Task 1. Here is a question set for a previous homework on proof. Try it first yourself.

Q1. Assume m and n are both integers. Prove by contraposition, if $mn < 120$ then $m < 10$ or $n < 12$. You may assume that, if $a \geq b$ and $c \geq d$, then $ac \geq bd$ for $a, b, c, d \in \mathbb{N}$ and c > 0. (4 marks)

Once you have decided on your own proof, discuss together in your group the submissions below. Which elements make a good proof? What would you consider errors? In the original homework this question was marked out of 4. How many marks would you give each of these attempts?

A

If $mn < 120$ then $m < 10$ or $n < 12$.

The contraposition of that proposition is
if $m \geqslant 10$ and $n \geqslant 12$ then $mn \geqslant 120$

Let $a \geqslant m$ for some integer a

Let $b \geqslant n$ for some integer b

$\therefore$ Trivially $ab \geqslant mn$

Thus by proving the contraposition, the statement is true.

B

Q1

Claim: For m and n that are both integers, if $mn < 120$, then $m < 10$ or $n < 12$.
Proof:

Proof by contraposition.

Contraposition of the statement: For m and n that are both integers, if $m \geqslant 10$ and $n \geqslant 12$, then $mn \geqslant 120$.

Since $m \geqslant 10$ and $n \geqslant 12$ by substitution $mn \geqslant 10 * 12 = 120$.

We have proved the contrapositive is true, hence the original claim is also true.

C

Prove by contraposition

If $mn < 120$ then $m < 10$ or $n < 12$, for $m, n \in \mathbb{Z}$

- Contrapositive: "If $n \nless 12$ and $m \nless 10$, then $mn \nless 120$"
$\equiv$ "If $m \geqslant 10$ and $n \geqslant 12$, then $mn \geqslant 120$"

If we assume that, if $a \geqslant b$ and $c \geqslant d$, then $ac \geqslant bd$ for $a, b, c, d \in \mathbb{N}$ and $c > 0$, then if $m \geqslant 10$ and $n \geqslant 12$, then $m.n \geqslant 10(12) = mn \geqslant 120$ ∎

D

If $mn < 120 \rightarrow m < 10$ or $n < 12$ then
$\neg(m < 10$ or $n < 12) \rightarrow \neg(mn < 120)$
$= m \geq 10$ and $n \geq 12 \rightarrow mn \geq 120$.

If $m \geq 10$ and $n \geq 12$, then let $m = a$,
$10 = b$, $n = c$, $12 = d$
$mn \geq 120$ for $a, b, c, d \in \mathbb{N}$ and $c > 0$.

E

Assume m and n are both integers. Prove by contraposition,
if $mn < 120$ then $m < 10$ or $n < 12$. You may assume that,
if $a \geq b$ and $c \geq d$, then $ac \geq bd$ for $a, b, c, d \in \mathbb{N}$ and
$c > 0$.
First I am going to take the contrapositive of the statement
to be:
Prove, if $m \geq 10$ and $n \geq 12$ then $mn \geq 120$.

To prove this I will let $m = 10$ and $n = 12$ and therefore
$mn = 120$ which is greater or equal to 120 as required.
Since $m \geq 10$ and $n \geq 12$ and we use the assumption in the
question, the contrapositive holds and since contrapositive
statements are equivalent the following is true:
If $mn \leq 120$ then $m < 10$ or $n < 12$.

Task 2. Prove that if n is any integer then 4 either divides n^2 or $n^2 - 1$.

Task 3 Discuss the mistakes in these ‘proofs’ below.

These are from Section 4.2 page 182 in your Textbook and copied here for convenience

Discuss Q18 and 19 and find the mistakes in these ‘proofs’

18. **Theorem:** The product of any even integer and any odd integer is even.

“**Proof:** Suppose m is any even integer and n is any odd integer. If $m \cdot n$ is even, then by definition of even there exists an integer r such that $m \cdot n = 2r$. Also since m is even, there exists an integer p such that $m = 2p$, and since n is odd there exists an integer q such that $n = 2q + 1$. Thus

$$mn = (2p)(2q + 1) = 2r,$$

where r is an integer. By definition of even, then, $m \cdot n$ is even, as was to be shown.”

19. **Theorem:** The sum of any two even integers equals $4k$ for some integer k.

“**Proof:** Suppose m and n are any two even integers. By definition of even, $m = 2k$ for some integer k and $n = 2k$ for some integer k. By substitution,

$$m + n = 2k + 2k = 4k.$$

This is what was to be shown.”

This is from Section 4.3 P189 in your textbook discuss and find the mistake in the 'proof'

In 35–39 find the mistakes in the "proofs" that the sum of any two rational numbers is a rational number.

35. "**Proof:** Any two rational numbers produce a rational number when added together. So if r and s are particular but arbitrarily chosen rational numbers, then $r+s$ is rational."

36. "**Proof:** Let rational numbers $r = \frac{1}{4}$ and $s = \frac{1}{2}$ be given. Then $r+s = \frac{1}{4}+\frac{1}{2} = \frac{3}{4}$, which is a rational number. This is what was to be shown."

37. "**Proof:** Suppose r and s are rational numbers. By definition of rational, $r = a/b$ for some integers a and b with $b \neq 0$, and $s = a/b$ for some integers a and b with $b \neq 0$. Then

$$r+s = \frac{a}{b}+\frac{a}{b} = \frac{2a}{b}.$$

Let $p = 2a$. Then p is an integer since it is a product of integers. Hence $r+s = p/b$, where p and b are integers and $b \neq 0$. Thus $r+s$ is a rational number by definition of rational. This is what was to be shown."

38. "**Proof:** Suppose r and s are rational numbers. Then $r = a/b$ and $s = c/d$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$ (by definition of rational). Then

$$r+s = \frac{a}{b}+\frac{c}{d}.$$

But this is a sum of two fractions, which is a fraction. So $r-s$ is a rational number since a rational number is a fraction."

39. "**Proof:** Suppose r and s are rational numbers. If $r+s$ is rational, then by definition of rational $r+s = a/b$ for some integers a and b with $b \neq 0$. Also since r and s are rational, $r = i/j$ and $s = m/n$ for some integers i, j, m, and n with $j \neq 0$ and $n \neq 0$. It follows that

$$r+s = \frac{i}{j}+\frac{m}{n} = \frac{a}{b},$$

which is a quotient of two integers with a nonzero denominator. Hence it is a rational number. This is what was to be shown."

Task 4 Try this question working collaboratively

A Diophantine equation is an equation for which you seek integer solutions. For example, the so-called pythagorean triples (x, y, z) are positive integer solutions to the equation $x^2 + y^2 = z^2$. Here is another statement to prove using Proof by Contradiction.

There are no positive integer solutions to the Diophantine equation $x^2 - y^2 = 1$