This homework runs from Thursday 2 November 2023 until 12 noon on Thursday 9 November 2023. Submission is to Gradescope Homework 4.

Question 1

(a) In how many ways can the integers 1 through 7 be permuted so that no odd integer will be in its natural position?

(b) If all the possibilities of part (a) are equally likely, what is the probability that exactly two of the seven integers are in their natural position?

[*4 marks*]

Question 2

A art box has in it 20 pencils: 5 red, 5 blue, and 10 green. You cannot distinguish between pencils of the same colour. You take out three pencils at random, one at a time, without replacement. Suppose X is the random variable 'the total number of red pencils taken out'.

(a) Copy the following table and fill in the probability distribution for X.

x	0	1	2	3
$P(X = x)$				

(b) Calculate the expected number of red pencils $\mathrm{E}(X)$.

(c) Calculate the standard deviation of X.

For each part include your working as well as the final answer.

[*6 marks*]

Solution 1

(a) We start by calculating the number of permutations where at least one odd number is in its natural position. Write A_1 for the set of permutations of 1–7 such that 1 is in position 1; write A_3 for the set of permutations of 1–7 such that 3 is in position 3. Define A_5 and A_7 similarly. By the principle of inclusion-exclusion, then, we can reason as follows.

$$\begin{aligned}
|A_1 \cup A_3 \cup A_5 \cup A_7| &= |A_1| + |A_3| + |A_5| + |A_7| \\
&\quad - |A_1 \cap A_3| - |A_1 \cap A_5| - |A_1 \cap A_7| \\
&\quad - |A_3 \cap A_5| - |A_3 \cap A_7| - |A_5 \cap A_7| \\
&\quad + |A_1 \cap A_3 \cap A_5| + |A_1 \cap A_3 \cap A_7| \\
&\quad + |A_1 \cap A_5 \cap A_7| + |A_3 \cap A_5 \cap A_7| \\
&\quad - |A_1 \cap A_3 \cap A_5 \cap A_7| \\
&= \binom{4}{1} 6! - \binom{4}{2} 5! + \binom{4}{3} 4! - \binom{4}{4} 3! \\
&= 4 \cdot 720 - 6 \cdot 120 + 4 \cdot 24 - 1 \cdot 6 = 2250
\end{aligned}$$

So the answer is $7! - 2250 = 2790$.

(b) Let the sample space Ω consist of all permutations as in part (a), so Ω has 2790 elements. Let E be the event that exactly two of the seven integers are in their natural position. These two fixed integers cannot be odd, so we choose two integers from 2, 4, and 6. There are $\binom{3}{2} = 3$ ways to do this. Now we permute the other five integers so that none are in their natural position, following the same inclusion-exclusion calculation as in part (a).

$$\begin{aligned}
|E| &= \binom{3}{2} \cdot \left(\binom{5}{0} 5! - \binom{5}{1} 4! + \binom{5}{2} 3! - \binom{5}{3} 2! + \binom{5}{4} 1! - \binom{5}{5} 0! \right) \\
&= 3 \cdot (1 \cdot 120 - 5 \cdot 24 + 10 \cdot 6 - 10 \cdot 2 + 5 \cdot 1 - 1 \cdot 1) \\
&= 132
\end{aligned}$$

Assuming that all permutations are equally likely, the probability of E is then:

$$P(E) = \frac{132}{2790} = \frac{22}{465} \approx 0.047\,.$$

Solution 2

(a) We take the ordered approach. Write R_i for the event that the ith pencil is red, and O_i for it being blue or green.

$$\begin{aligned}
P(X=0) &= P(O_1O_2O_3) = \frac{15}{20}\cdot\frac{14}{19}\cdot\frac{13}{18} = \frac{91}{228} \\
P(X=1) &= P(R_1O_2O_3)+P(O_1R_2O_3)+P(O_1O_2R_3) \\
&= \frac{5}{20}\cdot\frac{15}{19}\cdot\frac{14}{18}+\frac{15}{20}\cdot\frac{5}{19}\cdot\frac{14}{18}+\frac{15}{20}\cdot\frac{5}{19}\cdot\frac{14}{18} = \frac{35}{76} \\
P(X=2) &= P(R_1R_2O_3)+P(R_1O_2R_3)+P(O_1R_2R_3) \\
&= \frac{5}{20}\cdot\frac{4}{19}\cdot\frac{14}{18}+\frac{5}{20}\cdot\frac{15}{19}\cdot\frac{4}{18}+\frac{15}{20}\cdot\frac{5}{19}\cdot\frac{4}{18} = \frac{5}{38} \\
P(X=3) &= P(R_1R_2R_3) = \frac{5}{20}\cdot\frac{4}{19}\cdot\frac{3}{18} = \frac{1}{114}
\end{aligned}$$

Hence the table is:

x	0	1	2	3
$P(X=x)$	$\frac{91}{228}$	$\frac{35}{76}$	$\frac{5}{38}$	$\frac{1}{114}$

(b) Recall the definition of expected value and compute:

$$E(X) = \sum_{\text{all } x} x\cdot P(X=x) = 0\left(\frac{91}{228}\right)+1\left(\frac{35}{76}\right)+2\left(\frac{5}{38}\right)+3\left(\frac{1}{114}\right) = \frac{3}{4}$$

(c) Recall that by definition:

$$\mathrm{Var}(X) = \mathrm{E}(X^2)-(\mathrm{E}(X))^2$$

Next, compute:

$$\mathrm{E}(X^2) = \sum_{\text{all } x} x^2\cdot P(X=x) = 0\left(\frac{91}{228}\right)+1\left(\frac{35}{76}\right)+4\left(\frac{5}{38}\right)+9\left(\frac{1}{114}\right) = \frac{81}{76}$$

Hence the variance is $\mathrm{Var}(X) = \frac{81}{76}-(\frac{3}{4})^2 = \frac{153}{304}$. We conclude that the standard deviation is $\sqrt{\frac{153}{304}} \approx 0.709$.