Lecture on week 2: SO(3) representations

$r \in SO(3)$ rotation that keeps distances / angles / center

$\simeq R \in \mathbb{R}^{3\times 3}$

s.t. $R^T R = I$

$\det(R) = +1 \Rightarrow$ no reflexion

$r(x) = Rx$ almost a map, can directly use to move objects

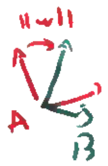
I] Angle axis representation

$r \in SO(3) \simeq$ angular velocity

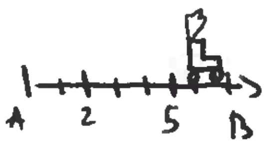
$\omega \in \mathbb{R}^3 \simeq so(3)$ little so(3). Lie algebra (developed later)

- How does velocity represents a rotation?

r is obtained by integrating ω for a duration of 1



To align A on B we integrate a constant velocity ~~‖ω‖~~ during a time of ~~1~~ ~~t_0~~



Easy to see in linear space, not so interesting.

Howerver, SO(3) is spherical, but so(3) is linear so this will be useful.

Q. Can we recover R from ω? (Not seen in class, for general culture) $\boxed{\text{You can skip until end of ----}}$

Assume a function $\mathbb{R} \xrightarrow{\text{smooth}} SO(3)$

$t \longrightarrow r(t)$

$\simeq R : t \longrightarrow R(t) \in \mathbb{R}^{3\times 3}$

s.t. $\forall t \quad R(t)\, R(t)^T = I$

what if we differentiate ?

$R(t)^T R(t) = I$. Now omitting (t) for clarity

$\Rightarrow \dot{\overline{RR^T}} = 0$ derivative of a product says:

$$\dot{R}R^T + \underbrace{R\dot{R}^T}_{(\dot{R}R^T)^T} = 0 \Rightarrow \dot{R}R^T = (-\dot{R}R^T)^T \quad (1)$$

we can show that equation (1) imamplies:

$$\dot{R}R^T = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \Rightarrow \text{anti-symmetric matrix}$$

now let's write $\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -c \\ b \\ -a \end{bmatrix}$

(just arbitrary notation)

$$\dot{R}R^T = \underbrace{\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}_{\omega_\times}.$$

This matrix is called the skew-symmetric matrix

Interestingly, $\omega_\times a = \omega \times a \quad \forall a \in \mathbb{R}^3$. $\omega_\times$ encodes cross-product.

(2) $\dot{R}(t) R^T(t) = \omega_\times$. constant angular velocity $\forall t$.

Can we recover R from ω then?

(2) has the form $\dot{x} = \alpha x$. The solution to this equation is $x = e^{\alpha t}$

$\Rightarrow R(t) = \mathrm{Exp}(t\omega_\times)$ with Exp the matrix exponential.

with more development we can obtain the Rodrigues formula:

$$\boxed{\begin{aligned} &R(\omega) = I + \sin\theta\, \omega_\times + (1-\cos\theta)\, \omega_\times^2 \\ &\text{where } \omega = \theta \vec{v} \end{aligned}}$$

We can go from ω to R using $R = \mathrm{Exp}(\omega_\times)$

Similarly a log function is defined and says

$\omega_\times = \log(R)$.

These 2 methods are available in Pinocchio.

Conclusion : $\omega_\times$ is a minimal representation for rotations, but requires recovering a matrix form to act as a map.

Quaternions : It was proven that no 3D map exists that can represent the ~~rotation~~ 3D rotation.

We define the Hamiltonian space $\mathbb{H}$ (in honour of Hamilton) :

$q \in \mathbb{H} \simeq \mathbb{R}^4$

s.t $\|q\| = 1$. $\Leftarrow$ $\boxed{\text{Unit norm, very important!}}$

If $\omega = \theta \vec{u}$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

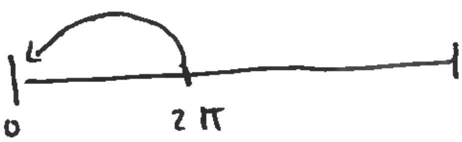
we write

$$q = \begin{bmatrix} \sin\theta/2 \; u_1 \\ \sin\theta/2 \; u_2 \\ \sin\theta/2 \; u_3 \\ \cos\theta_2 \end{bmatrix}$$

why do we need 4 dimensions?

- go back to 2D and SO(2)

$\theta \equiv \theta + 2\pi$



we are defining angles on $\mathbb{R}$ but need to hande discontinuities at 0 on 2π.

This is avoided with complex number

$r \in \mathbb{C}$

$r = x + iy$, with i s.t. $i^2 = 1$

If we add $\|r\| = 1$.

Then r represents the rotation of angle θ:

$r = \cos\theta + i\sin\theta$ st. $x' = rx$ applies a rotation of θ to x into x'

2 advantages.

- continuous representation
- composition is easy:

  $r_1 = \cos\theta_1 + i\sin\theta_1$

  $r_2 = \cos\theta_2 + i\sin\theta_2$

Exercise: show that

$r_1 r_2$ encodes a rotation by $\theta = \theta_1 + \theta_2$

ie $r_1 r_2 = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

$\Rightarrow$ SO(2) is of dim 1, represented by $\mathbb{C}$, a 2D space with 1 constraint

• Can we generalise complex numbers to 3D?

As for $i^2 = -1$, we need rules to make this work

Def: j s.t. $j^2 = -1$

$ij = -ji$

by convenience, we define $k = ij$

as a result,

- $k^2 = ijij = -ijji = -i(-1)i = -1$
- $ik = -ki$ • $jk = -kj$

why? Because with these rules $\mathbb{H}$ is a group and not without

$\underbrace{\mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}}$ $\Rightarrow$ $\mathbb{H}$ the set of quaternions

⚠ sometimes first, sometimes last in literature

Cor: $\mathbb{H} = \mathbb{C} + j\mathbb{C}$

- quaternion multiplication has non-trivial representation. But
- $q_2 \times q_1$ is a composition of rotations
- $q\, r_p\, \bar{q}$ is the operator that rotates p by q, with

$r_p = [p_x, p_y, p_z, 0]$

$q = [x, y, z, w]$

$\bar{q} = [-x, -y, -z, w]$