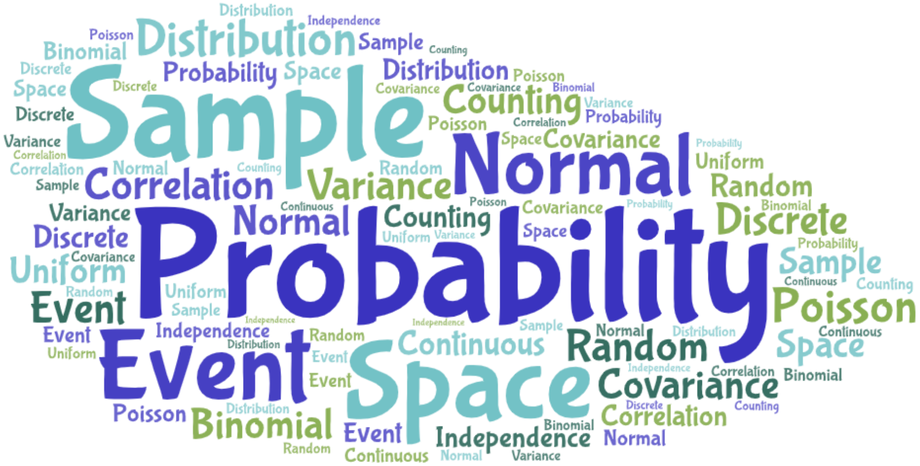
Discrete Mathematics and Probability

Week 7



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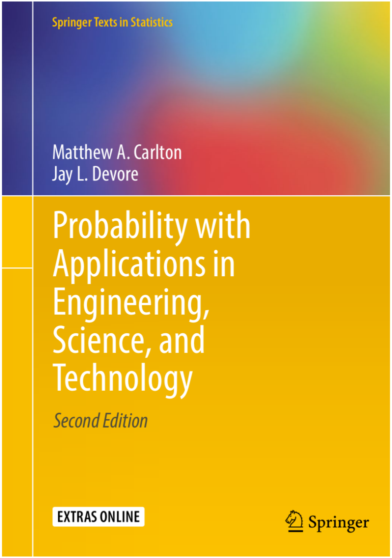
Springer Texts in Statistics
Matthew A. Carlton
Jay L. Devore
Probability with Applications in Engineering, Science, and Technology
Second Edition
EXTRAS ONLINE
Springer

Topics

- Counting: thinking algorithmically
- Events: what could happen in principle
- Experiments: how can events interact
- Probability: quantifying what could happen

Counting

Counting

Basic principles of combinatorics:

- if an experiment has n outcomes;
  and another experiment has m outcomes,
- then the two experiments jointly have $n \cdot m$ outcomes.

Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *permutations* of H are the different orders in which you can write all of its elements.

Permutations with repetitions

Definition

Let $H = \{h_1 \dots h_1,\ h_2 \dots h_2,\ \dots,\ h_r \dots h_r\}$ be a set of r different types of repeated objects: n_1 many of h_1, n_2 of h_2, $\dots n_r$ of h_r. The *permutations with repetitions* of H are the different orders in which you can write all of its elements.

k-Permutations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-permutations* of H are the different ways in which one can pick and write k of its elements of H in order.

k-Permutations with repetitions

Definition

Let $H = \{h_1 \ldots, h_2 \ldots, \ldots, h_r \ldots\}$ be a set of r different types of repeated objects, each of infinite supply. The *k-permutations with repetitions* of H are the different orders in which one can write an ordered sequence of length k using the elements of H.

k-Combinations

Definition

Let $H = \{h_1, h_2, \ldots, h_n\}$ be a set of n different objects. The *k-combinations* of H are the different ways in which one can pick k of its elements without order.

Events

Events

A mathematical model for experiments:

- *Sample space*: the set Ω of all possible outcomes
- An *event* is a collection[1] of possible outcomes: $E \subseteq \Omega$
- *Union* $E \cup F$ and *intersection* $E \cap F$ of events make sense

[1] Sometimes Ω is too large, and not all subsets are events. Ignore this now.

Examples

Union and intersection

Union

Union $E \cup F$ of events E and F means E and F.
Infinite union $\bigcup_i E_i$ of events E_i means at least one of the E_i's.

Intersection

Intersection $E \cap F$ of events E and F means E and F.
Infinite intersection $\bigcap_i E_i$ of events E_i means each of the E_i's.

Definition

If $E \cap F = \emptyset$, we call events E and F *mutually exclusive*.
If events $E_1, E_2, \ldots$ satisfy $E_i \cap E_j = \emptyset$ whenever $i \neq j$, we call them *mutually exclusive*. They cannot happen at the same time.

Inclusion and implication

Remark

If the event E is a *subset* of the event F, written $E \subseteq F$, then the occurrence of E *implies* that of F.

Complementarity

Definition

The *complement* of an event E is $E^c = \Omega - E$.
This is the event that E does *not* occur.

Experiments

Experiments

How events can interact:

- Commutativity
- Distributivity
- Associativity
- De Morgan's Law

Properties of events

- Commutativity: $E \cup F = F \cup E$
  $E \cap F = F \cap E$

- Associativity: $E \cup (F \cup G) = (E \cup F) \cup G$
  $E \cap (F \cap G) = (E \cap F) \cap G$

Properties of events

- Distributivity: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$
  $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$

De Morgan's law

- De Morgan's law: $(E \cup F)^c = E^c \cap F^c$

Probability

- Definition by axioms
- How to compute probabilities
- Inclusion-exclusion principle
- Equally likely outcomes

Axioms of probability

Definition

The *probability* $\mathbf{P}$ on a sample space Ω assigns *numbers* to *events* of Ω in such a way that:

1. the probability of any event is non-negative: $\mathbf{P}(E) \geq 0$;
2. the probability of the sample space is one: $\mathbf{P}(\Omega) = 1$;
3. for countably many *mutually exclusive* events $E_1, E_2, \ldots$:

$$\mathbf{P}\Big(\bigcup_i E_i\Big) = \sum_i \mathbf{P}(E_i)$$

How to compute probabilities

Proposition

For any event, $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$.

Corollary

We have $\mathbf{P}(\emptyset) = \mathbf{P}(\Omega^c) = 1 - \mathbf{P}(\Omega) = 1 - 1 = 0$.
For any event, $\mathbf{P}(E) = 1 - \mathbf{P}(E^c) \leq 1$.

How to compute probabilities

Proposition

For any two events, $\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$.

Proposition (*Boole's inequality*)

For any events $E_1, E_2, \ldots, E_n$*:*

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mathbf{P}(E_i).$$

Inclusion-exclusion

Proposition

For any events:

$$\mathbf{P}(E \cup F \cup G) = \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) - \mathbf{P}(E \cap F) - \mathbf{P}(E \cap G) - \mathbf{P}(F \cap G) + \mathbf{P}(E \cap F \cap G).$$

Inclusion-exclusion

Proposition

For any events:

$$\begin{aligned}\mathbf{P}(E \cup F \cup G) = \mathbf{P}(E) + \mathbf{P}(F) + \mathbf{P}(G) \\ - \mathbf{P}(E \cap F) - \mathbf{P}(E \cap G) - \mathbf{P}(F \cap G) \\ + \mathbf{P}(E \cap F \cap G).\end{aligned}$$

$$\begin{aligned}\mathbf{P}(E_1 \cup E_2 \cup \cdots \cup E_n) = &\sum_{1 \le i \le n} \mathbf{P}(E_i) \\ &- \sum_{1 \le i_1 < i_2 \le n} \mathbf{P}(E_{i_1} \cap E_{i_2}) \\ &+ \sum_{1 \le i_1 < i_2 < i_3 \le n} \mathbf{P}(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \\ &- \cdots \\ &+ (-1)^{n+1} \mathbf{P}(E_1 \cap E_2 \cap \cdots \cap E_n).\end{aligned}$$

Example

Example

In a sports club,

36 members play tennis,	22 play tennis and squash,
28 play squash,	12 play tennis and badminton,
18 play badminton,	9 play squash and badminton,

4 play tennis, squash and badminton.

How many play at least one of these games?

How to compute probabilities

Proposition

If $E \subseteq F$, *then* $\mathbf{P}(F - E) = \mathbf{P}(F) - \mathbf{P}(E)$.

Corollary

If $E \subseteq F$, *then* $\mathbf{P}(E) \leq \mathbf{P}(F)$.

Equally likely outcomes

The return of counting

Finite sample space, $|\Omega| = N < \infty$, has special important case where each experiment outcome has *equal probability*:

$$\mathbf{P}(\omega) = \frac{1}{N} \qquad \text{for all } \omega \in \Omega$$

Definition

Outcomes $\omega \in \Omega$ are also called *elementary events*.

Example

Example

Rolling two dice, what is the probability that the sum of the numbers shown is 7?

What's wrong with this solution? "The number 7 is one out of the possible values $2, 3, \ldots, 12$ for the sum, and the answer is $\frac{1}{11}$."

Summary

- Counting: permutations, combinations, repetitions
- Events: sample space, union, intersection, complement
- Experiments: distributivity, De Morgan's law
- Probability: axioms, how to compute, equally likely outcomes