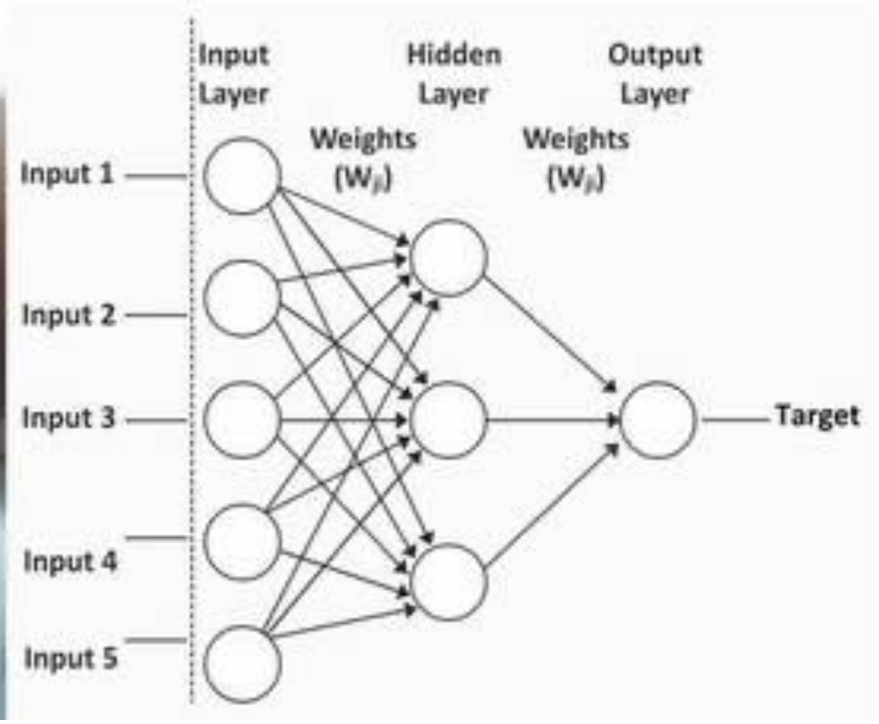
Network models - I

"I work with **models**"

Others: Me:



LEARNING OUTCOMES

Learn about different **generative network models**

Compare these models' **theoretical outputs** with **real-world networks**

Step-up your network analysis game

HOW WOULD YOU BUILD A NETWORK?

RANDOM NETWORKS

Two slightly different versions proposed independently, one by **Gilbert** and the other one by **Erdos and Rényi**

For reasons, both are called **Erdos-Rényi networks (ER networks)**

The most common formulation, however, is the one by **Gilbert**

RANDOM NETWORKS

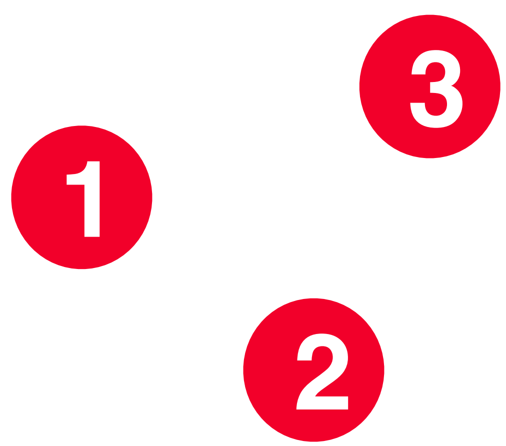
Gilbert model

Parameters: **n** number of nodes and a probability **p**
Algorithm:
For each pair of nodes i,j:

    Generate a random number r between **0** and **1** (**uniformly distributed**)
    If r<p make a link between i,j

RANDOM NETWORKS

Gilbert model

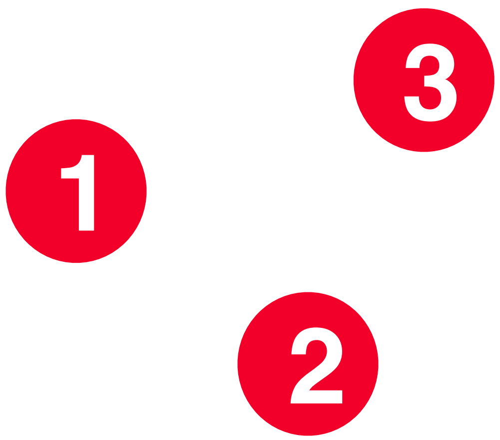


$p=0.5$

RANDOM NETWORKS

Gilbert model

i,j	r
1,2	0.12
1,3	
2,3	

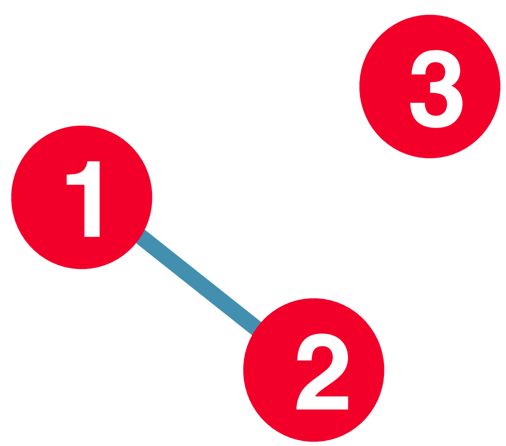


p=0.5

RANDOM NETWORKS

Gilbert model

i,j	r
1,2	0.12
1,3	
2,3	

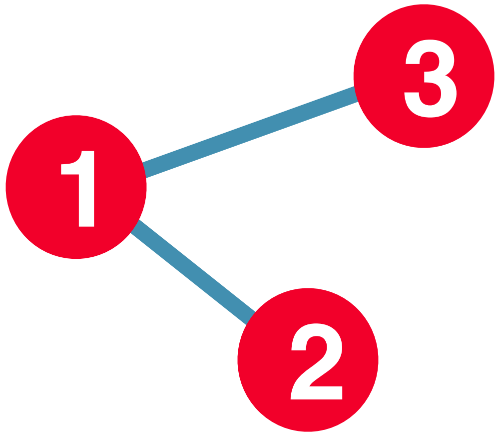


p=0.5

RANDOM NETWORKS

Gilbert model

i,j	r
1,2	0.12
1,3	0.29
2,3	

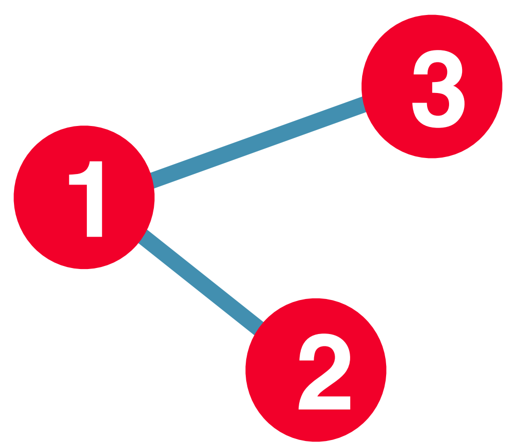


p=0.5

RANDOM NETWORKS

Gilbert model

i,j	r
1,2	0.12
1,3	0.29
2,3	0.72



p=0.5

RANDOM NETWORKS

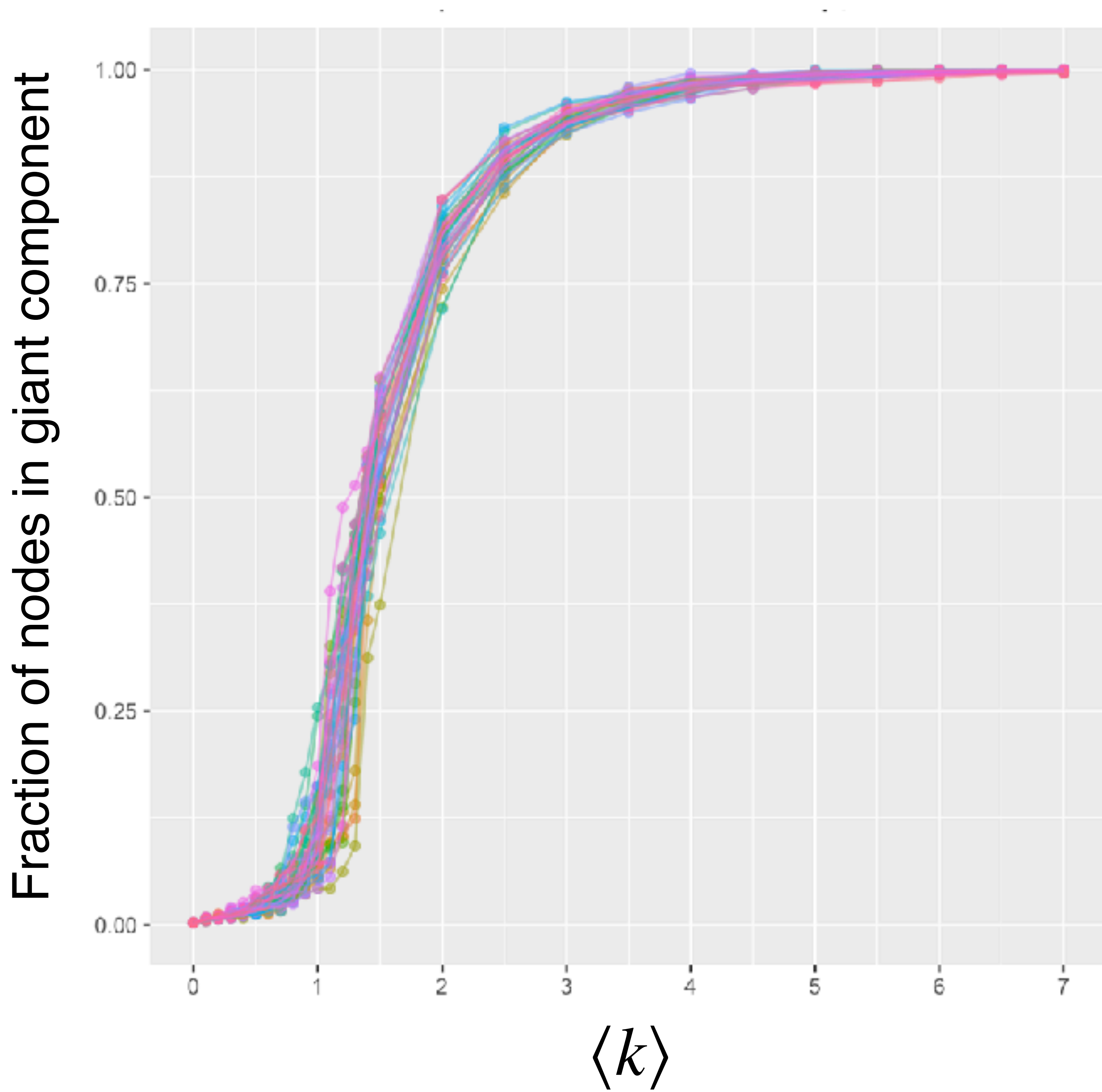
Erdos-Renyi model

Parameters: **n** number of nodes and **L** number of links

Algorithm:
Build all possible networks with n nodes and **L** links
Pick one randomly

Remarks:
For large values of **L**, these two formulations are (almost) equivalent

PHASE TRANSITION



DENSITY

$$\langle L \rangle = ?$$

DENSITY

$$\langle L \rangle = p \binom{N}{2} = p \frac{N(N-1)}{2}$$

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$$\langle k \rangle = ?$$

DENSITY

$$\langle L \rangle = p \binom{N}{2} = p \frac{N(N-1)}{2}$$

$$\langle k \rangle = \frac{2\langle L \rangle}{N} = p(N-1)$$

DENSITY

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$$\langle k \rangle = \frac{2\langle L \rangle}{N} = p(N-1)$$

$$\langle d \rangle = ?$$

DENSITY

$$\langle L \rangle = p\binom{N}{2} = p\frac{N(N-1)}{2}$$

$$\langle k \rangle = \frac{2\langle L \rangle}{N} = p(N-1)$$

$$\langle d \rangle = \frac{L}{L_{max}}$$

DENSITY

$$\langle L \rangle = p \binom{N}{2} = p \frac{N(N-1)}{2}$$

$$\langle k \rangle = \frac{2\langle L \rangle}{N} = p(N-1)$$

$$\langle d \rangle = \frac{\langle L \rangle}{L_{max}} = \frac{p \frac{N(N-1)}{2}}{L_{max}}$$

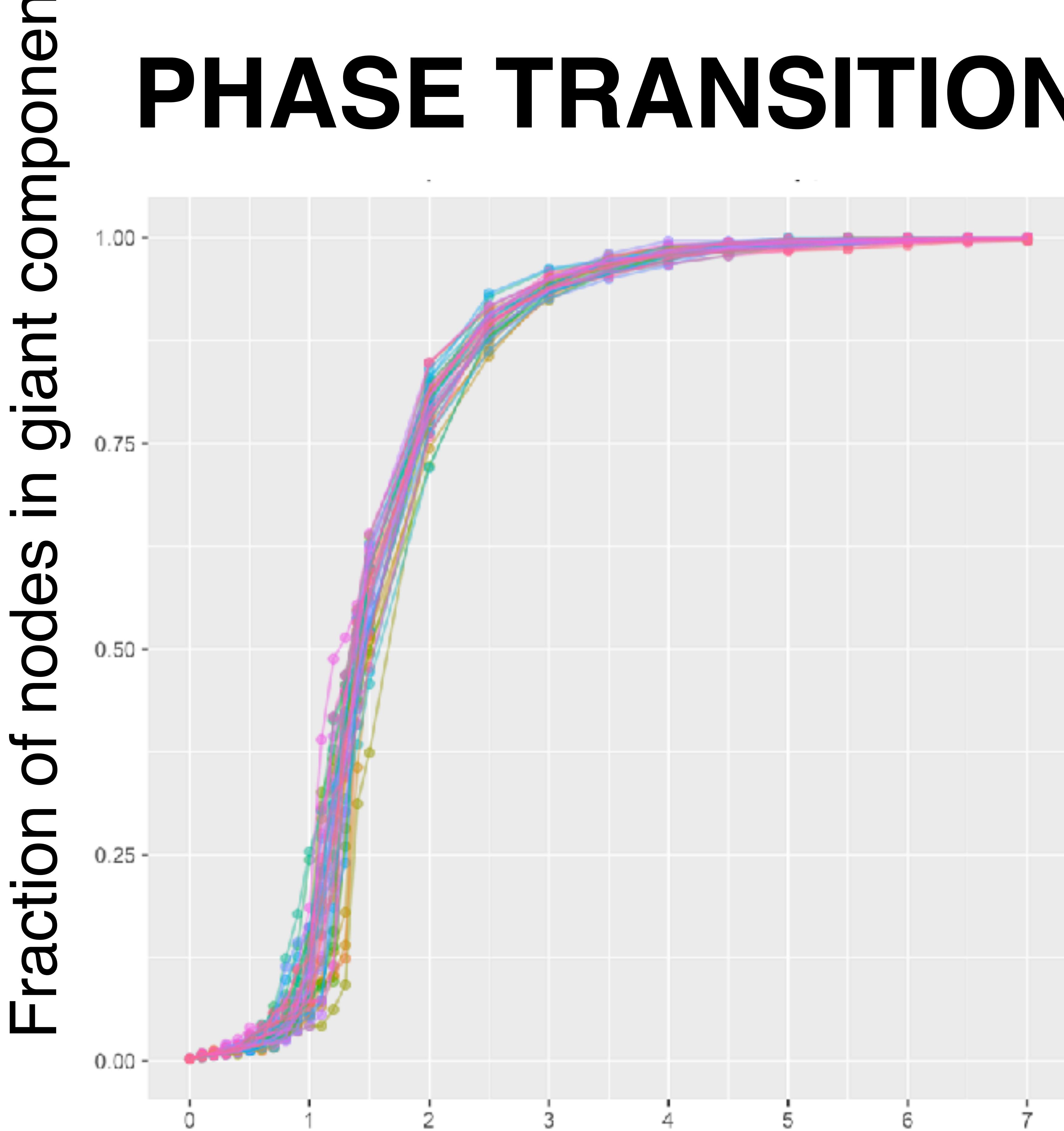
DENSITY

$$\langle L \rangle = p \binom{N}{2} = p\frac{N(N-1)}{2}$$

$$\langle k \rangle = \frac{2\langle L \rangle}{N} = p(N-1)$$

$$\langle d \rangle = \frac{\langle L \rangle}{L_{max}} = \frac{p\frac{N(N-1)}{2}}{L_{max}} = \frac{p\frac{N(N-1)}{2}}{\frac{N(N-1)}{2}} = p$$

PHASE TRANSITION



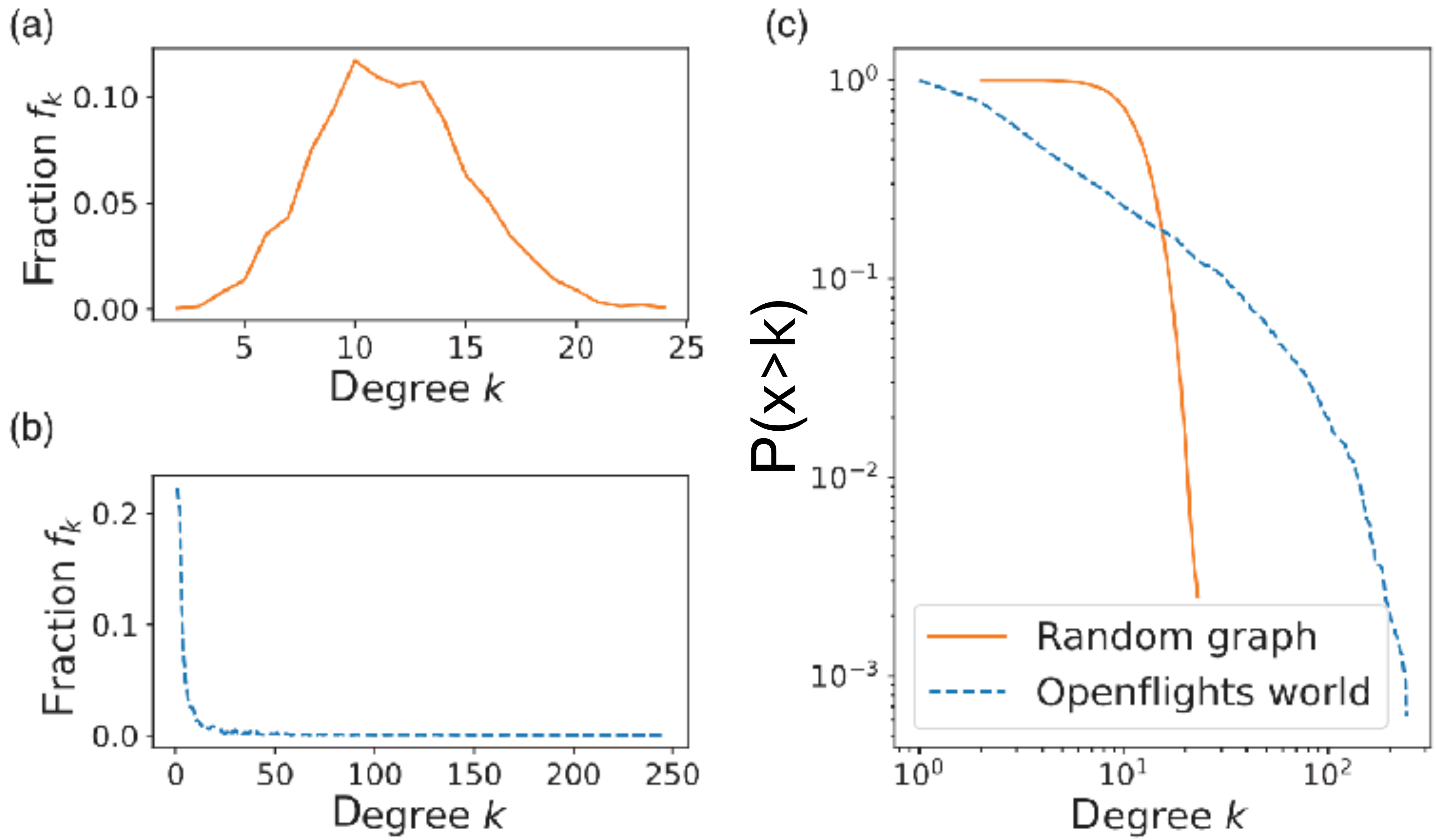
$\langle k \rangle = p(N-1)$

DEGREE DISTRIBUTION

Binomial distribution

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

DEGREE DISTRIBUTION



SHORT PATHS

assumption:
all nodes have degree k

Path length	**Nodes reached**
$\ell = 1$	k

SHORT PATHS

assumption:
all nodes have degree k

Path length	Nodes reached
$\ell = 1$	k
$\ell = 2$	$k(k-1)$

SHORT PATHS

assumption:
all nodes have degree k

Path length	Nodes reached
$\ell = 1$	k
$\ell = 2$	$k(k-1)$
$\ell = 3$	$k(k-1)^2$

SHORT PATHS

assumption:
all nodes have degree k

Path length	**Nodes reached**
$\ell = 1$	k
$\ell = 2$	$k(k-1)$
$\ell = 3$	$k(k-1)^2$

$$k(k-1)^{\ell-1} \approx k^{\ell}$$

If k large

SHORT PATHS

$$k^{\ell_{max}} = N$$

$$\ell_{max} = \log_k N = \frac{\log N}{\log k}$$

Diameter grows with logarithm of nodes, so distances are fairly small despite the network size

CLUSTERING COEFFICIENT

$$C(i) = ?$$

CLUSTERING COEFFICIENT

$$C(i) = p$$

Real-world networks are **sparse**, so p is small, so **clustering is small**

However, real-world networks have actually higher clustering coefficients

CONCLUSIONS

DO YOU THINK ERDOS-RENYI NETWORKS ARE GOOD APPROXIMATIONS OF REAL-WORLD NETWORKS? IF SO, WHICH TYPE OF NETWORK?

SMALL WORLDS

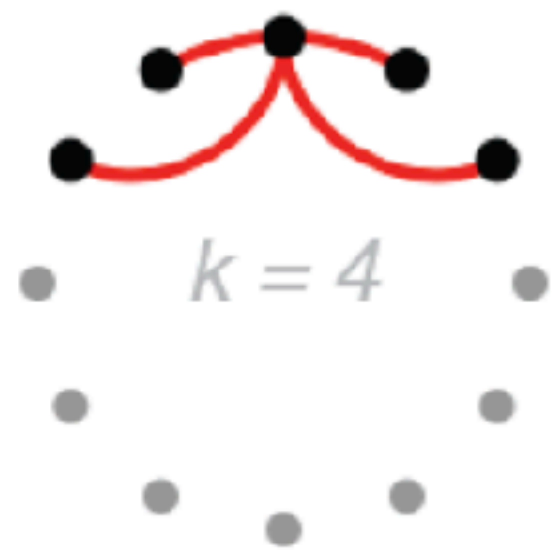
Proposed by **WATTS AND STROGATZ** (sometimes called WS networks)

Solves the problem of low **clustering coefficients**

SMALL WORLDS

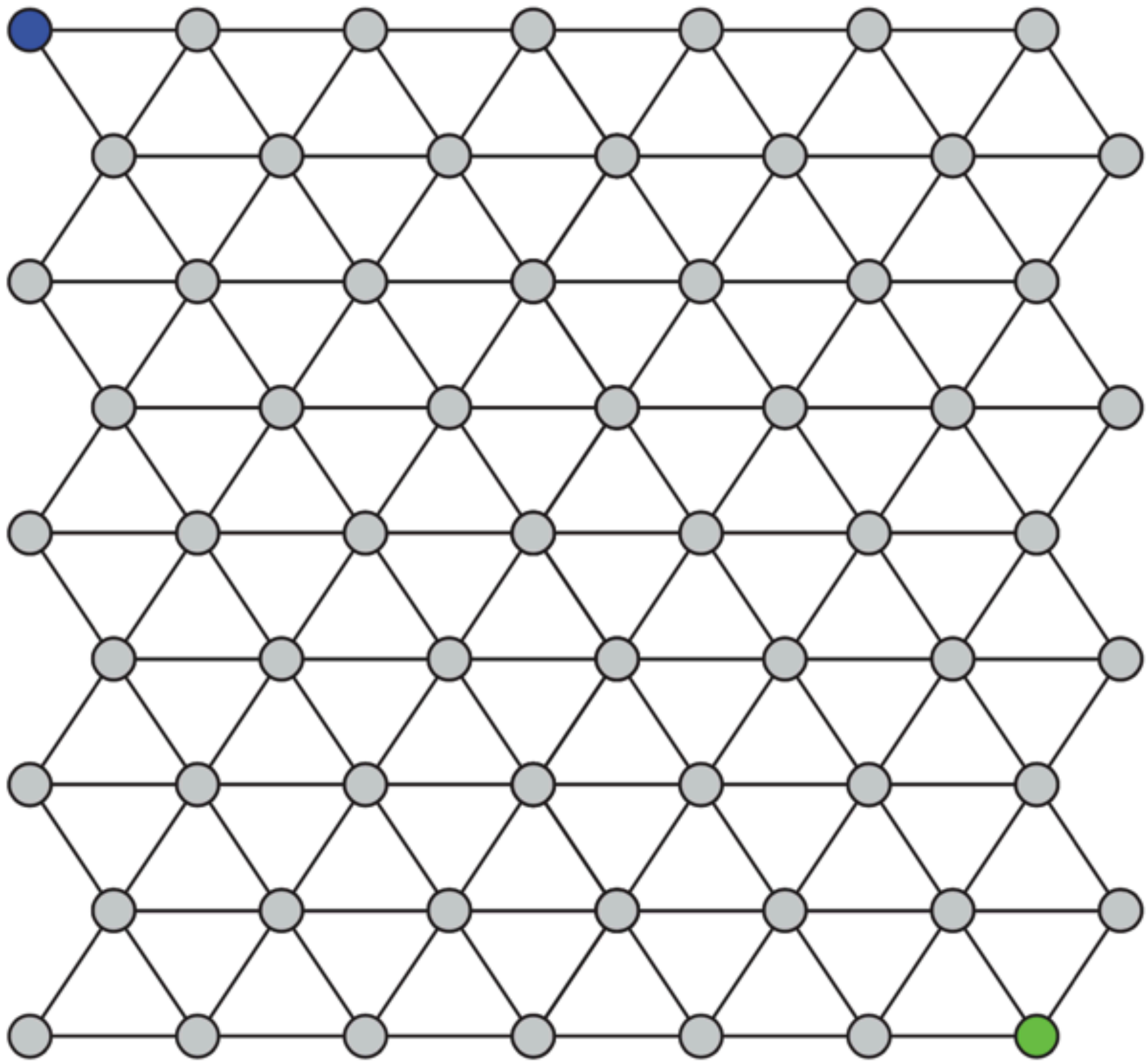
Start with n nodes

Connect a node to its
k nearest neighbours

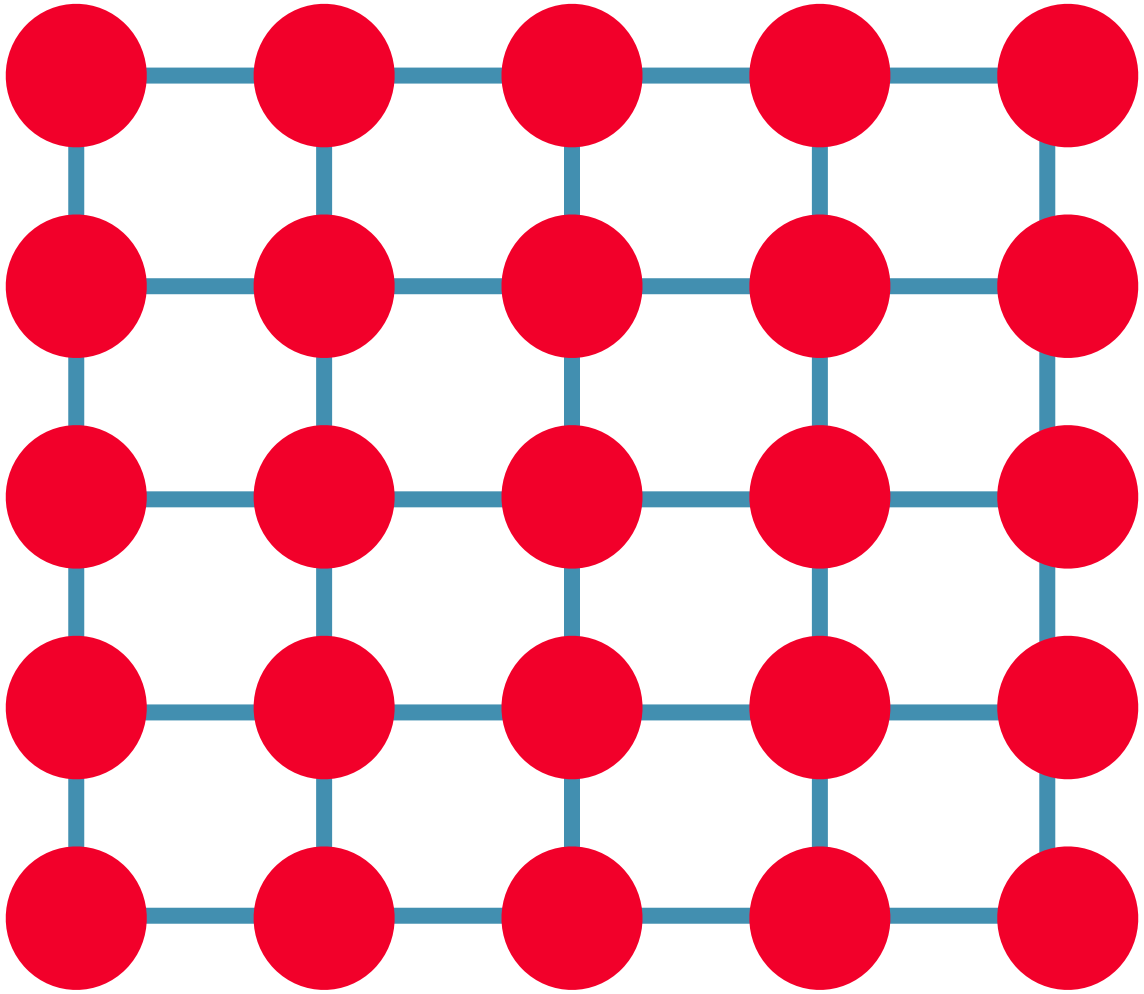


Do the same for all
the nodes in the network

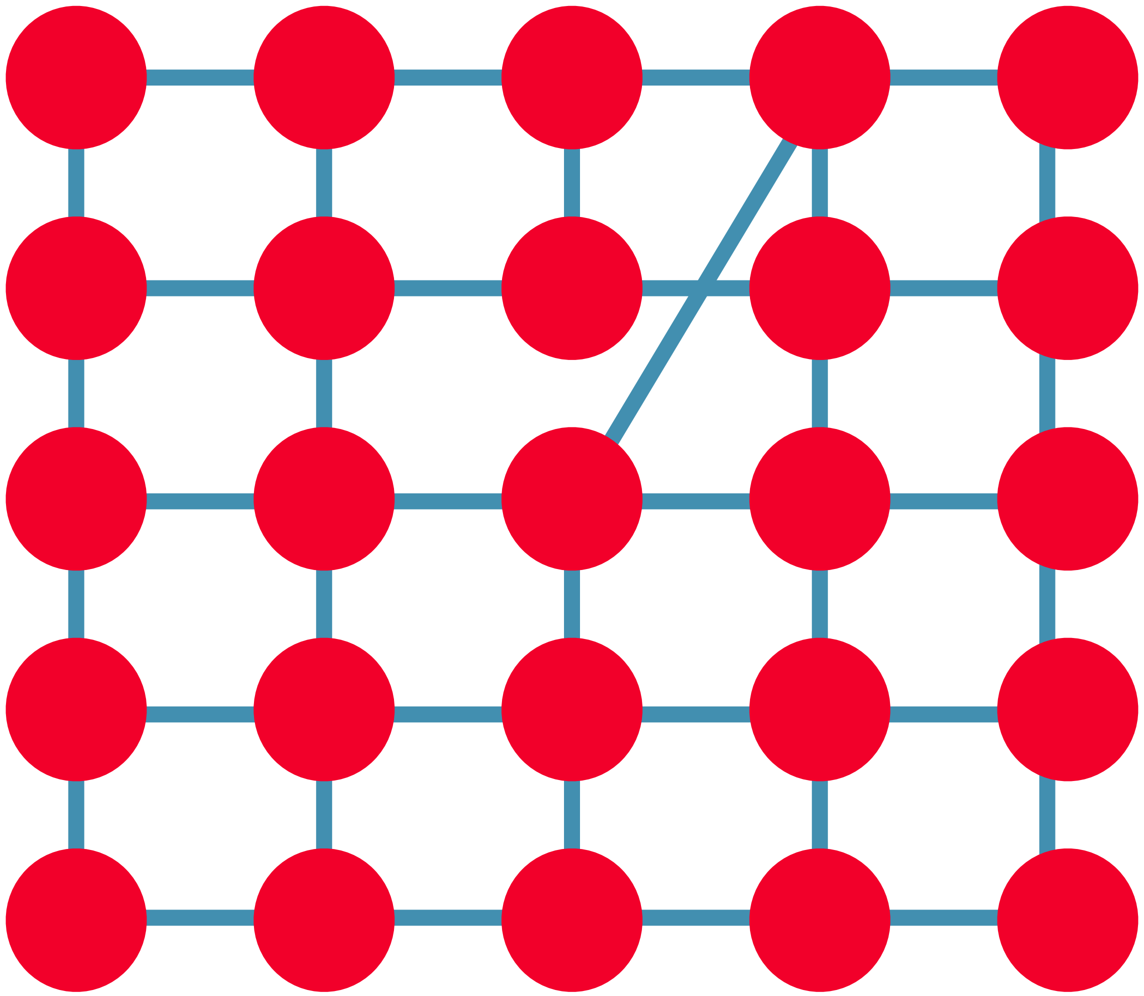
SMALL WORLDS



SMALL WORLDS



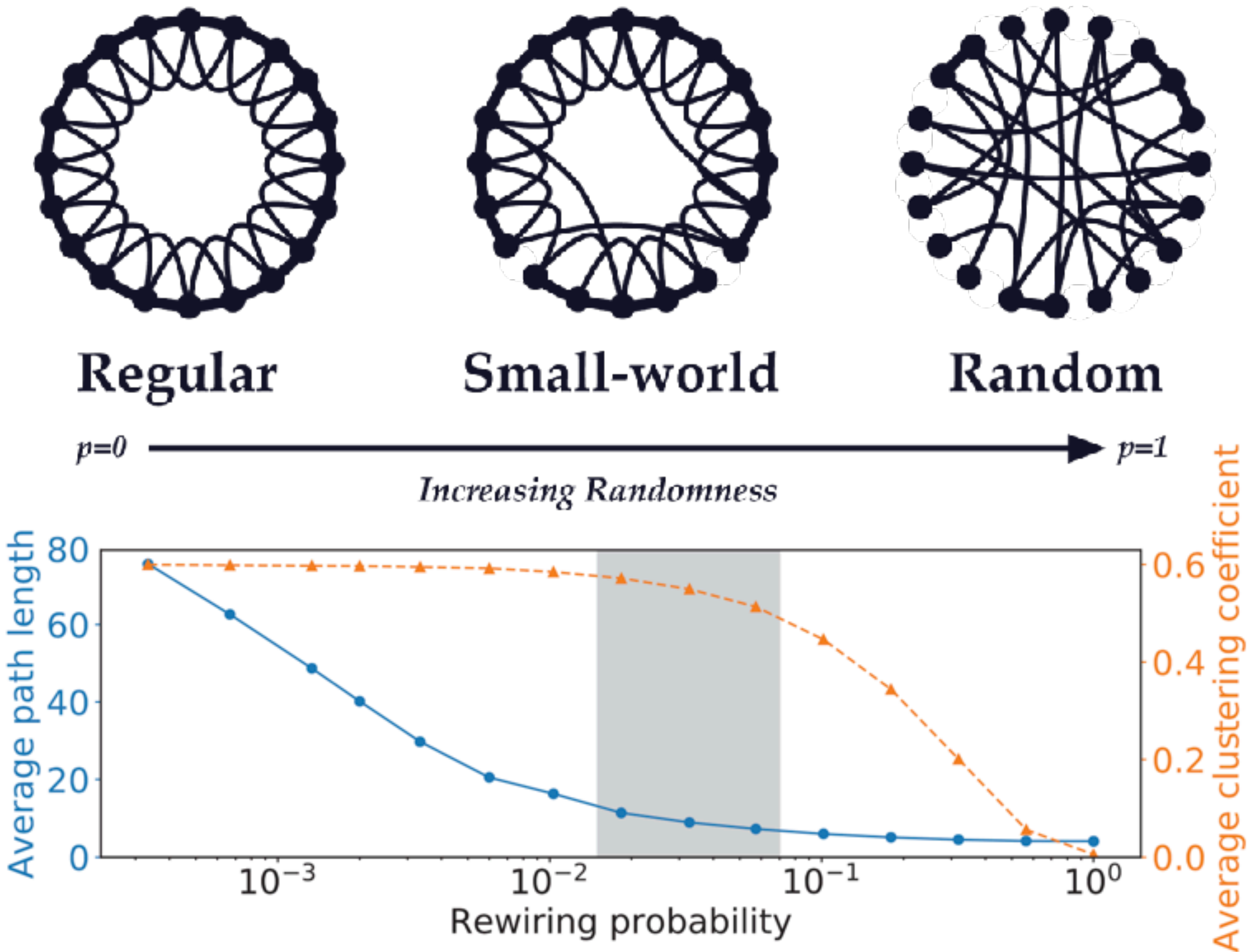
SMALL WORLDS



we rewire every link in the network with probability p

SMALL WORLDS

SMALL WORLDS



SMALL WORLDS

Desirable properties such as clustering and short paths

What is missing?

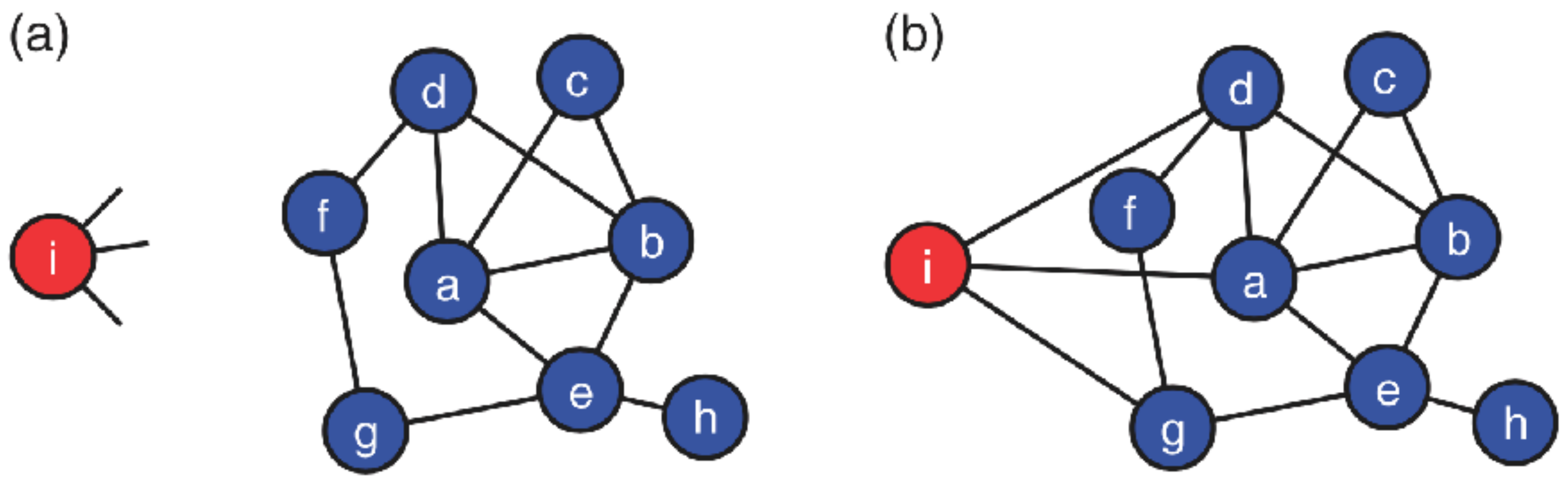
SCALE-FREE NETWORKS

PROPOSED BY REKA **ALBERT** AND LASZLO **BARABASI** (ALSO KNOWN AS BA NETWORK)

BASED ON THE IDEA OF **PREFERENTIAL ATTACHMENT**

INTRODUCES **DYNAMIC** GENERATION OF NETWORK, AND **EXPLAINS HUBS**

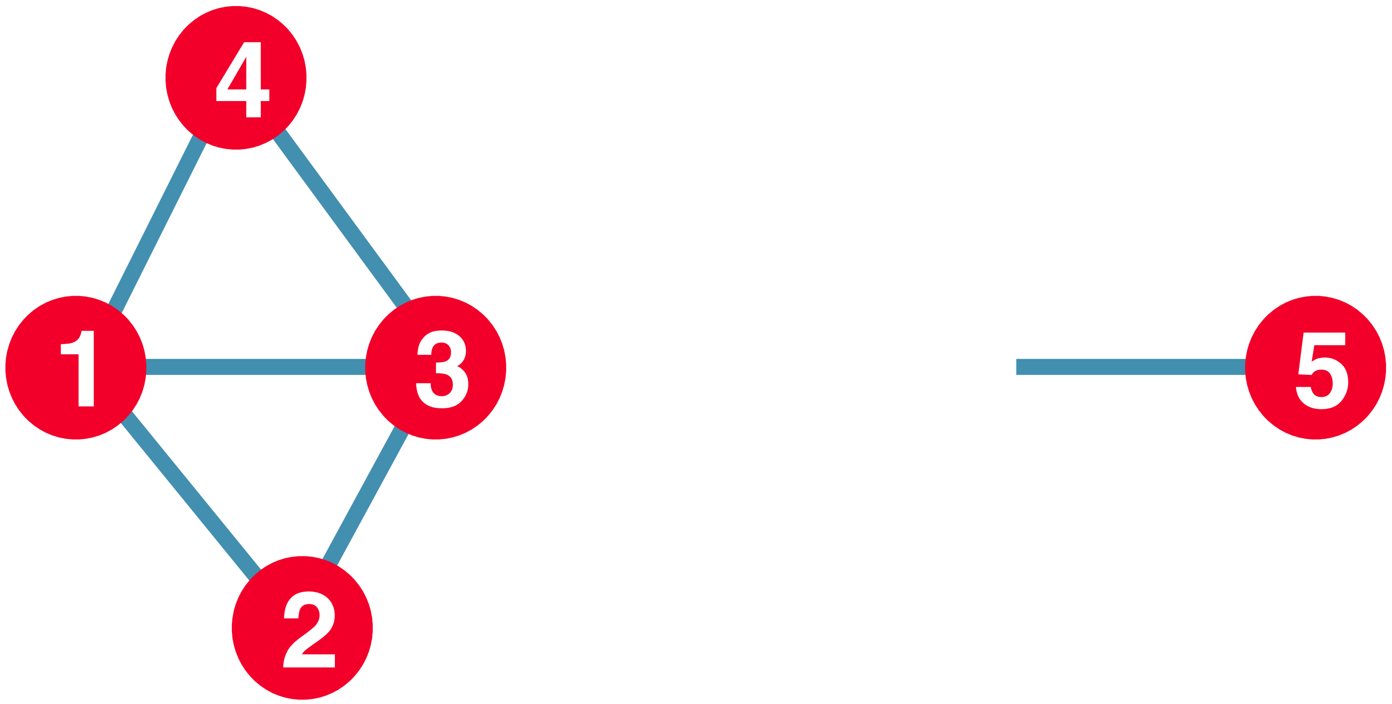
PREFERENTIAL ATTACHMENT



NEW NODES MAKES SOME CONNECTIONS

CONNECTIONS ARE MADE BASED
ON A GIVEN RULE (OR SET OF RULES)

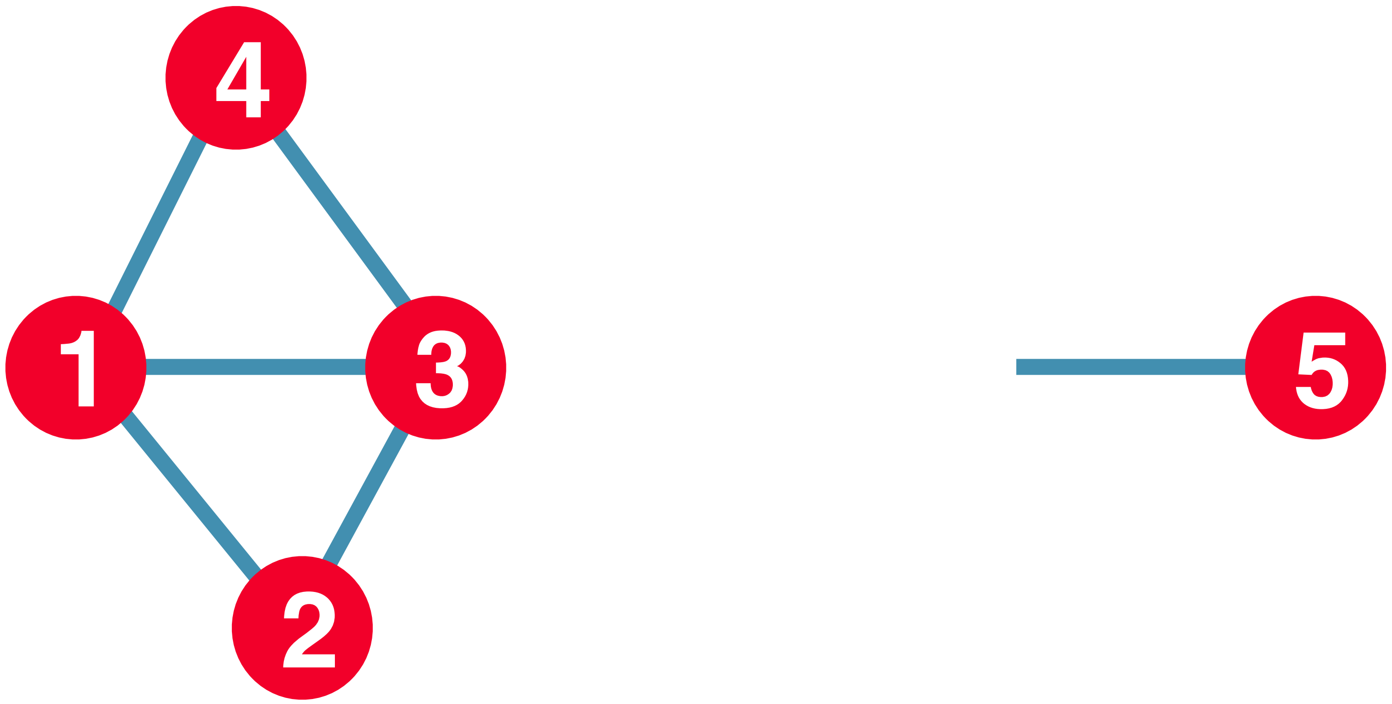
BARABASI-ALBERT MODEL



$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}$$

Probability a new node makes a connection to node i

BARABASI-ALBERT MODEL



$$\Pi(2) = \frac{2}{2+2+3+3} = \frac{1}{5}$$

For nodes 2 and 4

$$\Pi(3) = \frac{3}{2+2+3+3} = \frac{3}{10}$$

For nodes 1 and 3

BARABASI-ALBERT MODEL

Parameters:

number of initial nodes m_o

Number of links of new nodes $m \leq m_0$

Number of total nodes at the end of the process N

Algorithm:

For each new node i , with i from m_o to: N

Compute the probability of every node to get attached to i based on degree

Choose m nodes randomly based on this probability

form link between the new node and the chosen nodes

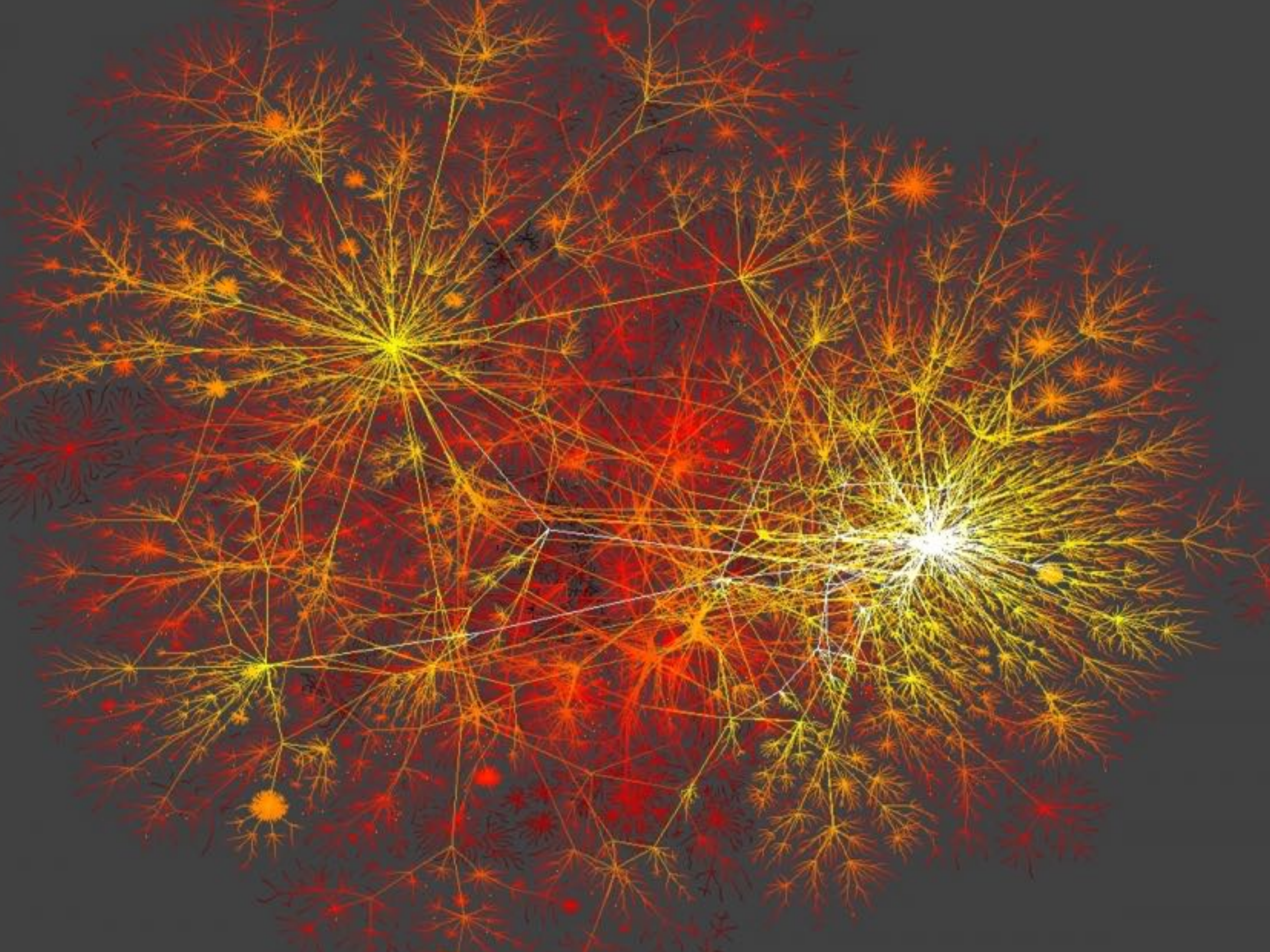
METRICS

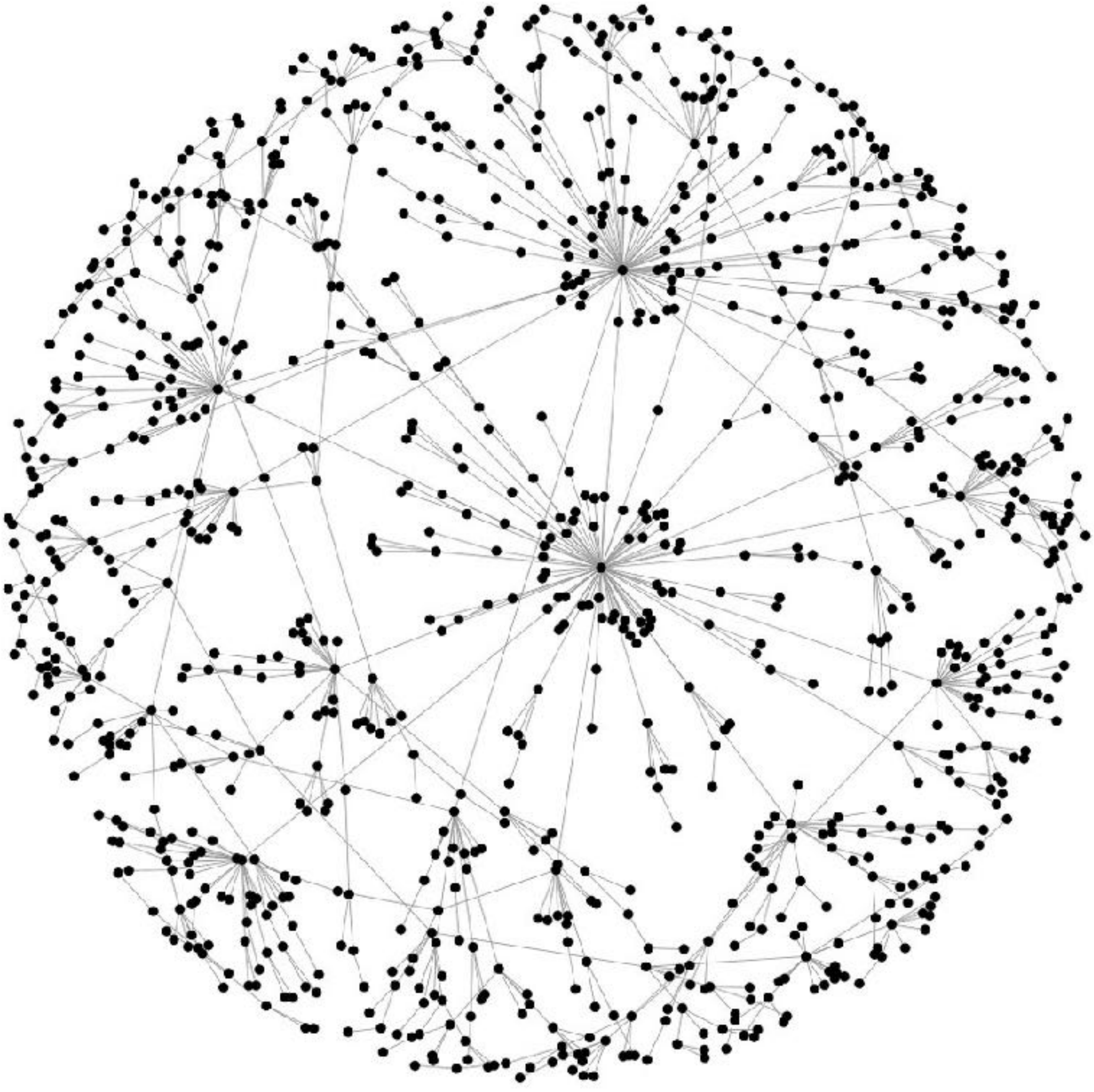
Parameters:

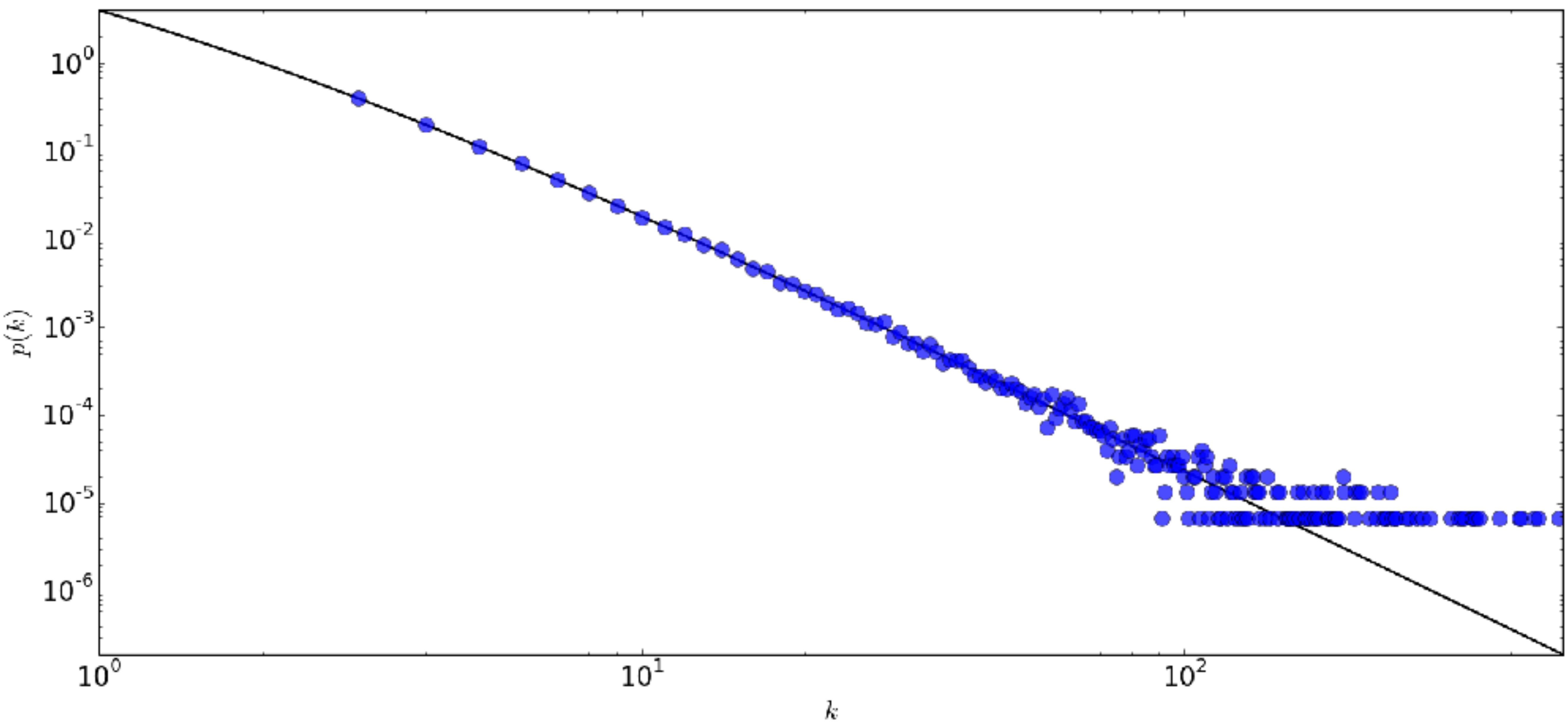
Degree distribution $P(k) \approx k^{-3}$

Clustering coefficient $C(k) \approx k^{-1}$

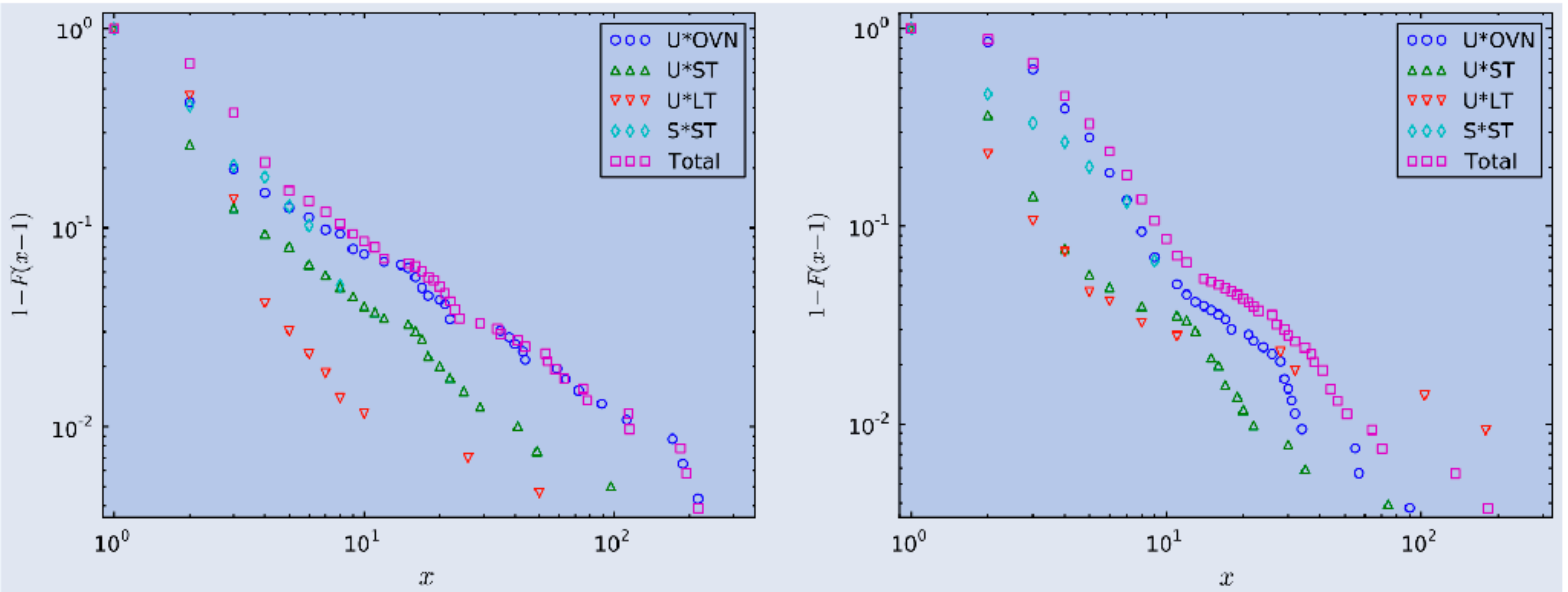
Diameter $\ell(k) \approx \dfrac{ln(N)}{lnln(N)}$







10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$p(k)$
10^0
10^1
10^2
k



(a) In-degree, 2012 (b) Out-degree, 2012

SUMMARY

We can **generate networks** with algorithms

Most **real-world networks** are compatible with one of these models

We can **compare the properties** of a real-world networ with that of a model to understand how they are formed