Discrete Mathematics and Probability

Lecture 16
Continuous Probability Distributions

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Week 9 Topics

Continuous Random Variables and Probability Distributions

Carlton & Devore: Chapter 3

- §3.1 Probability Density Functions and Cumulative Distribution Functions
- §3.2 Expected Values (but not Moment Generating Functions)
- §3.3 The Normal (Gaussian) Distribution
- §3.4 The Exponential Distribution (but not the Gamma Distribution)

The study guide and accompanying videos indicate the examinable course content. The corresponding sections in the book are to support this. Additional sections in the book are useful extension material but not required for this course.

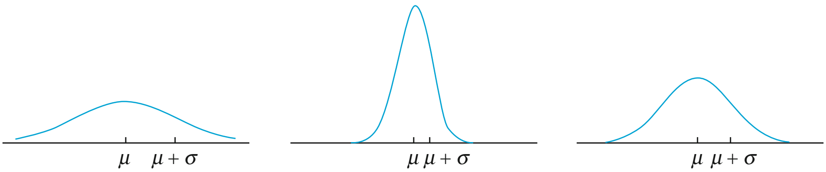
Normal (Gaussian) Distribution

Definition

A continuous random variable X has *normal distribution* (or *Gaussian distribution*) with parameters μ and σ if it has the following probability density function.

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

We write this as $X \sim N(\mu, \sigma)$.



Carlton & Devore Figure 3.13

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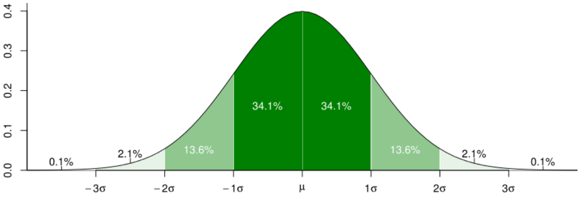


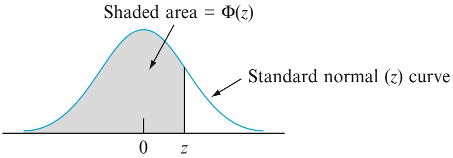
Standard Normal Distribution

Definition

The normal distribution with parameters $\mu = 0$ and $\sigma = 1$ is the *standard normal distribution* and a random variable with that distribution is a *standard normal random variable*, usually named Z and with the following probability density function.

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

The corresponding cumulative distribution function is written $\Phi(z)$.

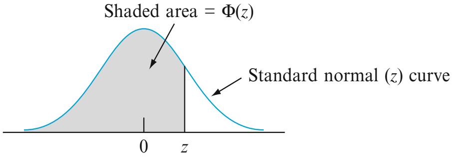


Carlton & Devore Figure 3.14

Tabulating the Standard Normal Distribution

The CDF $\Phi(z)$ for the standard normal distribution can be expressed as an integral.

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



Carlton & Devore Figure 3.14

However, this does not resolve into any convenient algebraic form and calculating values requires some method of computational approximation. Most statistical software packages provide functions to do this and it's also standard to have precomputed tables for $\Phi(z)$.

Carlton & Devore provide one at the back of the book, Appendix A.3, pp. 601–602.

Table A.3 (continued)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621

Example

Find the table in Carlton & Devore Appendix A.3, pp. 601–602.

Suppose Z is a continuous random variable with the standard normal distribution: $Z \sim N(0, 1)$. Use the table of standard normal values to calculate the following.

1. $P(Z < 0.93)$
2. $P(Z > 0.10)$

Standardizing a Normally-Distributed Random Variable

Proposition

If continuous random variable $X \sim N(\mu, \sigma)$ then random variable Z defined as

$$Z = \frac{X - \mu}{\sigma}$$

has standard normal distribution: $Z \sim N(0, 1)$.

This is then useful to calculate probabilities involving X using the standard normal CDF $\Phi(z)$.

$$P(X \leqslant a) = \Phi\left(\frac{a - \mu}{\sigma}\right) \qquad P(a \leqslant X \leqslant b) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

$$P(X \geqslant b) = 1 - \Phi\left(\frac{b - \mu}{\sigma}\right) \qquad (100p)\text{th percentile } \eta_p = \mu + \Phi^{-1}(p) \cdot \sigma$$

Standardizing a Normally-Distributed Random Variable

Proposition

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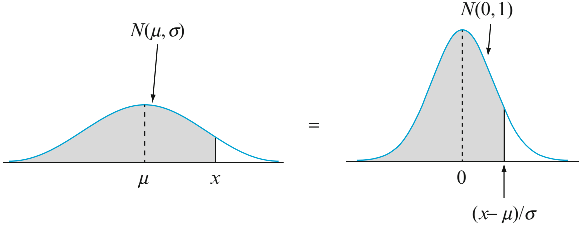


Carlton & Devore Figure 3.19

Example

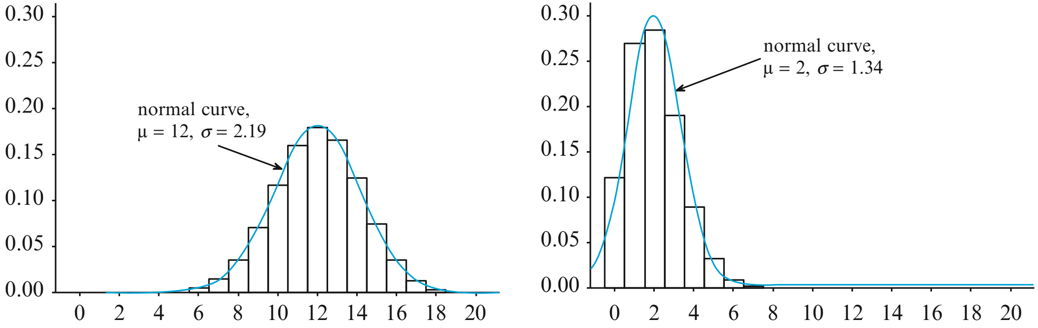
Continuous random variable X is normally distributed with mean 5 and standard deviation 2. Use the table of standard normal values to calculate the following.

(a) $P(X < 5.9)$

(b) $P(X > 6.5)$

Approximating the Binomial Distribution

Probability histograms for the binomial distributions Binom$(20, 0.6)$ and Binom$(20, 0.1)$.



Carlton & Devore Figure 3.23

The blue curves are the normal distributions with the same mean and standard deviation. For the distribution on the left the normal distribution is a good fit; for the one on the right, a poor fit.

Approximating the Binomial Distribution

Proposition

Suppose X is a binomial random variable counting successes in n trials each with probability p. If the distribution is not too skewed to left or right then this can be approximated by the normal distribution with w mean $\mu = np$ and standard deviation $\sigma = \sqrt{npq}$, where $q = (1 - p)$.

$$P(X \leqslant x) \;=\; B(x; n, p) \;\approx\; \Phi\left(\frac{x + 0.5 - np}{\sqrt{npq}}\right)$$

This approximation is adequate in practice $np \geqslant 10$ and $nq \geqslant 10$.

Example

I have an unfair six-sided die that is biased to give higher numbers: the probability of throwing a six is $1/3$. Suppose I throw it ten times and want to know the probability of throwing a six at least four times.

(a) Calculate this exactly using the binomial distribution.

(b) Calculate an approximation using the normal distribution.

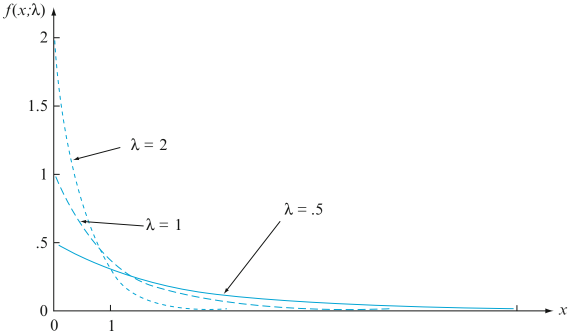
Exponential Distribution

Definition

A continuous random variable X has *exponential distribution* with parameter λ, for some $\lambda > 0$, if it has the following probability density function.

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

We write this as $X \sim \mathrm{Exp}(\lambda)$.



Carlton & Devore Figure 3.24

Exponential Distribution

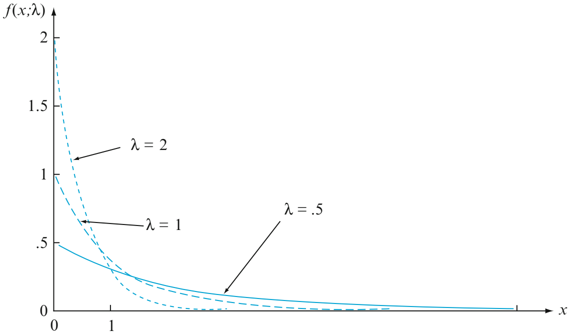
$X \sim \mathrm{Exp}(\lambda)$

PDF $$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

CDF $$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean $E(X) = 1/\lambda$

Standard deviation $SD(X) = 1/\lambda$



Carlton & Devore Figure 3.24

Example

A large database routinely experiences disk failure: thanks to storage redundancy the system can keep going, but the disks do need to be replaced. Suppose that D is the number of days between disk failures, and that this is a continuous random variable with exponential distribution $D \sim \text{Exp}(0.25)$. Calculate the following.

(a) The expected time between disk failures.

(b) $P(D \leqslant 1)$, the chance of a disk failure within one day.

(c) $P(D \leqslant 3 \mid D \geqslant 2)$, the chance a disk will fail on the third day if we have already had two clear days.

$X \sim \text{Exp}(\lambda)$

PDF $$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

CDF $$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean $E(X) = 1/\lambda$

Standard deviation $SD(X) = 1/\lambda$

Exponential Distribution is Memoryless

Proposition

The exponential distribution is *memoryless*: if $X \sim \mathrm{Exp}(\lambda)$ represents the waiting time until something happens, then as time passes the amount of time remaining always has the same distribution.

$$P(X \geqslant s + t \mid X \geqslant s) = P(X \geqslant t) \qquad \text{for all } s, t \in \mathbb{R}_{\geqslant 0}$$

$$P(X \leqslant s + t \mid X \geqslant s) = P(X \leqslant t) \qquad \text{for all } s, t \in \mathbb{R}_{\geqslant 0}$$

$$P((s + a) \leqslant X \leqslant (s + b) \mid X \geqslant s) = P(a \leqslant X \leqslant b) \qquad \text{for all } a, b, s \in \mathbb{R}_{\geqslant 0}$$

In fact the exponential distribution is the *only* continuous distribution with this property.

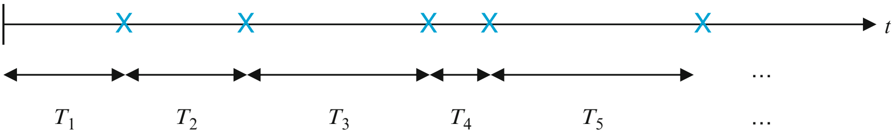
Poisson Distribution and Exponential Distribution

The memoryless property means that the exponential distribution occurs naturally in many situations where multiple things happen independently at random over time: emails arriving; calls to a customer service centre; molecules colliding in a chemical reaction.

This also links it to the discrete Poisson distribution, which arises when counting such things.

Let continuous random variable T be the time in minutes between successive arrivals at a drive-through business; and discrete random variable N be the number of arrivals each minute.

If T has exponential distribution with parameter λ, then N has Poisson distribution with the same parameter λ.



Carlton & Devore Figure 7.18

Summary

Topics

- Normal (Gaussian) distribution
- Standard normal distribution
- Standardizing a normally-distributed random variable
- Approximating the binomial distribution
- Exponential distribution
- Memoryless property
- Connection to Poisson distribution

Reading

Chapter 3, §§3.3.1, 3.3.2, 3.3.4, 3.3.5, and 3.4.1; pp. 171–178, 179–181, and 187–189.

Exercises

Chapter 3, Exercises 39–65, 71–74, and 79–82; pp. 182–186 and 194–196.