Answers to Task 1

(1) Assume m and n are both integers. Prove by contraposition, if $mn < 120$, then $m < 10$ or $n < 12$.
You may assume that, if $a \geq b$ *and* $c \geq d$ *then* $ac \geq bd$ *for* $a, b, c, d, \in \mathbb{N}$ *and* $c > 0$. (*)

[4 marks]

Marking Guide

The contrapositive is: if $m \geq 10$ and $n \geq 12$ then $mn \geq 120$. [2 marks all correct]

We will use proof by contraposition.

Using the given property (*) with $a = m$, $b = 10$, $c = n$ and $d = 12$ then $m \geq 10$ *and* $n \geq 12$, giving $mn \geq 120$. [1 mark for using the property given]

As the contrapositive is true then the given proposition is true, 'if $mn < 120$, then $m < 10$ or $n < 12$'.

[final statement 1 mark]

Notes on sample answers

A What do a and b relate to here? The proof needs to be explicit about using the property given.

B The "by substitution" isn't clear about what is being substituted where. If it is "by substitution in the property given" then it would be good to say so.

C The proof finishes without saying what has now been proved.

D The negation of $<$ is $\geq$, not $>$ as used here.

E The sentence setting m = 10 and n = 12 doesn't seem to contribute anything.

Task 2

2. Prove that if n is any integer then 4 either divides n^2 or $n^2 - 1$
(6 marks) **Solution:**
There are two main ways of using Proof by Cases.
1 mark for stating the type of proof that will be used
(1) Method: Proof by Cases -integers can be divided up into even and odd.
2 marks for clearly and correctly stating all cases in the chosen method

(a) Case 1: n is even: Let $n = 2k$ where k is any integer, then $n^2 = 4k^2$ and $4|n^2$.
(b) Case 2: n is odd: Let $n = 2k + 1$ then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$ and $4|(n^2 - 1)$ since $n^2 - 1 = 4(k^2 + k), \quad \forall k \in \mathbb{Z}$.

2 marks for correct working for all cases
Therefore, in all cases if n is any integer $4|n^2$ or $4|n^2 - 1$.
1 mark for a final statement of what has been shown
[Total for Q2: 6 marks]

(2) The set of remainders when dividing by four are $\{0,1,2,3\}$. These are in fact equivalence classes which you will learn about later. This means we can use Proof by Cases.
1 mark for stating the type of proof that will be used
Our four cases are $n = 4k, 4k + 1, 4k + 2$ and $4k + 3$, where k is any integer.
2 marks for clearly and correctly stating all cases in the chosen method

(a) Case 1: $n = 4k$ so $n^2 = 16k^2 = 4(4k^2)$ and $4|n^2$
(b) Case 2:$n = 4k + 1$ so $n^2 = 16k^2 + 8k + 1 = 4(4k^2 + 2k) + 1$ and $4|n^2 - 1$
(c) Case 3: $n = 4k + 2$ so $n^2 = 16k^2 + 16k + 4 = 4(4k^2 + 4k + 1)$ and $4|n^2$
(d) Case 4: $n = 4k + 3$ so $n^2 = 16k^2 + 24k + 9 = 4(4k^2 + 6k + 2) + 1$ and $4|n^2 - 1$

2 marks for correct working for all cases
Therefore, in all cases if n is any integer $4|n^2$ or $4|n^2 - 1$.
1 mark for a final statement of what has been shown
[Total for Q2: 6 marks]

Total 10 marks

Answers to Incorrect proofs

Section 4.2

18. This incorrect "proof" assumes what is to be proved. The second sentence states a conclusion that would follow from the assumption that $m \cdot n$ is even, and the next-to-last sentence states this conclusion as if it were known to be true. But it is not known to be true at this point in the proof. In fact, it is the main task of a genuine proof to derive this conclusion, not from the assumption that it is true but from the hypothesis of the theorem.

19. The mistake in the "proof" is that the same symbol, k, is used to represent two different quantities. By setting both m and n equal to $2k$, the "proof" specifies that $m = n$, and, therefore, it only deduces the conclusion in case $m = n$. If $m \neq n$, the conclusion is often false. For instance, $6 + 4 = 10$ but $10 \neq 4k$ for any integer k.

Section 4. 3

35. This "proof" assumes what is to be proved.

36. This incorrect proof just shows the theorem to be true in the one case where one of the rational numbers is $1/4$ and the other is $1/2$. It is an example of the mistake of arguing from examples. A correct proof must show the theorem is true for *any* two rational numbers.

37. By setting both r and s equal to a/b, this incorrect proof violates the requirement that r and s be arbitrarily chosen rational numbers. If both r and s equal a/b, then $r = s$.

38. The fourth sentence claims that $r + s$ is a fraction because it is a sum of two fractions. But the statement that the sum of two fractions is a fraction is a restatement of what is to be proved. Hence this proof assumes what is to be proved.

    To complete a proof it would be necessary to go on and show that the sum of two fractions $(a/b+c/d)$ is a fraction $((ad+bc)/bd)$ noting that $(ad+bc)$ is an integer and (bd) is a nonzero integer.

39. This incorrect proof assumes what is to be proved. The second sentence asserts that a certain conclusion follows when it is assumed that $r + s$ is rational, and the rest of the proof uses that conclusion to deduce that $r + s$ is rational.

Task 4: A Diophantine equation is an equation for which you seek integer solutions. For example, the so-called pythagorean triples (x, y, z) are positive integer solutions to the equation $x^2 + y^2 = z^2$. Here is theorem about a Diophantine equation which you can prove using Proof by Contradiction.

Theorem. There are no **positive** integer solutions to the Diophantine equation $x^2 - y^2 = 1$. (*)

Proof. (Proof by Contradiction.) Assume to the contrary that there is a solution (x, y) where x and y are positive integers. If this is the case, we can factor the left side: $x^2 - y^2 = (x - y)(x + y) = 1$. Since x and y are integers, it follows that either $x - y = 1$ and $x + y = 1$ or $x - y = -1$ and $x + y = -1$.

We use Proof by Cases.

Case 1: both factors are +1.

In the first case we can add the two equations to get $x = 1$ and $y = 0$, contradicting our assumption that x and y are positive.

Case 2: Both factors are -1

The second case is similar, getting $x = -1$ and $y = 0$, again contradicting our assumption.

Hence, in both cases our assumption is contradicted and the given Theorem (*) is true.