DMP Class Test

Solutions
25 October 2023

1. We prove this by contradiction.

   Suppose that there exist different positive integers x and y such that $x/y^2 = x^2/y$. Multiplying both sides by y^2, and dividing by x, using that $x \neq 0$, gives $1 = xy$. Using that x and y are positive integers, the only way one can have $xy = 1$ is when $x = 1$ and $y = 1$. But then $x = y$, contradicting the assumption that x and y are different.

   Thus the assumption must have been false, and there are no different positive integers x and y such that $x/y^2 = x^2/y$.

2. We prove the even stronger statement that $s_n = 2^{n-1}$, for all $n \geq 1$. The proof is by induction. Let $P(n)$ be the statement "$s_n = 2^{n-1}$".

   Base case: $s_1 = \sum_{i<1} s_i = s_0 = 1 = 2^0$. So $P(1)$ holds.

   Induction step: Assuming $P(k)$ for some $k \geq 1$, we must obtain $P(k{+}1)$. So we assume that $s_k = 2^{k-1}$. Now $s_{k+1} = \sum_{i<k+1} s_i = \sum_{i<k} s_i + s_k = s_k + s_k = 2^{k-1} + 2^{k-1} = 2 \cdot 2^{k-1} = 2^k$. This is $P(k{+}1)$.

   Thus, by induction, $s_n = 2^{n-1}$ for all $n \geq 1$ and in particular for all $n \geq 3$.

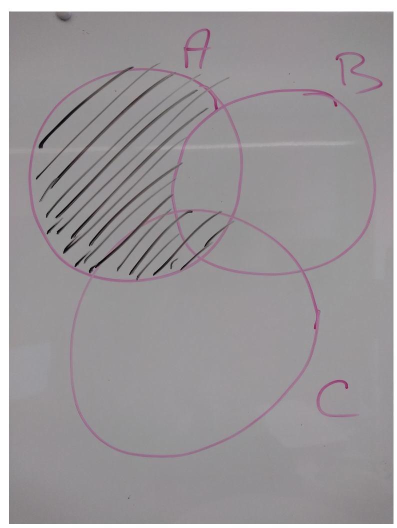
3. To prove that $A - (B - C) = (C \cap A) \cup (A - B)$ by the element method, first suppose that $x \in A - (B - C)$. Then $x \in A$ and $\notin B - C$. The latter implies that either $x \notin B$ or $x \in C$. In the first case we get $x \in A - B$, which implies $x \in (C \cap A) \cup (A - B)$. In the second case we get $x \in C \cap A$, which also implies $x \in (C \cap A) \cup (A - B)$. Thus, in either case $x \in (C \cap A) \cup (A - B)$. It follows that $A - (B - C) \subseteq (C \cap A) \cup (A - B)$

   Next suppose that $x \in (C \cap A) \cup (A - B)$. Then either $x \in C \cap A$ or $x \in A - B$. In the first case $x \in C$ and $x \in A$. Thus $x \notin B - C$, and hence $x \in A - (B - C)$. In the second case $x \in A$ and $x \notin B$. Thus $x \notin B - C$, and hence $x \in A - (B - C)$. So in either case $x \in A - (B - C)$. It follows that $(C \cap A) \cup (A - B) \subseteq A - (B - C)$.

   Together, we obtain $A - (B - C) = (C \cap A) \cup (A - B)$.

   Here is an algebraic proof of the same identity:

$$
\begin{aligned}
A - (B - C) &= && \text{(Law 12)} \\
A - (B \cap C^c) &= && \text{(Law 12)} \\
A \cap (B \cap C^c)^c &= && \text{(Law 9(b))} \\
A \cap (B^c \cup (C^c)^c) &= && \text{(Law 6)} \\
A \cap (B^c \cup C) &= && \text{(Law 3(b))} \\
(A \cap B^c) \cup (A \cap C) &= && \text{(Law 3(b))} \\
(A \cap C) \cup (A \cap B^c) &= && \text{(Law 1(a))} \\
(C \cap A) \cup (A \cap B^c) &= && \text{(Law 1(b))} \\
(C \cap A) \cup (A - B) &= && \text{(Law 12)}
\end{aligned}
$$



4. The function f is not injective, since $f(1) = f(-1) = 1$. It moreover is not surjective, since there is no x with $f(x) = -1$. Thus, is is not bijective either.

   To test whether g is surjective, we try to solve the equation $g(x) = y$. We obtain $\frac{3x}{x-1} = y$, so $xy - y = 3x$ and $x(y-3) = y$. Hence $x = \frac{y}{y-3}$. There is no solution for the case $y = 3$. In fact, if $x > 1$ then $g(x) > 3$ and if $x < 1$ then $g(x) < 3$. Hence g is not surjective. Thus, is is not bijective either.

   The above derivation does show that g is injective, because for any given $y \neq 3$ the only x with $g(x) = y$ is $x = \frac{y}{y-3}$.

   Since $h(x) = g(x) \neq 3$ for any $x \neq 1$, and g is injective, h is also injective. Moreover, h repairs the only point in which g fails to be surjective. Hence h is surjective, and thus also bijective.

5. It suffice provide an injective mapping from the words to the natural numbers. Such a mapping is obtained by seeing each word as a number is base 27, where the letters *a*–*z* are the non-zero digits. Such a map is clearly injective.

   Using base 26 and all digits is not quite right, because if *a* is the 0-digit, then *aab* maps to 001, which is the same number 01 obtained by *ab*, so we lose injectivity.