Task 1. Here is a question set for a previous homework on proof. Try it first yourself.

Q1. Assume m and n are both integers. Prove by contraposition, if \( mn < 120 \) then \( m < 10 \) or \( n < 12 \). You may assume that, if \( a \geq b \) and \( c \geq d \), then \( ac \geq bd \) for \( a, b, c, d \in \mathbb{N} \) and \( c > 0 \). (4 marks)

Once you have decided on your own proof, discuss together in your group the submissions below. Which elements make a good proof? What would you consider errors? In the original homework this question was marked out of 4. How many marks would you give each of these attempts?

A

\[
\text{If } mn < 120 \text{ then } m < 10 \text{ or } n < 12. \\
\text{The contraposition of that proposition is} \\
\text{if } m \geq 10 \text{ and } n \geq 12 \text{ then } mn \geq 120 \\
\text{Let } a \nmid m \text{ for some integer } a \\
\text{Let } b \nmid n \text{ for some integer } b \\
\therefore \text{Trivially } ab \nmid mn \\
\text{Thus by proving the contraposition, the} \\
\text{Statement is true.}
\]

B

Q1

Claim: For m and n that are both integers, if \( mn < 120 \), then \( m < 10 \) or \( n < 12 \).

Proof:

Proof by contraposition.

Contraposition of the statement: For m and n that are both integers, if \( m \geq 10 \) and \( n \geq 12 \), then \( mn \geq 120 \).

Since \( m \geq 10 \) and \( n \geq 12 \) by substitution \( mn \geq 10 \times 12 = 120 \).

We have proved the contrapositive is true, hence the original claim is also true.

C

\[
\text{Prove by contraposition} \\
\text{If } mn < 120 \text{ then } m < 10 \text{ or } n < 12, \text{ for } m, n \in \mathbb{Z} \\
\text{Contraposition: "If } n \notdivides 12 \text{ and } m \notdivides 10, \text{ then } mn \notdivides 120" \\
\equiv \text{"If } m \notdivides 10 \text{ and } n \notdivides 12, \text{ then } mn \notdivides 120" \\
\text{If we assume that, if } a \notdivides b \text{ and } c \notdivides d, \text{ then } ac \notdivides bd \\
\text{for } a, b, c, d \in \mathbb{N} \text{ and } c > 0, \text{ then if } m \notdivides 10 \text{ and } n \notdivides 12, \\
\text{then } mn \notdivides 10(12) = mn \notdivides 120. 
\]
If \( mn < 120 \) then \( m < 10 \) or \( n < 12 \), then

\[
\text{If } m > 10 \text{ and } n > 12, \text{ then let } m = a, \quad 10 = b, \; n = c, \quad 12 = d.
\]

\[
mn > 120 \quad \text{for } a, \; b, \; c, \; d \in \mathbb{N} \text{ and } c > 0.
\]

Assume \( m \) and \( n \) are both integers. Prove by contraposition, if \( mn < 120 \) then \( m < 10 \) or \( n < 12 \). You may assume that, if \( a \leq b \) and \( c \geq d \), then \( ac \geq bd \) for \( a, b, c, d \in \mathbb{N} \) and \( c > 0 \).

First I am going to take the contrapositive of the statement to be:

Prove, if \( m \neq 10 \) and \( n \neq 12 \) then \( mn \neq 120 \).

To prove this, I will let \( m = 10 \) and \( n = 12 \) and therefore \( mn = 120 \) which is greater or equal to 120 as required. Since \( m \neq 10 \) and \( n \neq 12 \) and we use the assumption in the question, the contrapositive holds and since contrapositive statements are equivalent the following is true:

If \( mn \leq 120 \) then \( m \leq 10 \) or \( n \leq 12 \).

**Task 2.** Prove that if \( n \) is any integer then 4 either divides \( n^2 \) or \( n^2 - 1 \).
Task 3 Discuss the mistakes in these ‘proofs’ below.

These are from Section 4.2 page 182 in your Textbook and copied here for convenience.

Discuss Q18 and 19 and find the mistakes in these ‘proofs’

18. Theorem: The product of any even integer and any odd integer is even.

   ‘Proof:’ Suppose $m$ is any even integer and $n$ is any odd integer. If $m \cdot n$ is even, then by definition of even there exists an integer $r$ such that $m \cdot n = 2r$. Also since $m$ is even, there exists an integer $p$ such that $m = 2p$, and since $n$ is odd there exists an integer $q$ such that $n = 2q + 1$. Thus

   \[ mn = (2p)(2q + 1) = 2r, \]

   where $r$ is an integer. By definition of even, then, $m \cdot n$ is even, as was to be shown.”

19. Theorem: The sum of any two even integers equals $4k$ for some integer $k$.

   ‘Proof:’ Suppose $m$ and $n$ are any two even integers. By definition of even, $m = 2k$ for some integer $k$ and $n = 2k$ for some integer $k$. By substitution,

   \[ m + n = 2k + 2k = 4k. \]

   This is what was to be shown.”
In 35–39 find the mistakes in the “proofs” that the sum of any two rational numbers is a rational number.

35. “Proof:” Any two rational numbers produce a rational number when added together. So if \( r \) and \( s \) are particular but arbitrarily chosen rational numbers, then \( r + s \) is rational.”

36. “Proof:” Let rational numbers \( r = \frac{1}{4} \) and \( s = \frac{1}{2} \) be given. Then \( r + s = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \), which is a rational number. This is what was to be shown.”

37. “Proof:” Suppose \( r \) and \( s \) are rational numbers. By definition of rational, \( r = \frac{a}{b} \) for some integers \( a \) and \( b \) with \( b \neq 0 \), and \( s = \frac{a}{b} \) for some integers \( a \) and \( b \) with \( b \neq 0 \). Then

\[
\frac{r}{b} + \frac{s}{b} = \frac{2a}{b}.
\]

Let \( p = 2a \). Then \( p \) is an integer since it is a product of integers. Hence \( r + s = p/b \), where \( p \) and \( b \) are integers and \( b \neq 0 \). Thus \( r + s \) is a rational number by definition of rational. This is what was to be shown.”

38. “Proof:” Suppose \( r \) and \( s \) are rational numbers. Then \( r = \frac{a}{b} \) and \( s = \frac{c}{d} \) for some integers \( a \), \( b \), \( c \), and \( d \) with \( b \neq 0 \) and \( d \neq 0 \) (by definition of rational). Then

\[
\frac{r}{b} + \frac{s}{d} = \frac{c}{d},
\]

But this is a sum of two fractions, which is a fraction. So \( r - s \) is a rational number since a rational number is a fraction.”

39. “Proof:” Suppose \( r \) and \( s \) are rational numbers. If \( r + s \) is rational, then by definition of rational \( r + s = \frac{a}{b} \) for some integers \( a \) and \( b \) with \( b \neq 0 \). Also since \( r \) and \( s \) are rational, \( r = \frac{i}{j} \) and \( s = \frac{m}{n} \) for some integers \( i \), \( j \), \( m \), and \( n \) with \( j \neq 0 \) and \( n \neq 0 \). It follows that

\[
\frac{r}{j} + \frac{s}{n} = \frac{a}{b},
\]

which is a quotient of two integers with a nonzero denominator. Hence it is a rational number. This is what was to be shown.”

Task 4 Try this question working collaboratively

A Diophantine equation is an equation for which you seek integer solutions. For example, the so-called pythagorean triples \( (x, y, z) \) are positive integer solutions to the equation \( x^2 + y^2 = z^2 \). Here is another statement to prove using Proof by Contradiction.

There are no positive integer solutions to the Diophantine equation \( x^2 - y^2 = 1 \)