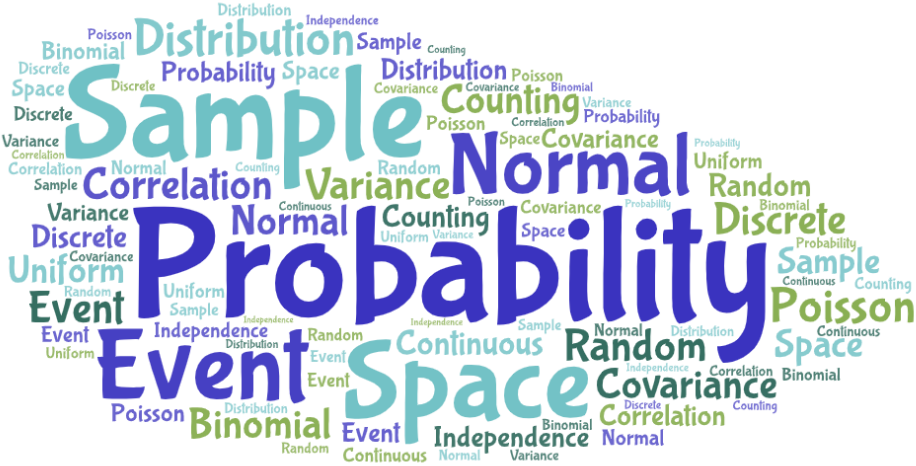
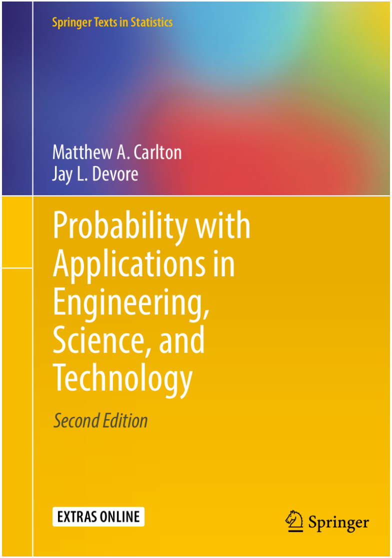
Discrete Mathematics and Probability

Week 7



Chris Heunen


Springer Texts in Statistics
Matthew A. Carlton
Jay L. Devore
Probability with Applications in Engineering, Science, and Technology
Second Edition
EXTRAS ONLINE
Springer

Topics

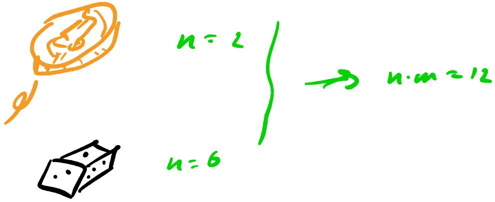
- Counting: thinking algorithmically
- Events: what could happen in principle
- Experiments: how can events interact
- Probability: quantifying what could happen

Counting

Counting

Basic principles of combinatorics:

- if an experiment has n outcomes;
  and another experiment has m outcomes,
- then the two experiments jointly have $n \cdot m$ outcomes.



Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *permutations* of H are the different orders in which you can write all of its elements.

Example: Horse race $\{A, B, C, D, E\}$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ permutations

e.g. (E, B, A, D, C)

gold silver bronze

How many? $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$

$0! = 1$

Permutations with repetitions

Definition

Let $H = \{h_1 \dots h_1,\ h_2 \dots h_2,\ \dots,\ h_r \dots h_r\}$ be a set of r different types of repeated objects: n_1 many of h_1, n_2 of h_2, … n_r of h_r. The *permutations with repetitions* of H are the different orders in which you can write all of its elements.

How many? $n = n_1 + n_2 + \dots + n_r$ total nr objects

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! \cdot n_2! \cdots n_r!}$$

"multinomial coefficient"

Example: words with letters A,B,R,A,C,A,D,A,B,R,A

$$\binom{11}{5, 2, 1, 1, 2} = 83160$$

k-Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-permutations* of H are the different ways in which one can pick and write k of its elements of H in order.

How many? $\quad P_{n,k} = \dfrac{n!}{(n-k)!}$

Example: top 3 in horse race $\{A, B, C, D, E\}$

e.g. (E, B, A)

$\dfrac{5!}{(5-3)!} = 20$ possibilities

k-Permutations with repetitions

Definition

Let $H = \{h_1 \ldots, h_2 \ldots, \ldots, h_r \ldots\}$ be a set of r different types of repeated objects, each of infinite supply. The *k-permutations with repetitions* of H are the different orders in which one can write an ordered sequence of length k using the elements of H.

Example: # 3-letter words of letters of English alphabet $= 26^3 = 17576$

bit strings of length k $= 2^k$

k-Combinations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The k-*combinations* of H are the different ways in which one can pick k of its elements without order.

How many?

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

"binomial coefficient" $\quad C_{n,k} = \frac{P_{n,k}}{k!}$

Example: # ways to form committee of 5 students from class of 30

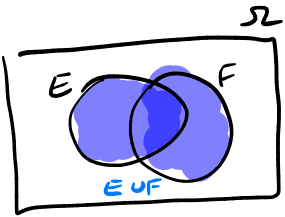
$$= \binom{30}{5} = \frac{30!}{5!\,25!} = 142506$$

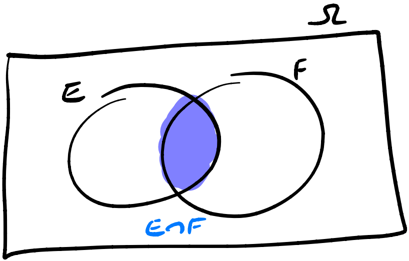
Events

Events

A mathematical model for experiments:

- *Sample space*: the set Ω of all possible outcomes
- An *event* is a collection[1] of possible outcomes: $E \subseteq \Omega$
- *Union* $E \cup F$ and *intersection* $E \cap F$ of events make sense





[1] Sometimes Ω is too large, and not all subsets are events. Ignore this now.

Examples

Experiment	Sample space	Event
Will it rain today?	$\Omega = \{r, n\}$	e.g. $E = \{r\}$
5-horse race	Ω = permutations of $\{A, B, C, D, E\}$	e.g. $E = \{B \text{ wins}\}$ $F = \{E \text{ wins}, A \text{ third}\}$ notice $E \cap F = \emptyset$ — never happens
Flip 2 coins	$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$	e.g. E = coins different $= \{(H,T), (T,H)\}$ F = at least one heads $= \{(H,H), (H,T), (T,H)\}$
roll die until get 6	Ω = {sequences of 1–5 and then 6}	e.g. E = first 4, then 6 on 3rd roll $= \{(4,k,6) \mid k \in \{1,\dots,5\}\}$

Union and intersection

Union

Union $E \cup F$ of events E and F means E and F.
Infinite union $\bigcup_i E_i$ of events E_i means at least one of the E_i's.

Intersection

Intersection $E \cap F$ of events E and F means E and F.
Infinite intersection $\bigcap_i E_i$ of events E_i means each of the E_i's.

Definition

If $E \cap F = \emptyset$, we call events E and F *mutually exclusive*.
If events $E_1, E_2, \ldots$ satisfy $E_i \cap E_j = \emptyset$ whenever $i \neq j$, we call them *mutually exclusive*. They cannot happen at the same time.

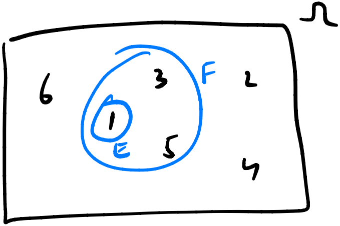
Inclusion and implication

Remark

If the event E is a *subset* of the event F, written $E \subseteq F$, then the occurrence of E *implies* that of F.

Experiment = rolling a die

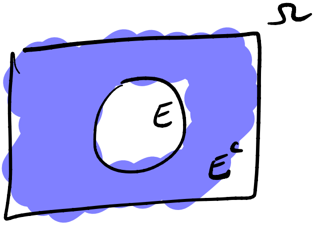
$E = \{\text{rolling } 1\} \subseteq \{\text{rolling odd nr}\} = F$



Complementarity

Definition

The *complement* of an event E is $E^c = \Omega - E$.
This is the event that E does *not* occur.



$E \cap E^c = \emptyset$ $E \cup E^c = \Omega$

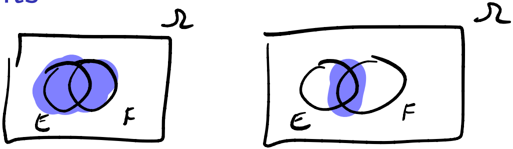
Experiments

Experiments

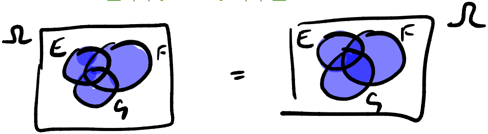
How events can interact:

- Commutativity
- Distributivity
- Associativity
- De Morgan's Law

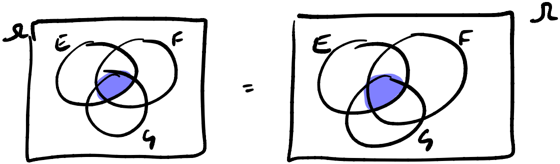
Properties of events



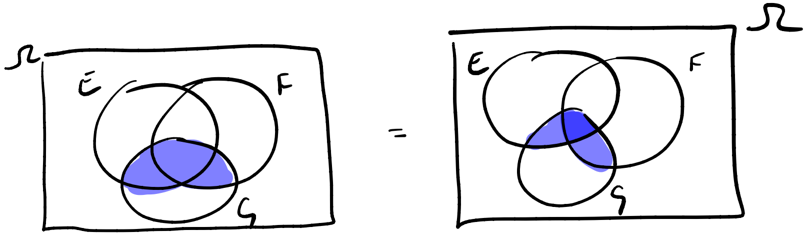
- Commutativity: $E \cup F = F \cup E$
  $E \cap F = F \cap E$



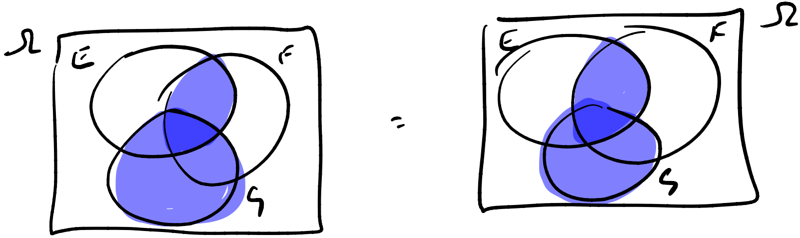
- Associativity: $E \cup (F \cup G) = (E \cup F) \cup G$
  $E \cap (F \cap G) = (E \cap F) \cap G$



Properties of events



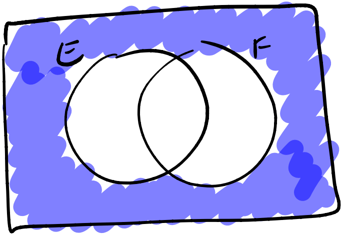
- Distributivity: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$
  $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$



De Morgan's law

E = you have umbrella
F = I have umbrella

- De Morgan's law: $(E \cup F)^c = E^c \cap F^c$ = not true that one of us has umbrella = both have no umbrella



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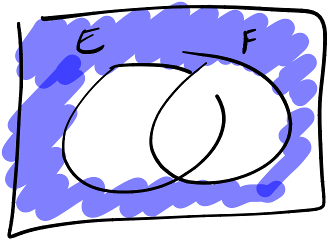
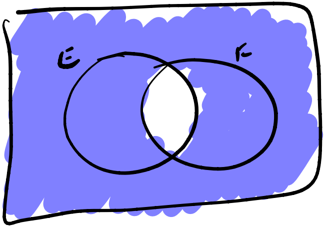


$(E \cap F)^c = E^c \cup F^c$ = not true both have umbrella = one doesn't have umbrella



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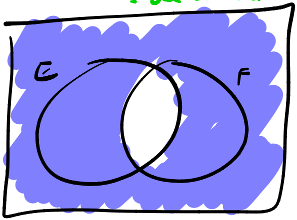


Probability

Probability

- Definition by axioms
- How to compute probabilities
- Inclusion-exclusion principle
- Equally likely outcomes

Axioms of probability

Definition

The *probability* $\mathbf{P}$ on a sample space Ω assigns *numbers* to *events* of Ω in such a way that:

1. the probability of any event is non-negative: $\mathbf{P}(E) \geq 0$;
2. the probability of the sample space is one: $\mathbf{P}(\Omega) = 1$;
3. for countably many *mutually exclusive* events $E_1, E_2, \ldots$:

$$\mathbf{P}\Big(\bigcup_i E_i\Big) = \sum_i \mathbf{P}(E_i)$$

How to compute probabilities

Proposition

For any event, $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$.

Proof: E and E^c are mutually exclusive, $E \cap E^c = \emptyset$

so by axiom 3: $P(E) + P(E^c) = P(\Omega)$

by axiom 2: $= 1$

Corollary

We have $\mathbf{P}(\emptyset) = \mathbf{P}(\Omega^c) = 1 - \mathbf{P}(\Omega) = 1 - 1 = 0$.

For any event, $\mathbf{P}(E) = 1 - \mathbf{P}(E^c) \leq 1$.

How to compute probabilities

Proposition

For any two events, $\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$.

Proposition (*Boole's inequality*)

For any events $E_1, E_2, \ldots, E_n$:

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mathbf{P}(E_i).$$

Proof by induction on n:

base case n=2: $P(E \cup F) = P(E) + P(F) - P(E \cap F) \leq P(E) + P(F)$

induction step: $P(\bigcup_{i=1}^{n+1} E_i) = P((\bigcup_{i=1}^{n} E_i) \cup E_{n+1})$

$= P(\bigcup_{i=1}^{n} E_i) + P(E_{n+1}) - P(\cdots \cap \cdots)$

$\leq \sum_{i=1}^{n+1} P(E_i)$

Inclusion-exclusion

Proposition

For any events:

$$\begin{aligned}\mathbf{P}(E \cup F \cup G) = {} & \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\ & - \mathbf{P}(E \cap F) - \mathbf{P}(E \cap G) - \mathbf{P}(F \cap G) \\ & + \mathbf{P}(E \cap F \cap G).\end{aligned}$$

Inclusion-exclusion

Proposition

For any events:

$$\begin{aligned}\mathbf{P}(E \cup F \cup G) = \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\ - \mathbf{P}(E \cap F) - \mathbf{P}(E \cap G) - \mathbf{P}(F \cap G) \\ + \mathbf{P}(E \cap F \cap G).\end{aligned}$$

$$\begin{aligned}\mathbf{P}(E_1 \cup E_2 \cup \cdots \cup E_n) &= \sum_{1 \le i \le n} \mathbf{P}(E_i) \\ &- \sum_{1 \le i_1 < i_2 \le n} \mathbf{P}(E_{i_1} \cap E_{i_2}) \\ &+ \sum_{1 \le i_1 < i_2 < i_3 \le n} \mathbf{P}(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \\ &- \cdots \\ &+ (-1)^{n+1} \mathbf{P}(E_1 \cap E_2 \cap \cdots \cap E_n).\end{aligned}$$

Example

Example

In a sports club,

36 members play tennis, 22 play tennis and squash,
28 play squash, 12 play tennis and badminton,
18 play badminton, 9 play squash and badminton,
4 play tennis, squash and badminton.

How many play at least one of these games?

$$P(T \cup S \cup B) = P(T) + P(S) + P(B) - P(T \cap S) - P(T \cap B) - P(S \cap B) + P(T \cap S \cap B)$$

$$= \frac{36}{N} + \frac{28}{N} + \frac{18}{N} - \frac{22}{N} - \frac{12}{N} - \frac{9}{N} + \frac{4}{N} = \frac{43}{N}$$

How to compute probabilities

Proposition

If $E \subseteq F$*, then* $\mathbf{P}(F - E) = \mathbf{P}(F) - \mathbf{P}(E)$.

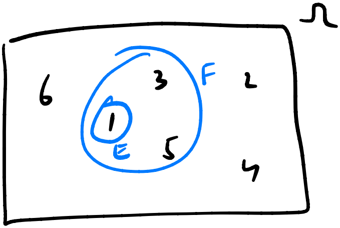
Corollary

If $E \subseteq F$*, then* $\mathbf{P}(E) \leq \mathbf{P}(F)$.

Experiment = rolling a die

$E = \{\text{rolling } 1\} \subseteq \{\text{rolling odd no.}\} = F$

$\frac{1}{6} = P(E) \leq P(F) = \frac{1}{2}$



Equally likely outcomes

The return of counting

Finite sample space, $|\Omega| = N < \infty$, has special important case where each experiment outcome has *equal probability*:

$$\mathbf{P}(\omega) = \frac{1}{N} \qquad \text{for all } \omega \in \Omega$$

Definition

Outcomes $\omega \in \Omega$ are also called *elementary events*.

If $|E| = k$ then $P(E) = \frac{|E|}{|\Omega|} = \frac{k}{N}$

Example

Example

Rolling two dice, what is the probability that the sum of the numbers shown is 7?

What's wrong with this solution? "The number 7 is one out of the possible values $2, 3, \ldots, 12$ for the sum, and the answer is $\frac{1}{11}$."

Sums are not equally likely: 12 is only 1 case out of 36.

$\Omega = \{(i,j) \mid i,j \in \{1,2,\ldots,6\}\}$

E = sum is 7 = $\{(i,j) \mid i+j=7\} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$P(E) = \frac{6}{36} = \frac{1}{6}$

Summary

- Counting: permutations, combinations, repetitions
- Events: sample space, union, intersection, complement
- Experiments: distributivity, De Morgan's law
- Probability: axioms, how to compute, equally likely outcomes