This homework runs from 3pm Thursday 28 September 2023 until 12 noon on Thursday 5 October 2023. Submission is to Gradescope Homework 2.

Question 1

Assume n is a positive integer with $n \geq 1$. Prove by mathematical induction that

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}\,.$$

[*7 marks*]

Question 2

Suppose a sequence of integers $a_1, a_2, a_3, \ldots$ is defined recursively as follows:

$$a_1 = 4 \quad \text{and} \quad a_{n+1} = a_n + 18n + 3 \quad \text{for } n \geq 1.$$

Prove by induction that $a_n = (3n-1)^2$ for all $n \geq 1$. [*3 marks*]

Solution 1

Let $P(n)$ be the statement given in the question:

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}. \qquad P(n)$$

We wish to prove by induction that $P(n)$ holds for all integers $n \geq 1$.

For the base case $P(1)$ we must show the following.

$$\sum_{r=1}^{1} r^3 = \frac{1^2(1+1)^2}{4} \qquad P(1)$$

We can expand out both sides.

$$\sum_{r=1}^{1} r^3 = 1^3 = 1 \qquad \frac{1^2(1+1)^2}{4} = \frac{1 \cdot 2^2}{4} = 1$$

Since both sides are equal to the same value they are equal to each other and $P(1)$ holds.

For inductive step we assume that $P(k)$ holds for some value $k \geq 1$ and seek to show $P(k+1)$.

$$\sum_{r=1}^{k} r^3 = \frac{k^2(k+1)^2}{4} \qquad \text{Induction Hypothesis } P(k)$$

We expand out the left-hand side of $P(k+1)$ and calculate.

$$\begin{aligned}
P(k+1) \text{ LHS} \qquad \sum_{r=1}^{k+1} r^3 &= \left(\sum_{r=1}^{k} r^3\right) + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \qquad \text{by } P(k) \\
&= \frac{1}{4}(k^2(k+1)^2 + 4(k+1)^3) = \frac{1}{4}(k^2 + 4(k+1))(k+1)^2 \\
&= \frac{1}{4}(k^2 + 4k + 4))(k+1)^2 = \frac{1}{4}(k+2)^2(k+1)^2 \\
&= \frac{(k+1)^2((k+1)+1)^2}{4} \qquad P(k+1) \text{ RHS}
\end{aligned}$$

This shows $P(k+1)$ holds if we know that $P(k)$ does.

From the base and step case we deduce by mathematical induction that $P(n)$ holds for all integers $n \geq 1$.

Solution 2

For the base case $n = 1$ we check that $(3 \cdot 1 - 1)^2 = 2^2 = 4 = a_1$ as required.

For the step case we assume as induction hypothesis that $a_k = (3k-1)^2$ for integer $k \geq 1$ and try to show that $a_{k+1} = (3(k+1)-1)^2$. We can start by expanding both sides of this equation.

$$\begin{aligned}
a_{k+1} &= a_k + 18k + 3 \quad \text{by definition of sequence} \\
&= (3k-1)^2 + 18k + 3 \quad \text{by induction hypothesis} \\
&= (9k^2 - 6k + 1) + 18k + 3 \\
&= 9k^2 + 12k + 4
\end{aligned}
\qquad
\begin{aligned}
(3(k+1)-1)^2 &= (3k+2)^2 \\
&= 9k^2 + 12k + 4
\end{aligned}$$

Both sides are equal and we have shown $a_{k+1} = (3(k+1)-1)^2$ as required.

From the base and step case we deduce by mathematical induction that $a_n = (3n-1)^2$ for all integers $n \geq 1$.