DMP Class Test

Discrete Mathematics

October 26th 2022

1. (a) For $n \in \mathbb{Z}^+$ prove by contradiction the statement 'if $5n+4$ is even, then n is even'.

[marks 3]

   (b) Prove that the simultaneous equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

have rational solutions x, y when a, b, c, d, e, f are all non-zero integers and $ad \neq bc$.

[marks 5]

2. Use the principle of strong induction to show that if u_n is defined recursively as

$$u_1 = 3, \qquad u_2 = 5, \qquad u_k = 3u_{k-1} - 2u_{k-2} \quad \text{for } k \in \mathbb{Z}^+, k \geq 3 \ ,$$

then the sequence can be represented by $u_n = 2^n + 1$ for every integer $n \geq 1$.

[marks 7]

3. We define the symmetric difference of two sets A and B as the set

$$A \mathbin{\Delta} B = x : (x \in A \textbf{ and } x \notin B) \textbf{ or } (x \in B \textbf{ and } x \ \notin A) \ .$$

   (a) Write the symmetric difference in set notation using $-$ and $\cup$.

[marks 1]

   (b) Draw a Venn diagram illustrating $A \mathbin{\Delta} B$

[marks 1]

   (c) Use the algebraic method to prove

$$A \mathbin{\Delta} A \mathbin{\Delta} A = A \ .$$

[marks 5]

4. (a) Each of the following following describes a function where each function has domain and codomain equal to $\mathbb{Z}$. In each case show whether or not the function is one-to-one (injective) or onto (surjective). Also comment on any that are bijective (one-to-one correspondence).

   i. $f(n) = 2n + 1$

   ii. $g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd} \end{cases}$

   iii. $h(n) = \begin{cases} n + 1 & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd} \end{cases}$

[marks 5]

(b) Show that the set of all nonnegative integers is countable by showing a bijection between $\mathbb{Z}^+$ and $\mathbb{Z}^{nonneg}$ using an explicit function.

[marks 3]