Question 1
Assume $n$ is a positive integer with $n \geq 1$. Prove by mathematical induction that

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}.$$  

[7 marks]

Question 2
Suppose a sequence of integers $a_1, a_2, a_3, \ldots$ is defined recursively as follows:

$$a_1 = 4 \quad \text{and} \quad a_{n+1} = a_n + 18n + 3 \quad \text{for} \ n \geq 1.$$  

Prove by induction that $a_n = (3n - 1)^2$ for all $n \geq 1$.  

[3 marks]
Solution 1

Let $P(n)$ be the statement given in the question:

$$
\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}. \quad P(n)
$$

We wish to prove by induction that $P(n)$ holds for all integers $n \geq 1$.

For the base case $P(1)$ we must show the following.

$$
\sum_{r=1}^{1} r^3 = 1^3 = 1 \quad P(1)
$$

We can expand out both sides.

$$
\sum_{r=1}^{1} r^3 = 1^3 = 1 \quad \frac{1^2(1+1)^2}{4} = \frac{1 \cdot 2^2}{4} = 1
$$

Since both sides are equal to the same value they are equal to each other and $P(1)$ holds.

For inductive step we assume that $P(k)$ holds for some value $k \geq 1$ and seek to show $P(k+1)$.

$$
\sum_{r=1}^{k} r^3 = \frac{k^2(k+1)^2}{4} \quad \text{Induction Hypothesis } P(k)
$$

We expand out the left-hand side of $P(k+1)$ and calculate.

$$
P(k+1) \text{ LHS } \sum_{r=1}^{k+1} r^3 = \left( \sum_{r=1}^{k} r^3 \right) + (k+1)^3
= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \text{by } P(k)
= \frac{1}{4}(k^2(k+1)^2 + 4(k+1)^3) = \frac{1}{4}(k^2 + 4(k+1))(k+1)^2
= \frac{1}{4}(k^2 + 4k + 4)(k+1)^2 \quad = \frac{1}{4}(k+2)^2(k+1)^2
= \frac{(k+1)^2((k+1)+1)^2}{4} \quad P(k+1) \text{ RHS}
$$

This shows $P(k+1)$ holds if we know that $P(k)$ does.

From the base and step case we deduce by mathematical induction that $P(n)$ holds for all integers $n \geq 1$.

Solution 2

For the base case $n = 1$ we check that $(3 \cdot 1 - 1)^2 = 2^2 = 4 = a_1$ as required.

For the step case we assume as induction hypothesis that $a_k = (3k - 1)^2$ for integer $k \geq 1$ and try to show that $a_{k+1} = (3(k + 1) - 1)^2$. We can start by expanding both sides of this equation.

$$
a_{k+1} = a_k + 18k + 3 \quad \text{by definition of sequence} \quad (3(k + 1) - 1)^2 = (3k + 2)^2
= (3k - 1)^2 + 18k + 3 \quad \text{by induction hypothesis} \quad = 9k^2 + 12k + 4
= (9k^2 - 6k + 1) + 18k + 3 \quad = 9k^2 + 12k + 4
= 9k^2 + 12k + 4
$$

Both sides are equal and we have shown $a_{k+1} = (3(k + 1) - 1)^2$ as required.

From the base and step case we deduce by mathematical induction that $a_n = (3n - 1)^2$ for all integers $n \geq 1$. 