1. (a) For \( n \in \mathbb{Z}^+ \) prove by contradiction the statement ‘if \( 5n + 4 \) is even, then \( n \) is even’.

Solution:
Proof by Contradiction
We assume that \( 5n + 4 \) is even and on the contrary \( n \) is odd.
Let the odd number \( n = 2k + 1 \) where \( k \) is some non-negative integer. Then
\[
5n + 4 = 5(2k + 1) + 4 = 10k + 9 = 2(5k + 4) + 1.
\]
Since \( m = 5k + 4 \) is an integer as the sums and products of integers are integers, \( 2m + 1 \) is an odd integer by definition.
Hence \( 5n + 4 \) is odd and this is a contradiction to the original statement that \( 5n + 4 \) is even. Thus the original statement is true.

(b) Prove that the simultaneous equations

\[
ax + by = e\\
\]
\[
\]
\[
ax + dy = f
\]

have rational solutions \( x, y \) when \( a, b, c, d, e, f \) are all non-zero integers and \( ad \neq bc \).

Solution:
Solving gives \( (ad - bc)x = de - bf \) and \( (ad - bc)y = af - ce \).
Thus
\[
x = \frac{de - bf}{ad - bc},\\
y = \frac{af - ce}{ad - bc}.
\]
We are given \(ad - bc \in \mathbb{Z}^{nonzero}\) and since integers are closed by multiplication and subtraction (Epp p161) then both \(de - bf\) and \(af - ce\) are also integers.

By the definition of rationals (Epp, p183) \(x, y \in \mathbb{Q}\). □

2. Use the principle of strong induction to show that if \(u_n\) is defined recursively as

\[ u_1 = 3, \quad u_2 = 5, \quad u_k = 3u_{k-1} - 2u_{k-2} \quad \text{for} \quad k \in \mathbb{Z}^+, k \geq 3, \]

then the sequence can be represented by \(u_n = 2^n + 1\) for every integer \(n \geq 1\).

[marks 7]

Solution:

Let \(u_1, u_2, \ldots\) be the sequence defined by specifying that \(u_1 = 3, u_2 = 5\) and \(u_k = 3u_{k-1} - 2u_{k-2}\) for \(k \in \mathbb{Z}^+, k \geq 3\). Let \(P(n) : u_n = 2^n + 1\). We will prove that \(P(n)\) is true for every integer \(n \geq 1\).

Then \(P(1) : u_1 = 2^1 + 1 = 3\) and \(P(2) : u_2 = 2^2 + 1 = 5\) which agree with the initial values of the sequence.

We will show that for every integer \(k \geq 2\), if \(P(i)\) is true for each integer from 1 through \(k\), then \(P(k+1)\) is also true.

Let \(k\) be any integer with \(k \geq 2\) and suppose that

\[ \text{IH} : u_i = 2^i + 1 \quad \text{for each integer} \quad i \quad \text{with} \quad 1 \leq i \leq k. \]

We must show that

\[ P(k+1) : u_{k+1} = 2^{k+1} + 1. \]

Since \(k \geq 2\), we have \(k + 1 \geq 3\) and \(k - 1 \geq 1\) which is all in the range of \(i\). So

\[ \text{IS:} \quad u_{k+1} = 3u_k - 2u_{k-1} \]

\[ = 3(2^k + 1) - 2(2^{k-1} + 1) \quad \text{using IH twice} \]

\[ = 2 \cdot 2^k + 2^k + 3 - 2 \cdot 2^{k-1} - 2 \quad \text{using laws of exponents} \]

\[ = 2^{k+1} + 2^k + 3 - 2^k - 2 \]

\[ = 2^{k+1} + 1. \quad \text{as was to be shown.} \]

Since we have proved both the basis and the inductive step, by PSMI then \(P(n)\) is true. □

3. We define the symmetric difference of two sets \(A\) and \(B\) as the set

\[ A \Delta B = x : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A). \]

(a) Write the symmetric difference in set notation using \(-\) and \(\cup\).

[marks 1]

(b) Draw a Venn diagram illustrating \(A \Delta B\)

[marks 1]
(c) Use the algebraic method to prove

\[ A \Delta A \Delta A = A. \]

[marks 5]

**Solution:**

(a) \((A - B) \cup (B - A)\)

(b)

(c) Starting with \(A \Delta A\) we see

\[
A \Delta A = (A - A) \cup (A - A) \quad \text{(by definition)}
\]

\[= A - A \quad \text{(Idempotent law)}
\]

\[= A \cap A^c \quad \text{(by definition of \(-\))}
\]

\[= \emptyset. \quad \text{(Complement law)}
\]

Then

\[ (A \Delta A) \Delta A = \emptyset \Delta A \quad \text{(by previous result)}
\]

\[= (\emptyset - A) \cap (A - \emptyset). \quad \text{(by definition of \(\Delta\))}
\]

\[= (\emptyset \cap A^c) \cup (A \cap \emptyset^c) \quad \text{(by definition of \(-\))}
\]

\[= \emptyset \cup A \quad \text{(Complement, commutative and identity laws)}
\]

\[= A. \quad \text{(Identity law)}
\]

4. (a) Each of the following following describes a function where each function has domain and codomain equal to \(\mathbb{Z}\). In each case show whether or not the function is one-to-one (injective) or onto (surjective). Also comment on any that are bijective (one-to-one correspondence).

i. \(f(n) = 2n + 1\)

**Solution:**

Let \(a_1, a_2\) be any two integers in the domain. Then if \(f(a_1) = f(a_2), 2a_1 + 1 = 2a_2 + 1 \iff a_1 = a_2\). Therefore, \(f\) is a one-to-one function.

The range of \(f\) consists of only odd integers so \(f\) is not onto.
ii. \( g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd} \end{cases} \)

**Solution:**
The function \( g \) is not injective since, for example, \( g(4) = 2 \) and \( g(1) = 2 \).
Since \( g(2m) = m \) for any integer \( m \), \( g \) is an onto function. \( \square \)

iii. \( h(n) = \begin{cases} n + 1 & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd} \end{cases} \)

**Solution:**
Note that \( h(n) \) is an odd integer when \( n \) is even and even integer when \( n \) is odd.
Hence, if \( h(a_1) = h(a_2) \) then either \( a_1 + 1 = a_2 + 1 \) or \( a_1 - 1 = a_2 - 1 \) and in both cases \( a_1 = a_2 \). Therefore, \( h \) is one-to-one.
If \( m \) is an odd integer, the integer \( m - 1 \) is even and so \( h(m - 1) = (m - 1) + 1 = m \).
If \( m \) is an even integer, the integer \( m + 1 \) is odd and so \( h(m + 1) = (m + 1) - 1 = m \).
Therefore, the range of \( h \) is the whole set of integers and so \( h \) is surjective. Since \( h \) is both one-to-one and onto, \( h \) is bijective. \( \square \)

[marks 5]

(b) Show that the set of all nonnegative integers is countable by showing a bijection between \( \mathbb{Z}^+ \) and \( \mathbb{Z}_{nonneg} \) using an explicit function.

[marks 3]

**Solution:**
We define a function \( f : \mathbb{Z}^+ \to \mathbb{Z}_{nonneg} \) as \( f(n) = n - 1, \) \( \forall n \in \mathbb{Z}^+ \).
If \( n \geq 1 \) then \( n - 1 \geq 0 \) so \( f \) is well defined.
Also, \( f \) is one-to-one because for all positive integers \( n_1 \) and \( n_2 \), if \( f(n_1) = f(n_2) \) then \( n_1 - 1 = n_2 - 1 \) so \( n_1 = n_2 \).
Additionally \( f \) is onto because if \( m \in \mathbb{Z}_{nonneg} \) then with \( n - 1 = m \) giving \( n = m + 1 \), then we see that \( f(m + 1) = (m + 1) - 1 = m \) by definition of \( f \). Thus, because there is a function \( f : \mathbb{Z}^+ \to \mathbb{Z}_{nonneg} \) that is both one-to-one and onto (bijective) then \( \mathbb{Z}^+ \) has the same cardinality as \( \mathbb{Z}_{nonneg} \).
It follows that \( \mathbb{Z}_{nonneg} \) is countably infinite and hence countable. (Epp p473, 476) \( \square \)