This homework runs from 3pm Thursday 5 October 2023 until 12 noon on Thursday 12 October 2023. Submission is to Gradescope Homework 3.

Question 1

For two sets A and B their *symmetric difference* Δ can be described as the set of all elements from A and B that are not in the intersection of A and B. We write that as follows.

$$(A \cup B) - (A \cap B) \tag{1}$$

There following is an alternate description proposed for the same set Δ.

$$(A - B) \cup (B - A) \tag{2}$$

(a) Describe the set (2) in words, as was done earlier for the expression (1). [*1 mark*]

(b) Construct an algebraic proof that the two expressions (1) and (2) are equivalent. You can use any of the set identities listed in Theorem 6.2.2 of the Epp textbook. Make it clear which identity you are using at each stage in the proof. [*5 marks*]

Question 2

Suppose that J and K are subsets of $\mathbb{Z}$, the set of integers, with $12 \in K$ and $f : J \to K$ some (total) function. For each of the following situations say how many elements of J could be related by function f to the value 12.

(a) Function f is one-to-one (injective).

(b) Function f is onto (surjective).

(c) Function f is bijective.

In which of these cases can we be sure that f has a unique inverse function f^{-1} that is defined everywhere on K? [*4 marks*]

Solution 1

(a) Expression (2) is the set of elements that are either in A but not in B, or in B but not A.

(b) The following uses the names from Epp for the various set identities. The Commutativity Law is used throughout and rolled in with uses of the other laws.

$$
\begin{aligned}
&(A - B) \cup (B - A) && \\
&\quad = (A \cap B^c) \cup (B \cap A^c) && \text{Set Difference Law} \\
&\quad = ((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup A^c)) && \text{Distributive Law} \\
&\quad = ((A \cup B) \cap (B^c \cup B)) \cap ((A \cup A^c) \cap (B^c \cup A^c)) && \text{Distributive Law, twice} \\
&\quad = ((A \cup B) \cap U) \cap (U \cap (B^c \cup A^c)) && \text{Complement Law, twice} \\
&\quad = (A \cup B) \cap (B^c \cup A^c) && \text{Identity Law, twice} \\
&\quad = (A \cup B) \cap (B \cap A)^c && \text{De Morgan's Law} \\
&\quad = (A \cup B) - (B \cap A) && \text{Set Difference Law}
\end{aligned}
$$

Other presentations are possible and can sometimes make things easier to follow. For example, below we pick out an individual term and show a result for that on its own, which can then be incorporated in the main proof.

$$
\begin{aligned}
(A \cap B^c) \cup B &= (A \cup B) \cap (B^c \cup B) && \text{Set union distributes over set intersection} \\
&= (A \cup B) \cap U && \text{Set complement law } B^c \cup B = U \\
&= A \cup B && \text{Set union identity law}
\end{aligned}
$$

These use slightly different names to refer to the algebraic identities being used but are still clear.

Solution 2

(a) Zero or one element of J could be related to 12 by function f.

For example, if $J = K = \mathbb{Z}$ then the injective function "multiply by five" takes no integers to 12; while the function "subtract one" takes precisely one integer, the number 13, to 12. If two or more elements were mapped to 12 then the function would not be injective.

(b) Any number of elements of J from one upwards.

For example, if $J = \{1, \ldots, n\}$ for some integer $n \geq 1$ and $K = \{12\}$ then the function taking every value in J to 12 maps exactly n different integers onto 12. If no values were mapped to 12 then f would not be surjective.

(c) Exactly one element of J will be related to 12 by function f.

This follows from the previous two cases since a function is bijective exactly when it is both surjective and injective.

In case (c) function f will have a uniquely-defined inverse f^{-1}. Being injective is not on its own enough to ensure an inverse is defined everywhere; and being surjective does not guarantee an inverse will be uniquely-defined.