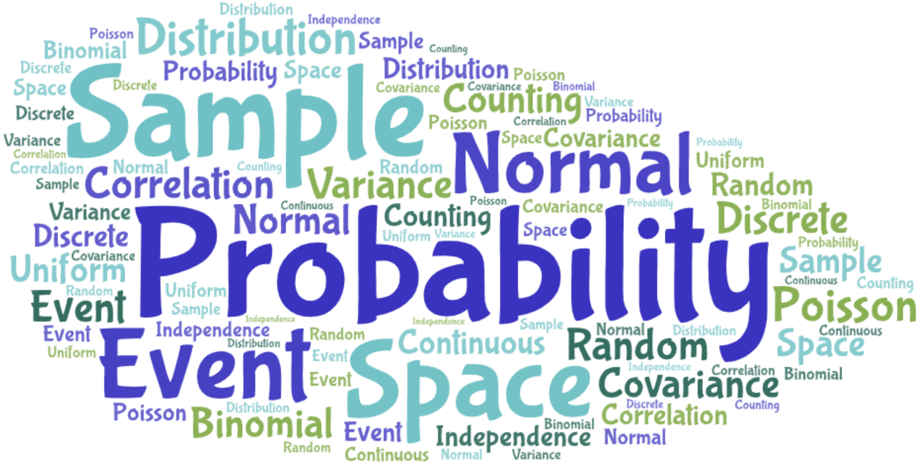
Discrete Mathematics and Probability

Week 7



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Topics

- Recap: examples with equally likely outcomes
- Conditional probability: how knowledge influences probability
- Bayes' theorem: link probabilities of related events

Recap

Permutations and combinations

Example

An urn contains 6 red balls and 5 blue balls.
Draw three balls at random, at once (that is, without replacement).
What is the chance of drawing one red and two blue balls?

$\Omega = \{$ordered choices of 3 out of 11$\} = \{$3-permutations of 11$\}$

$|\Omega| = \frac{11!}{(11-3)!} = 11 \cdot 10 \cdot 9$

$E = \{RBB, BRB, BBR\}$

$6 \cdot 5 \cdot 4 \qquad 5 \cdot 6 \cdot 4 \qquad 5 \cdot 4 \cdot 6$ ways

$|E| = 3 \cdot 4 \cdot 5 \cdot 6$

$P(E) = \frac{|E|}{|\Omega|} = \frac{3 \cdot 4 \cdot 5 \cdot 6}{9 \cdot 10 \cdot 11} = \frac{4}{11}$

$\Omega = \{$ <u>unordered</u> choices of 3 of 11 $\} = \{$ 3-combination of 11 $\}$

$|\Omega| = \binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{6}$

$E = \{$ pick 1 of 6 red balls, 2 of 5 blue balls $\}$

$|E| = \binom{6}{1} \cdot \binom{5}{2} = 6 \cdot 10$

$P(E) = \frac{|E|}{|\Omega|} = \frac{6 \cdot 10}{11 \cdot 10 \cdot 9 / 6} = \frac{4}{11}$

Both fine. Sometimes ordered easier.

Sometimes unordered. Never mix the two!

Permutations and combinations

Example

An urn contains n balls, one of which is red, all others are black. We draw k balls at random (without replacement). What is the chance that the red ball will be drawn?

$\Omega = \{ k\text{-combinations of } n \}$

$|\Omega| = \binom{n}{k}$

$E = \{ \text{pick red ball and } k-1 \text{ others} \}$

$|E| = \binom{1}{1} \cdot \binom{n-1}{k-1}$

$$P(E) = \frac{|E|}{|\Omega|} = \frac{\binom{1}{1} \cdot \binom{n-1}{k-1}}{\binom{n}{k}} = \frac{(n-1)! \; k! \; (n-k)!}{(k-1)! \; (n-1-k+1)! \; n!} = \frac{k}{n}$$

Draws in order:

$E_i = \{ i^{th} \text{ draw is red} \}$

$P(E_i) = \frac{1}{n}$

$\{E_i\}$ mutually exclusive.

$E = \bigcup_i E_i$

$P(E) = P(\bigcup_i E_i) = \sum_i P(E_i) = \sum_{i=1}^{k} \frac{1}{n} = \frac{k}{}$

Equally likely events

Example

Out of n people, what is the probability that no two share a birthday?

0 if $n > 365$ (pigeonhole principle)

$\Omega = \{$ choice of day for each of n people $\}$

$|\Omega| = 365^n$

$E = \{$ no coinciding birthdays $\}$

$|E| = 365 \cdot 364 \cdots (365-n+1) = \frac{365!}{(365-n)!}$

$P(E) = \frac{|E|}{|\Omega|} = \frac{365!}{(365-n)!\, 365^n}$

n	$P(E)$
10	88%
20	59%
30	29%
40	11%
50	3%
100	0.00003%

Inclusion-exclusion

Example

Flipping two fair coins, what is the probability of at least one coming up heads?

$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$

$E = \{(H,H), (H,T), (T,H)\}$

$P(E) = \frac{3}{4}$

or :

$E^c = \{\text{no heads}\} = \{(T,T)\}$

$P(E) = 1 - P(E^c) = 1 - \frac{1}{4} = 3/4$

or :

$F = \{\text{first coin } H\} = \{(H,H), (H,T)\}$

$G = \{\text{2nd coin } H\} = \{(H,H), (T,H)\}$

$P(F \cup G) = P(F) + P(G) - P(F \cap G) = \frac{2}{4} + \frac{2}{4} - \frac{1}{4} = \frac{3}{4}$

Conditional probability

Conditional probability

Often you have *partial information* about the outcome of an experiment. This alters the likelihoods for various outcomes.

Example

Roll two dice. What is the probability that the sum of the numbers is 8? What if we know that the first die shows a 5?

$\Omega = \{1, \ldots, 6\}^2$

$E = \{(2,6), (3,5), \ldots, (6,2)\}$

$P(E) = \frac{5}{36}$

but if first die is 5:

$\Omega = \{1, \ldots, 6\}$

$F = \{3\}$

$P(F) = \frac{1}{6}$

partial information can change probability!

Reduced sample space

We reduced our world to the event we were given:
$F = \{\text{first die shows } 5\} = \{(5, 1), (5, 2), \ldots, (5, 6)\}$

Definition

The event that is given to us is called a *reduced sample space*. We can simply work in this set to figure out the conditional probabilities given this event.

The event F has 6 equally likely outcomes. Only one of them, $(5, 3)$, provides a sum of 8. Hence the conditional probability is $\frac{1}{6}$.

Definition of conditional probability

The question can be reformulated.

$$E = \{\text{the sum is 8}\} = \{(2, 6), (3, 5), \ldots, (6, 2)\}$$

"In what proportion of cases in F will E also occur?"
"How does probability of 'E and F' compare to probability of F?"

Definition

Let F be an event with $\mathbf{P}(F) > 0$.
The *conditional probability of E given F* is:

$$\mathbf{P}(E \mid F) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)}$$

$E \cap F = \{(5,3)\}$

$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/36}{1/6} = \frac{1}{6}$

Axioms

Proposition

Conditional probability $\mathbf{P}(\cdot \,|\, F)$ *satisfies the axioms of probability:*

1. *conditional probability is non-negative:* $\mathbf{P}(E \,|\,) \geq 0$;
2. *conditional probability of sample space is one:* $\mathbf{P}(\Omega \,|\, F) = 1$;
3. *for countably many mutually exclusive events* $E_1, E_2, \ldots$:

$$\mathbf{P}\left(\bigcup_i E_i \,\middle|\, F\right) = \sum_i \mathbf{P}(E_i \,|\, F)$$

How to compute conditional probabilities

Corollary

- $\mathbf{P}(E^c \mid F) = 1 - \mathbf{P}(E \mid F)$
- $\mathbf{P}(\emptyset \mid F) = 0$
- $\mathbf{P}(E \mid F\} = 1 - \mathbf{P}(E^c \mid F) \leq 1$
- $\mathbf{P}(E \cup G \mid F) = \mathbf{P}(E \mid F) + \mathbf{P}(G \mid F) - \mathbf{P}(E \cap G \mid F)$
- *If* $E \subseteq G$, *then* $\mathbf{P}(G - E \mid F) = \mathbf{P}(G \mid F) - \mathbf{P}(E \mid F)$
- *If* $E \subseteq G$, *then* $\mathbf{P}(E \mid F) \leq \mathbf{P}(G \mid F)$

BUT: Don't change the condition!
$\mathbf{P}\{E \mid F\}$ and $\mathbf{P}\{E \mid F^c\}$ have nothing to do with each other.

Multiplication rule

Proposition (*Multiplication rule*)

$$\mathbf{P}(E_1 \cap \cdots \cap E_n) = \mathbf{P}(E_1) \cdot \mathbf{P}(E_2 \mid E_1) \cdot \mathbf{P}(E_3 \mid E_1 \cap E_2) \cdots \mathbf{P}(E_n \mid E_1 \cap \cdots \cap E_{n-1})$$

$$= P(\cancel{E_1}) \cdot \frac{P(E_2 \cap \cancel{E_1})}{\cancel{P(E_1)}} \cdot \frac{P(E_3 \cap \cancel{E_2 \cap E_1})}{P(\cancel{E_2 \cap E_1})} \cdots \frac{P(E_n \cap \cdots \cap E_1)}{P(\cancel{E_{n-1} \cap \cdots} \cap E_1)}$$

Example again

Example

An urn contains 6 red and 5 blue balls. We draw three balls at random, at once (that is, without replacement). What is the chance of drawing one red and two blue balls?

$$P(R_1 \cap B_2 \cap B_3) + P(B_1 \cap R_2 \cap B_3) + P(B_1 \cap B_2 \cap R_3)$$

$$= P(R_1) \cdot P(B_2 \mid R_1) \cdot P(B_3 \mid R_1 \cap B_2)$$
$$+ P(B_1) \cdot P(R_2 \mid B_1) \cdot P(B_3 \mid R_2 \cap B_1)$$
$$+ P(B_1) \cdot P(B_2 \mid B_1) \cdot P(R_3 \mid B_1 \cap B_2)$$

$$= \frac{6}{11} \cdot \frac{5}{10} \cdot \frac{4}{9} + \frac{5}{11} \cdot \frac{6}{10} \cdot \frac{4}{9} + \frac{5}{11} \cdot \frac{4}{10} \cdot \frac{6}{9} = \frac{4}{11}$$

Bayes' theorem

Bayes' Theorem

The aim is to say something about $\mathbf{P}(F \mid E)$, once we know $\mathbf{P}(E \mid F)$ (and other things. . .). This will be very useful, and serve as a fundamental tool in probability and statistics.

The Law of Total Probability

Theorem (*Partition Theorem*)

$$\mathbf{P}(E) = \mathbf{P}(E \mid F) \cdot \mathbf{P}(F) + \mathbf{P}(E \mid F^c) \cdot \mathbf{P}(F^c)$$

$$= \frac{P(E \cap F)}{\cancel{P(F)}} \cdot \cancel{P(F)} + \frac{P(E \cap F^c)}{\cancel{P(F^c)}} \cdot \cancel{P(F^c)}$$

$$= P((E \cap F) \cup (E \cap F^c))$$

$$= P(E \cap (F \cup F^c))$$

$$= P(E)$$

The Law of Total Probability

Theorem (*Partition Theorem*)

$$\mathbf{P}(E) = \mathbf{P}(E \mid F) \cdot \mathbf{P}(F) + \mathbf{P}(E \mid F^c) \cdot \mathbf{P}(F^c)$$

Definition

Countably many events $F_1, F_2, \ldots$ form a *partition* of Ω if $F_i \cap F_j = \emptyset$ and $\bigcup_i F_i = \Omega$.

Theorem (*Partition Theorem*)

For any event E and any partition $F_1, F_2, \ldots$:

$$\mathbf{P}(E) = \sum_i \mathbf{P}(E \mid F_i) \cdot \mathbf{P}(F_i)$$

Example

Example

According to an insurance company:

- 30% of population are *accident-prone*:
  they will have an accident in any given year with 0.4 chance.
- 70% of population are *careful*:
  they have an accident in any given year with 0.2 chance.

How likely is a new customer to have an accident in 2023?

$F = \{$new customer is error-prone$\}$
$A = \{$new customer has accident in 2023$\}$

$$P(A) = P(A \mid F) \cdot P(F) + P(A \mid F^c) \cdot P(F^c)$$
$$= 0.4 \cdot 0.3 + 0.2 \cdot 0.7$$
$$= 26\%$$

weighted average

Bayes' Theorem

Theorem (Bayes' Theorem)

$$\mathbf{P}\{F \mid E\} = \frac{\mathbf{P}\{E \mid F\} \cdot \mathbf{P}\{F\}}{\mathbf{P}\{E \mid F\} \cdot \mathbf{P}\{F\} + \mathbf{P}\{E \mid F^c\} \cdot \mathbf{P}\{F^c\}}$$

If $\{F_i\}_i$ *partitions* Ω, *then:*

$$\mathbf{P}\{F_i \mid E\} = \frac{\mathbf{P}\{E \mid F_i\} \cdot \mathbf{P}\{F_i\}}{\sum_j \mathbf{P}\{E \mid F_j\} \cdot \mathbf{P}\{F_j\}}$$

Pf: def of conditional + law of total prob.

Belief update

Example

Consider the insurance company again. Imagine it's now 2024. We learn that the new customer did have an accident in 2023. Now what is the chance that they are accident-prone?

$$P(F|A) = \frac{P(A|F)\cdot P(F)}{P(A|F)\cdot P(F) + P(A|F^c)\cdot P(F^c)}$$

$$= \frac{0.4\cdot 0.3}{0.4\cdot 0.3 + 0.2\cdot 0.7} = \frac{6}{13} \approx 46\%$$

$P(F) = 30\%$!

Summary

- Probability: multiple ways to compute
- Conditional probability: reduced sample space,
  multiplication rule
- Bayes' theorem: partition theorem, belief update