Topics

- Counting: thinking algorithmically
- Events: what could happen in principle
- Experiments: how can events interact
- Probability: quantifying what could happen
Counting
Counting

Basic principles of combinatorics:

- if an experiment has $n$ outcomes;
  and another experiment has $m$ outcomes,
- then the two experiments jointly have $n \cdot m$ outcomes.
Permutations

**Definition**

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of $n$ different objects. The *permutations* of $H$ are the different orders in which you can write all of its elements.

**Example:**

Horse race $\{A, B, C, D, E\}$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ permutations

E.g. $(E, B, A, D, C)$

- Gold
- Silver
- Bronze

How many? $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$

$0! = 1$
Permutations with repetitions

**Definition**

Let \( H = \{h_1 \ldots h_1, h_2 \ldots h_2, \ldots, h_r \ldots h_r\} \) be a set of \( r \) different types of repeated objects: \( n_1 \) many of \( h_1 \), \( n_2 \) of \( h_2 \), \ldots \( n_r \) of \( h_r \). The permutations with repetitions of \( H \) are the different orders in which you can write all of its elements.

How many? \( n = n_1 + n_2 + \cdots + n_r \) total \( n \) objects

\[
\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! \cdot n_2! \cdots n_r!}
\]

"multinomial coefficient"

Example: words with letters A,B,R,A,C,A,D,A,B,R,A

\[
(5, 2, 1, 1, 2) = 83,680
\]
**k-Permutations**

**Definition**

Let \( H = \{ h_1, h_2, \ldots, h_n \} \) be a set of \( n \) different objects. The \( k \)-permutations of \( H \) are the different ways in which one can pick and write \( k \) of its elements of \( H \) in order.

\[
P_{n,k} = \frac{n!}{(n-k)!}
\]

**Example:** top 3 in horse race \{A, B, C, D, E\}

- e.g. \( (E, B, A) \)

\[
\frac{5!}{(5-3)!} = 20 \text{ possibilities}
\]
$k$-Permutations with repetitions

**Definition**

Let $H = \{h_1, \ldots, h_2, \ldots, \ldots, h_r, \ldots\}$ be a set of $r$ different types of repeated objects, each of infinite supply. The $k$-permutations with repetitions of $H$ are the different orders in which one can write an ordered sequence of length $k$ using the elements of $H$.

Example:

- # 3-letter words of letters of English alphabet = $26^3 = 17576$
- # bit strings of length $k$ = $2^k$
**k-Combinations**

**Definition**

Let \( H = \{h_1, h_2, \ldots, h_n\} \) be a set of \( n \) different objects. The \( k \)-combinations of \( H \) are the different ways in which one can pick \( k \) of its elements **without order**.

**How many?**

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

"binomial coefficient"  \( \binom{n}{k} = \frac{P(n,k)}{k!} \)

**Example:**  # ways to form committee of 5 students from class of 30  
\[
= \binom{30}{5} = \frac{30!}{5!25!} = 142506
\]
Events
Events

A mathematical model for experiments:

- *Sample space*: the set $\Omega$ of all possible outcomes
- An *event* is a collection of possible outcomes: $E \subseteq \Omega$
- *Union* $E \cup F$ and *intersection* $E \cap F$ of events make sense

\[\begin{align*}
\Omega & \quad \Omega \\
E & \quad F \\
E \cup F & \\
E \cap F & \\
\end{align*}\]

\[\begin{align*}
\Omega & \quad \Omega \\
E & \quad F \\
E \cap F & \\
E \cup F & \\
\end{align*}\]

\[1\]Sometimes $\Omega$ is too large, and not all subsets are events. Ignore this now.
### Examples

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Sample space</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>Will it rain today?</td>
<td>(\mathcal{S} = {r, n})</td>
<td>e.g. (E = {r})</td>
</tr>
<tr>
<td>5-horse race</td>
<td>(\mathcal{S}) = permutations of ({A, B, C, D, E})</td>
<td>e.g. (E = {B \text{ wins}})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(F = {E \text{ wins, A third}})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>notice (E \cap F = \emptyset) never happens</td>
</tr>
<tr>
<td>Flip 2 coins</td>
<td>(\mathcal{S} = {(H,H), (H,T), (T,H), (T,T)})</td>
<td>e.g. (E = \text{ coins different})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(= {(H,T), (T,H)})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(F = \text{ at least one heads})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(= {(H,H), (H,T), (T,H)})</td>
</tr>
<tr>
<td>roll die until get 6</td>
<td>(\mathcal{S} = {\text{sequences of 1-5 and roll 6}})</td>
<td>e.g. (E = \text{first 4, then 6 on 3rd roll})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(= {(4,6,6) \mid k \in {1, \ldots, 3}})</td>
</tr>
</tbody>
</table>
## Union

*Union* $E \cup F$ of events $E$ and $F$ means $E$ and $F$.

Infinite union $\bigcup_i E_i$ of events $E_i$ means at least one of the $E_i$’s.

## Intersection

*Intersection* $E \cap F$ of events $E$ and $F$ means $E$ and $F$.

Infinite intersection $\bigcap_i E_i$ of events $E_i$ means each of the $E_i$’s.

## Definition

If $E \cap F = \emptyset$, we call events $E$ and $F$ **mutually exclusive**.

If events $E_1, E_2, \ldots$ satisfy $E_i \cap E_j = \emptyset$ whenever $i \neq j$, we call them **mutually exclusive**. They cannot happen at the same time.
Inclusion and implication

**Remark**

If the event $E$ is a *subset* of the event $F$, written $E \subseteq F$, then the occurrence of $E$ *implies* that of $F$.

**Experiment** = rolling a die

$E = \{\text{rolling } 1 \} \subseteq \{\text{rolling odd num} \} = F$
**Complementarity**

**Definition**

The *complement* of an event $E$ is $E^c = \Omega - E$. This is the event that $E$ does *not* occur.

\[
\Omega = E \cup E^c
\]

\[
E \cap E^c = \emptyset
\]
Experiments
Experiments

How events can interact:

- Commutativity
- Distributivity
- Associativity
- De Morgan’s Law
Properties of events

- **Commutativity:** $E \cup F = F \cup E$
  $E \cap F = F \cap E$

- **Associativity:**
  $E \cup (F \cup G) = (E \cup F) \cup G$
  $E \cap (F \cap G) = (E \cap F) \cap G$
Properties of events

- Distributivity: 
  \[(E \cup F) \cap G = (E \cap G) \cup (F \cap G)\]
  
  
  
  
- \[(E \cap F) \cup G = (E \cup G) \cap (F \cup G)\]
De Morgan’s law

- De Morgan’s law: $(E \cup F)^c = E^c \cap F^c = \text{not true that one of us has umbrella} \neq \text{both have umbrella}$

- $(E \cap F)^c = E^c \cup F^c = \text{not true that both have umbrella} \neq \text{one does not have umbrella}$
Probability
Probability

- Definition by axioms
- How to compute probabilities
- Inclusion-exclusion principle
- Equally likely outcomes
Axioms of probability

Definition

The probability $P$ on a sample space $\Omega$ assigns numbers to events of $\Omega$ in such a way that:

1. the probability of any event is non-negative: $P(E) \geq 0$;
2. the probability of the sample space is one: $P(\Omega) = 1$;
3. for countably many mutually exclusive events $E_1, E_2, \ldots$:

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$$
How to compute probabilities

**Proposition**

For any event, $P(E^c) = 1 - P(E)$.

**Proof:** $E$ and $E^c$ are mutually exclusive, $E \cap E^c = \emptyset$

so by axiom 3: $P(E) + P(E^c) = P(\Omega)$

by axiom 2: $1 = 1$.

**Corollary**

We have $P(\emptyset) = P(\Omega^c) = 1 - P(\Omega) = 1 - 1 = 0$.

For any event, $P(E) = 1 - P(E^c) \leq 1$. 
How to compute probabilities

**Proposition**

For any two events, \( P(E \cup F) = P(E) + P(F) - P(E \cap F). \)

**Proposition (Boole’s inequality)**

For any events \( E_1, E_2, \ldots, E_n: \)

\[
P \left( \bigcup_{i=1}^{n} E_i \right) \leq \sum_{i=1}^{n} P(E_i).
\]

**Proof by induction on \( n: \)**

- **Base case \( n = 2: \)** \( P(E \cup F) = P(E) + P(F) - P(E \cap F) \leq P(E) + P(F) \)
- **Induction step: \( P(\bigcup_{i=1}^{n} E_i) = P(\bigcup_{i=1}^{n-1} E_i) + P(E_n) \)
  \[= P(\bigcup_{i=1}^{n-1} E_i) + P(E_n) - P(\bigcup_{i=1}^{n-1} E_i \cap E_n) \]
  \[\leq \sum_{i=1}^{n-1} P(E_i) + P(E_n) - \sum_{i=1}^{n-1} P(E_i) \]
  \[= \sum_{i=1}^{n} P(E_i) \]
Inclusion-exclusion

Proposition

For any events:

\[ P(E \cup F \cup G) = P(E) + P(F) + P(G) \]
\[ - P(E \cap F) - P(E \cap G) - P(F \cap G) \]
\[ + P(E \cap F \cap G). \]
Inclusion-exclusion

**Proposition**

For any events:

\[ P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G). \]

\[ P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{1 \leq i \leq n} P(E_i) \]

\[ - \sum_{1 \leq i_1 < i_2 \leq n} P(E_{i_1} \cap E_{i_2}) \]

\[ + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \]

\[ - \cdots \]

\[ + (-1)^{n+1} P(E_1 \cap E_2 \cap \cdots \cap E_n). \]
In a sports club,

- 36 members play tennis,
- 22 play tennis and squash,
- 28 play squash,
- 12 play tennis and badminton,
- 18 play badminton,
- 9 play squash and badminton,
- 4 play tennis, squash and badminton.

How many play at least one of these games?

\[
P(T \cup S \cup B) = P(T) + P(S) + P(B)
- P(T \cap S) - P(T \cap B) - P(S \cap B)
+ P(T \cap S \cap B)
= \frac{36}{N} + \frac{20}{N} + \frac{18}{N} - \frac{12}{N} - \frac{9}{N} + \frac{4}{N} = \frac{48}{N}
\]
How to compute probabilities

Proposition
If $E \subseteq F$, then $P(F - E) = P(F) - P(E)$.

Corollary
If $E \subseteq F$, then $P(E) \leq P(F)$.
Equally likely outcomes
The return of counting

Finite sample space, $|\Omega| = N < \infty$, has special important case where each experiment outcome has equal probability:

$$P(\omega) = \frac{1}{N} \quad \text{for all } \omega \in \Omega$$

**Definition**

Outcomes $\omega \in \Omega$ are also called *elementary events*.

$$\text{If } |E| = k \quad \text{then} \quad P(E) = \frac{|E|}{|\Omega|} \cdot \frac{k}{N}$$
Example

Rolling two dice, what is the probability that the sum of the numbers shown is 7?

What’s wrong with this solution? “The number 7 is one out of the possible values 2, 3, . . . , 12 for the sum, and the answer is \( \frac{1}{11} \).”

Sums are not equally likely: 12 is only 1 out of 36.

\[
\begin{align*}
S &= \{(i,j) \mid i,j \in \{1,2,\ldots,6\}\} \\
E &= \{(i,j) \mid i+j=7\} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \\
P(E) &= \frac{6}{36} = \frac{1}{6}
\end{align*}
\]
Summary

- Counting: permutations, combinations, repetitions
- Events: sample space, union, intersection, complement
- Experiments: distributivity, De Morgan’s law
- Probability: axioms, how to compute, equally likely outcomes