ADS Tutorial 4

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Problem 1

Let $G = (V, E)$ be a connected graph with positive edge costs, and let G_T be a Minimum Spanning Tree (MST) on G. Let G' be a connected graph that is the same as G, but with different edge costs c'_e rather than c_e for each edge $e \in E$. In which of the following cases will G_T still be a minimum spanning tree of the connected graph G'?

(a) When $c'_e = c_e + 17$.

(b) When $c'_e = 17c_e$.

(c) When $c'_e = c_e^2$.

(d) All of the above.

(e) None of the above.

Problem 2

In this problem we will consider the *uniqueness* of MSTs on connected graphs.

(a) First consider connected graphs in which the edge costs might not be distinct. Draw an example of such a graph that has two different MSTs.

(b) Now consider any connected graph G for which the edge costs are all distinct. Prove that there is a unique MST on G.

Problem 3*

Suppose you are given a connected graph $G = (V, E)$ with edge costs which are all distinct, with $|V| = n$ and $|E| = m$, and a specified edge $e^* \in E$. Design an algorithm with running time $O(m + n)$ which returns "yes" if e^* is contained in a MST of G and "no" otherwise.
To prove the correctness of your algorithm, you may first want to prove the following statement:

> *An edge $e = \{v, w\}$ does not belong to a MST of G if and only if there is a path from v to w which consists entirely of edges that have smaller cost than e.*

Hint: You may want to use the Cut Property and the Cycle Property to prove the statement.

Problem 4

Consider an alternative algorithm for computing an MST G_T on a connected graph $G = (V, E)$ with distinct edge costs. At iteration t of the algorithm, let G_T^t be the graph consisting of the set of nodes V and the set of edges that have been added by the algorithm to the developing MST in iterations $1, \ldots, t-1$. Let $\mathcal{C}^t$ be the set of connected components of G_T^t. For each connected component $C \in \mathcal{C}^t$, add to the MST the edge with the smallest cost $e = \{v, u\}$ where v is in C and u is not in C. Update the graph to G_T^{t+1} and the set of connected components to $\mathcal{C}^{t+1}$, and repeat until there is a single connected component. Before the first iteration, G_T^1 contains no edges and $\mathcal{C}^1 = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$, i.e., every node is a connected component on its own.
Show that this algorithm terminates in $O(m \log n)$ time and that in the end it produces a MST of G. It might be useful to first prove the following statement, using the Cut Property for the proof:

> *Let $u \in V$ be any node in G. The MST of G must contain edge $\{u, w\}$, which has the minimum cost amongst all edges incident to u.*

Note: This algorithm was first proposed by Otakar Borůvka in 1926, in response to a request to construct an electrical network connecting several cities using the least amount of wire. The algorithm was rediscovered multiple times, in the late 1930s, the early 1950s and the mid-1960s.