This homework runs from 3pm Thursday 21 September 2023 until 12 noon on Thursday 28 September 2023. Submission is to Gradescope Homework 1.

Question 1

(a) Suppose $n \in \mathbb{Z}$ is an integer. Write down the contrapositive of the following statement.

$$\text{If } 7n - 2 \text{ is even then } n \text{ is even.}$$

[*2 marks*]

(b) Use proof by contraposition to prove the statement in part (a). [*4 marks*]

Question 2

If r and s are real numbers then their *geometric mean* is $\sqrt{rs}$, the square root of their product.

Suppose that r and s are non-zero real numbers with geometric mean g where g is rational. Prove that r is rational if and only if s is rational.

$$r \text{ is rational} \iff s \text{ is rational}$$

[*4 marks*]

Solution 1

(a) The contrapositive is "If n is odd then $7n - 2$ is odd".

(b) For proof by contraposition we proceed to check the statement in part (a).

Suppose that n is an odd integer. Then we know that $n = 2k + 1$ for some $k \in \mathbb{Z}$. From this we calculate as follows.

$$\begin{aligned} 7n - 2 &= 7(2k + 1) - 2 \\ &= 14k + 7 - 2 \\ &= 14k + 5 \\ &= 2(7k + 2) + 1 \end{aligned}$$

Taking integer $p = 7k + 2$ we then have $7n - 2 = 2p + 1$ and so $7n - 2$ is odd. This shows the contrapositive is true and therefore so is the original statement.

Solution 2

Suppose that r is rational. Then we can calculate as follows, using the fact that r is given to be non-zero.

$$\begin{aligned} g &= \sqrt{rs} \\ g^2 &= rs \\ s &= \frac{g^2}{r} \end{aligned}$$

Since g and r are both rational, and the product and quotient of rational numbers is rational, then s is also rational.

For the reverse, we assume that s is rational and then prove r is rational by the same method.

It's also possible to prove this by calculating explicitly with fractions. Suppose $g = a/b$ for integers a and b with b non-zero. Suppose $r = c/d$ is rational, with c and d non-zero integers. We can then calculate:

$$\begin{aligned} \frac{a}{b} &= \sqrt{\frac{c}{d}s} \\ \frac{a^2}{b^2} &= \frac{c}{d}s \\ s &= \frac{a^2 d}{b^2 c}. && \text{(Using } c \neq 0 \text{ from } r \neq 0\text{)} \end{aligned}$$

Since a, b, c, and d are all integers, and the set of integers is closed under multiplication, we know that $a^2 d$ and $b^2 c$ are integers. By the zero product property, since b and c are non-zero we know $b^2 c$ is too. All these together show that s is rational.

Again the reverse direction holds by symmetry.

Either method is valid here: working with rationals or explicit fractions. In each case it's essential to make clear the assumptions being used in the proof. For the first method, that's about product and quotients of rationals; for the second, it's that a product of integers is an integer. All methods rely on r and s being non-zero.

It's not enough just to use these assumptions: for a clear proof it's important to use them in the right place, to know you are using them, and to communicate to anyone reading the proof which assumptions you are using and when.