T A^T y$
2. Every term is a product of two numbers $\geqslant 0$.
3. Both optimal $\iff$ gap 0 $\iff$ all terms 0. $\square$

$x = (1, 0, 1)$, $y = (1, 1, 1)$

$y_1 w_1$	$y_2 w_2$	$y_3 w_3$	$x_1 s_1$	$x_3 s_3$
2	5	3	4	2

gap $= b^T y - c^T x = 16$

$x^* = (2, 0, 1)$, $y^* = (1, 0, 1)$

- all six products are 0, gap $= 0$

Each product has a zero factor: the pair is complementary.

Complementary slackness at work

Given the optimal vertex $x^* = (2, 0, 1)$: find y.

constraint	1	2	3
slack w_i	0	1	0
y_i	1	0	1

variable	x_1	x_2	x_3
x_j	2	0	1
s_j	0	3	0

The dual constraints:

(1) $2y_1 + 4y_2 + 3y_3 \geqslant 5$ **tight**

(2) $3y_1 + y_2 + 4y_3 \geqslant 4$ slack 3

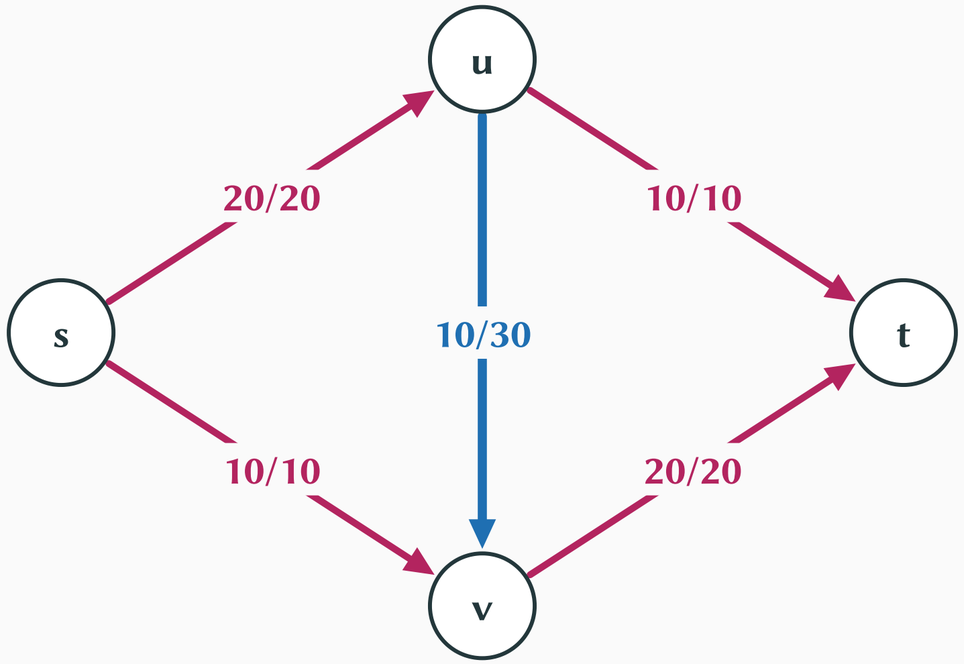
(3) $y_1 + 2y_2 + 2y_3 \geqslant 3$ **tight**

The remaining dual constraint holds: $s_2 = 3 \geqslant 0$.
Hence $y = (1, 0, 1)$ is feasible and complementary: **optimal**, $13 = 13$.

Flows

What is the dual of maximum flow?

Maximum flow is a linear program

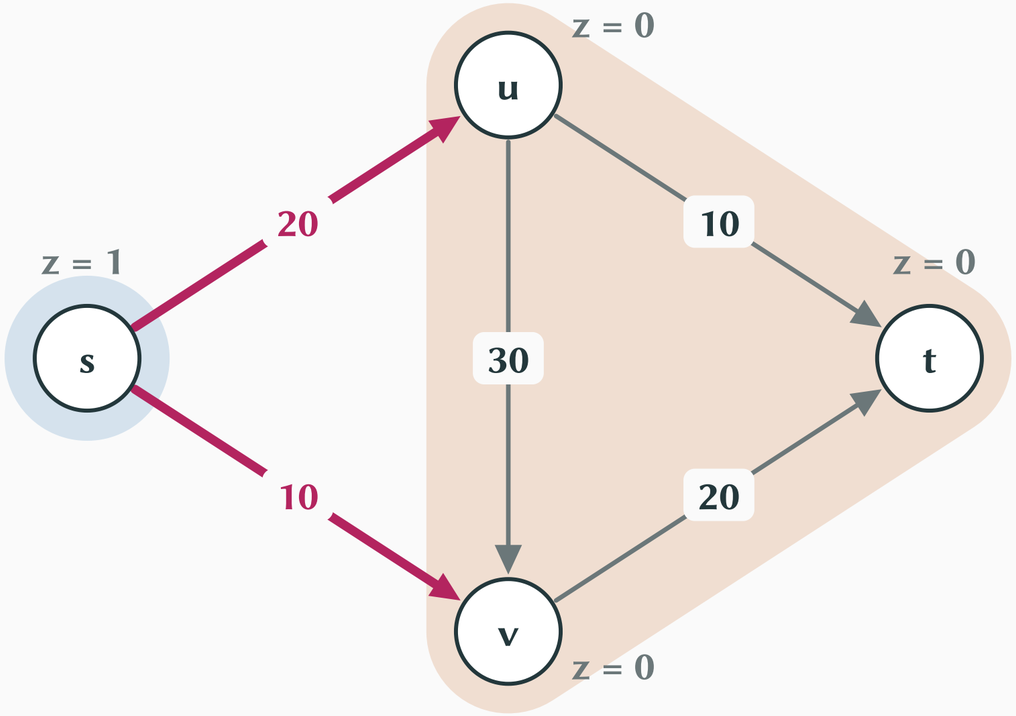


$$
\begin{aligned}
\textbf{maximise} \quad & f_{su} + f_{sv} \\
\text{subject to} \quad & f_{su} \leqslant 20 && \text{capacity} \\
& f_{sv} \leqslant 10 \\
& f_{uv} \leqslant 30 \\
& f_{ut} \leqslant 10 \\
& f_{vt} \leqslant 20 \\
& f_{su} - f_{uv} - f_{ut} = 0 && \text{conservation} \\
& f_{sv} + f_{uv} - f_{vt} = 0 \\
& f_{su}, f_{sv}, f_{uv}, f_{ut}, f_{vt} \geqslant 0
\end{aligned}
$$

In general: maximise $\sum_v f_{sv} - \sum_v f_{vs}$; $f_{uv} \leqslant c_{uv}$; $\sum_v f_{vu} = \sum_v f_{uv}$ for $u \neq s, t$; $f \geqslant 0$.

Optimum: $f^* = (20, 10, 10, 10, 20)$, $\mathrm{val}(f^*) = 30$

The dual of maximum flow is minimum cut



$S = \{s\}$, $z = (0, 0)$, $d = (1, 1, 0, 0, 0)$
objective $20 + 10 = 30$

$$
\begin{aligned}
\textbf{minimise} \quad & 20d_{su} + 10d_{sv} + 30d_{uv} + 10d_{ut} + 20d_{vt} \\
\text{subject to} \quad & d_{su} + z_u \geqslant 1 \\
& d_{sv} + z_v \geqslant 1 \\
& d_{uv} - z_u + z_v \geqslant 0 \\
& d_{ut} - z_u \geqslant 0 \\
& d_{vt} - z_v \geqslant 0 \\
& d_{su}, d_{sv}, d_{uv}, d_{ut}, d_{vt} \geqslant 0
\end{aligned}
$$

$d_e \geqslant 0$ for each capacity constraint;
z_u for each conservation constraint, any sign.
With $z_s = 1$ and $z_t = 0$ each line reads $d_e \geqslant z_u - z_v$ for $e = (u, v)$.

LP optimum $= 30 = \mathrm{val}(f^*)$.
Every cut is a feasible dual solution:
weak duality (Fact 3 of Lecture 11).

LP-relaxation: is the dual solution a cut?

Minimum cut as an ILP.
The dual constraints with $d_e, z_u \in \{0, 1\}$.
Its feasible solutions are exactly the cuts.

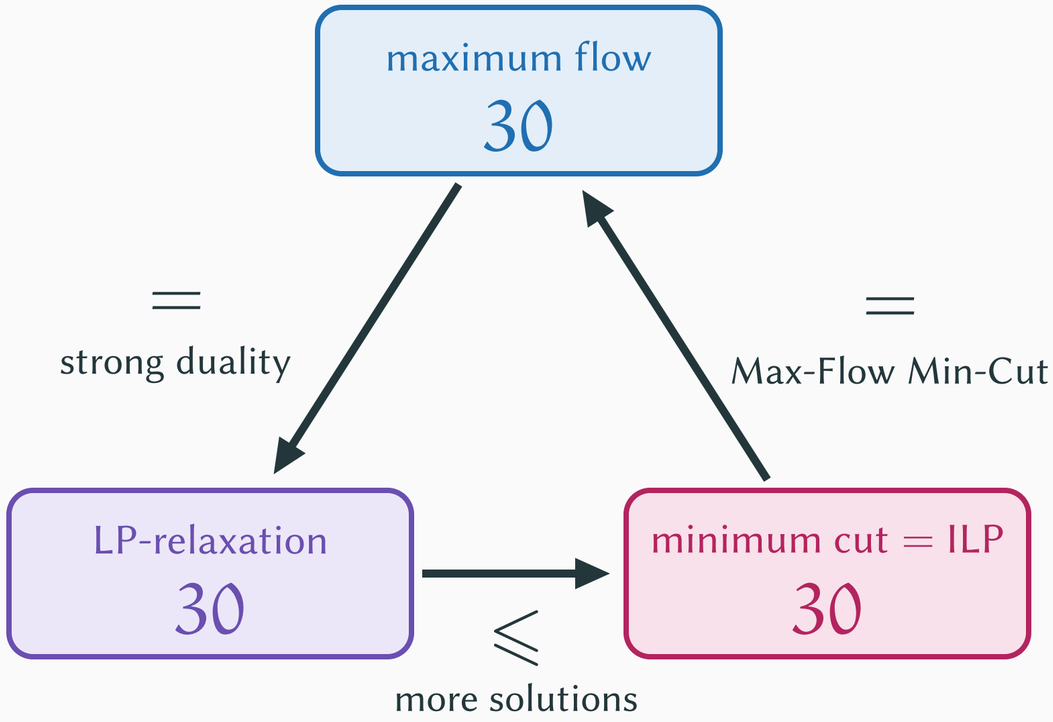
The **LP-relaxation** allows any real $d_e \geqslant 0$ and z_u.
It is exactly the dual LP.
It has more solutions than the ILP, so LP $\leqslant$ ILP.

? *Is the LP optimum smaller, or is it a cut?*

All three are equal:

$$\textbf{integrality gap} = \frac{\text{ILP optimum}}{\text{LP optimum}} = 1.$$

A minimum cut is optimal for the relaxation.



Total unimodularity

Totally unimodular: every square submatrix has determinant 0, $+1$ or -1.

Theorem. A totally unimodular, b integral: every vertex of $\{x : Ax \leqslant b,\ x \geqslant 0\}$ is integral.

Proof.

1. A vertex is the unique solution of n independent tight constraints: $A'x = b'$, with rows from A or from $-I$.
2. $\det A' = \pm 1$: expanding along the rows from $-I$ leaves $\pm$ a minor of A. It is 0 or ± 1, and not 0 because A' is invertible.
3. Cramer's rule: $x_j = \det A'_j / \det A'$ is an integer. □

Integer flows: the incidence matrix

The conservation rows of the max-flow LP: the incidence matrix of the network.

$$\begin{array}{c} \\ u \\ v \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} f_{su} & f_{sv} & f_{uv} & f_{ut} & f_{vt} \end{array} \\ \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

All 20 minors are 0, $+1$ or -1.

The ten 2×2 ones: $1,\ 1,\ 0,\ -1,\ 1,\ 1,\ 0,\ 1,\ 1,\ 1$

Lemma. The incidence matrix of a directed graph is totally unimodular.

Max flow: integral capacities give an **integral maximum flow**.

Summary

1. **Degeneracy.** A zero basic variable gives a zero step.
   Bland's rule prevents cycling.
2. **Speed.** Worst case: $2^n - 1$ pivots (Klee–Minty).
   In practice very fast.
3. **Duality.** The dual is the transpose, $c^T x \leqslant b^T y$.
   Read y off the final dictionary, and $y_i w_i = 0 = x_j s_j$.
4. **Flows.** Max flow and min cut are dual LPs:
   Max-Flow Min-Cut is strong duality.

> If the primal has an optimal solution, so does the dual,
> with equal optimal values.

Reference: Vanderbei, Chapters 3–5