DMP Class Test

Discrete Mathematics

October 26th 2022

1. (a) For $n \in \mathbb{Z}^+$ prove by contradiction the statement 'if $5n + 4$ is even, then n is even'.

[marks 3]

Solution:
Proof by Contradiction
We assume that $5n + 4$ is even and on the contrary n is odd.
Let the odd number $n = 2k + 1$ where k is some non-negative integer. Then

$$\begin{aligned} 5n + 4 &= 5(2k + 1) + 4 \\ &= 10k + 9 \\ &= 2(5k + 4) + 1\,. \end{aligned}$$

Since $m = 5k + 4$ is an integer as the sums and products of integers are integers, $2m + 1$ is an odd integer by definition.
Hence $5n + 4$ is odd and this is a contradiction to the original statement that $5n + 4$ is even.
Thus the original statement is true. □

(b) Prove that the simultaneous equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

have rational solutions x, y when a, b, c, d, e, f are all non-zero integers and $ad \neq bc$.

[marks 5]

Solution:
Solving gives $(ad - bc)x = de - bf$ and $(ad - bc)y = af - ce$.
Thus

$$\begin{aligned} x &= \frac{de - bf}{ad - bc}\,, \\ y &= \frac{af - ce}{ad - bc}\,. \end{aligned}$$

We are given $ad - bc \in \mathbb{Z}^{nonzero}$ and since integers are closed by multiplication and subtraction (Epp p161) then both $de - bf$ and $af - ce$ are also integers.

By the definition of rationals (Epp, p183) $x, y \in \mathbb{Q}$. □

2. Use the principle of strong induction to show that if u_n is defined recursively as

$$u_1 = 3, \qquad u_2 = 5, \qquad u_k = 3u_{k-1} - 2u_{k-2} \quad \text{for } k \in \mathbb{Z}^+, k \geq 3\ ,$$

then the sequence can be represented by $u_n = 2^n + 1$ for every integer $n \geq 1$.

[marks 7]

Solution:
Let $u_1, u_2, \ldots$ be the sequence defined by specifying that $u_1 = 3, u_2 = 5$ and $u_k = 3u_{k-1} - 2u_{n-2}$ for $k \in \mathbb{Z}^+, k \geq 3$. Let $P(n) : u_n = 2^n + 1$. We will prove that $P(n)$ is true for every integer $n \geq 1$.
Then $P(1) : u_1 = 2^1 + 1 = 3$ and $P(2) : u_2 = 2^2 + 1 = 5$ which agree with the initial values of the sequence.
We will show that for every integer $k \geq 2$, if $P(i)$ is true for each integer from 1 through k, then $P(k + 1)$ is also true.
Let k be any integer with $k \geq 2$ and suppose that

$$\text{IH} : u_i = 2^i + 1 \quad \text{for each integer } i \text{ with } 1 \leq i \leq k.$$

We must show that

$$P(k + 1) :\ u_{k+1} = 2^{k+1} + 1\,.$$

Since $k \geq 2$, we have $k + 1 \geq 3$ and $k - 1 \geq 1$ which is all in the range of i. So

$$\begin{aligned} \text{IS:}\quad u_{k+1} &= 3u_k - 2u_{k-1} \\ &= 3(2^k + 1) - 2(2^{k-1} + 1) \quad \text{using IH twice} \\ &= 2 \cdot 2^k + 2^k + 3 - 2 \cdot 2^{k-1} - 2. \quad \text{using laws of exponents} \\ &= 2^{k+1} + 2^k + 3 - 2^k - 2 \\ &= 2^{k+1} + 1. \quad \text{as was to be shown.} \end{aligned}$$

Since we have proved both the basis and the inductive step, by PSMI then $P(n)$ is true. □

3. We define the symmetric difference of two sets A and B as the set

$$A \,\Delta\, B = x : (x \in A \textbf{ and } x \notin B) \textbf{ or } (x \in B \textbf{ and } x \notin A)\,.$$

   (a) Write the symmetric difference in set notation using $-$ and $\cup$.

   [marks 1]

   (b) Draw a Venn diagram illustrating $A \,\Delta\, B$

   [marks 1]

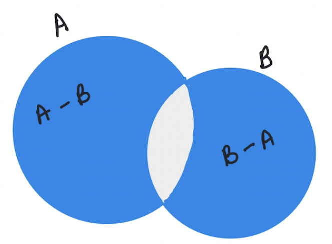
(c) Use the algebraic method to prove

$$A \,\Delta\, A \,\Delta\, A = A\,.$$

[marks 5]

Solution:

(a) $(A - B) \cup (B - A)$



(b)

(c) Starting with $A \,\Delta\, A$ we see

$$\begin{aligned} A \,\Delta\, A &= (A - A) \cup (A - A) \qquad \text{(by definition)} \\ &= A - A \qquad \text{(Idempotent law)} \\ &= A \cap A^c \qquad \text{(by definition of } -) \\ &= \emptyset. \qquad \text{(Complement law)} \end{aligned}$$

Then

$$\begin{aligned} (A \,\Delta\, A) \,\Delta\, A &= \emptyset \,\Delta\, A \qquad \text{(by previous result)} \\ &= (\emptyset - A) \cap (A - \emptyset). \qquad \text{(by definition of } \Delta) \\ &= (\emptyset \cap A^c) \cup (A \cap \emptyset^c) \qquad \text{(by definition of } -) \\ &= \emptyset \cup A \qquad \text{(Complement, commutative and identity laws)} \\ &= A. \qquad \text{(Identity law)} \end{aligned}$$

□

4. (a) Each of the following following describes a function where each function has domain and codomain equal to $\mathbb{Z}$. In each case show whether or not the function is one-to-one (injective) or onto (surjective). Also comment on any that are bijective (one-to-one correspondence).

   i. $f(n) = 2n + 1$
   Solution:
   Let a_1, a_2 be any two integers in the domain. Then if $f(a_1) = f(a_2)$, $2a_1 + 1 = 2a_2 + 1 \iff a_1 = a_2$. Therefore, f is a one-to-one function.
   The range of f consists of only odd integers so f is not onto. □

ii. $g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd} \end{cases}$

Solution:
The function g is not injective since, for example, $g(4) = 2$ and $g(1)$ =2.
Since $g(2m) = m$ for any integer m, g is an onto function. □

iii. $h(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$

Solution:
Note that $h(n)$ is an odd integer when n is even and even integer when n is odd.
Hence, if $h(a_1) = h(a_2)$ then either $a_1 + 1 = a_2 + 1$ or $a_1 - 1 = a_2 - 1$ and in both cases $a_1 = a_2$. Therefore, h is one-to-one.
If m is an odd integer, the integer $m - 1$ is even and so $h(m-1) = (m-1) + 1 = m$.
If m is an even integer, the integer $m + 1$ is odd and so $h(m+1) = (m+1) - 1 = m$
Therefore, the range of h is the whole set of integers and so h is surjective. Since h is both one-to-one and onto, h is bijective. □

[marks 5]

(b) Show that the set of all nonnegative integers is countable by showing a bijection between $\mathbb{Z}^+$ and $\mathbb{Z}^{nonneg}$ using an explicit function.

[marks 3]

Solution:
We define a function $f : \mathbb{Z}^+ \to \mathbb{Z}^{nonneg}$ as $f(n) = n - 1, \quad \forall n \in Z^+$.
If $n \geq 1$ then $n - 1 \geq 0$ so f is well defined.
Also, f is one-to-one because for all positive integers n_1 and n_2, if $f(n_1) = f(n_2)$ then $n_1 - 1 = n_2 - 1$ so $n_1 = n_2$.
Additionally f is onto because if $m \in \mathbb{Z}^{nonneg}$ then with $n - 1 = m$ giving $n = m + 1$, then we see that $f(m+1) = (m+1) - 1 = m$ by definition of f. Thus, because there is a function $f : \mathbb{Z}^+ \to \mathbb{Z}^{nonneg}$ that is both one-to-one and unto (bijective) then $\mathbb{Z}^+$ has the same cardinality as $\mathbb{Z}^{nonneg}$.
It follows that $\mathbb{Z}^{nonneg}$ is countably infinite and hence countable. (Epp p473, 476) □