Task A

a. Proof (by mathematical induction): Let the property $P(n)$ be the sentence

Any checkerboard with dimensions $2 \times 3n$ can be completely covered by L-shaped trominoes. $\leftarrow P(n)$

We will prove that $P(n)$ is true for every integer $n \geq 1$.

Show that $P(1)$ is true: The truth of $P(1)$ is shown in the following diagram:

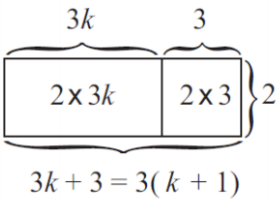
Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:
Let k be any integer with $k \geq 1$ and suppose that

Any checkerboard with dimensions $2 \times 3k$ can be completely covered by L-shaped trominoes. $\leftarrow$ $P(k)$ inductive hypothesis

We must show that

Any checkerboard with dimensions $2 \times 3(k+1)$ can be completely covered by L-shaped trominoes. $\leftarrow P(k+1)$

Observe that any checkerboard with dimensions $2 \times 3(k+1)$ can be split into two pieces: a checkerboard with dimensions $2 \times 3k$ and a checkerboard with dimensions 2×3.



The inductive hypothesis insures that the $2 \times 3k$ checkerboard can be completely covered by L-shaped trominoes, and the illustration for the basis step shows that the checkerboard with dimensions 2×3 can also be completely covered by L-shaped trominoes. Thus the entire checkerboard with dimensions $2 \times 3(k+1)$ can be completely covered by L-shaped trominoes *[as was to be shown]*.

Task B

Proof (by mathematical induction): Let the property $P(n)$ be the sentence

$$7^n - 2^n \text{ is divisible by } 5. \qquad \leftarrow P(n)$$

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that P(0) is true: $P(0)$ is true because $7^0 - 2^0 = 1 - 1 = 0$ and 0 is divisible by 5 (since $0 = 5 \cdot 0$).

Show that for every integer $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 0$, and suppose

$$7^k - 2^k \text{ is divisible by } 5. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$7^{k+1} - 2^{k+1} \text{ is divisible by } 5. \qquad \leftarrow P(k+1)$$

By definition of divisibility, the inductive hypothesis is equivalent to the statement $7^k - 2^k = 5r$ for some integer r. Then

$$\begin{aligned} 7^{k+1} - 2^{k+1} &= 7 \cdot 7^k - 2 \cdot 2^k \\ &= (5+2) \cdot 7^k - 2 \cdot 2^k \\ &= 5 \cdot 7^k + 2 \cdot 7^k - 2 \cdot 2^k \\ &= 5 \cdot 7^k + 2(7^k - 2^k) && \text{by algebra} \\ &= 5 \cdot 7^k + 2 \cdot 5r && \text{by inductive hypothesis} \\ &= 5(7^k + 2r) && \text{by algebra.} \end{aligned}$$

Now $7^k + 2r$ is an integer because products and sums of integers are integers. Therefore, by definition of divisibility, $7^{k+1} - 2^{k+1}$ is divisible by 5 *[as was to be shown]*.

Task C

Proof (by strong mathematical induction): Let the property $P(n)$ be the equation

$$f_n = 3\cdot 2^n + 2\cdot 5^n.$$

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $P(0)$ and $P(1)$ are true: By definition of $f_0, f_1, f_2, \ldots$, we have that $f_0 = 5$ and $f_1 = 16$. Since $3\cdot 2^0 + 2\cdot 5^0 = 3 + 2 = 5$ and $3\cdot 2^1 + 2\cdot 5^1 = 6 + 10 = 16$, both $P(0)$ and $P(1)$ are true.

Show that for every integer $k \geq 1$, if $P(i)$ is true for each integer i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose

$$f_i = 3\cdot 2^i + 2\cdot 5^i \text{ for every integer } i \text{ with } 0 \leq i \leq k. \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$f_{k+1} = 3\cdot 2^{k+1} + 2\cdot 5^{k+1}.$$

Now

$$\begin{aligned}
f_{k+1} &= 7f_k - 10f_{k-1} && \text{by definition of } f_0, f_1, f_2, \ldots \\
&= 7(3\cdot 2^k + 2\cdot 5^k) - 10(3\cdot 2^{k-1} + 2\cdot 5^{k-1}) && \text{by inductive hypothesis} \\
&= 7(6\cdot 2^{k-1} + 10\cdot 5^{k-1}) - 10(3\cdot 2^{k-1} + 2\cdot 5^{k-1}) && \text{since } 2^k = 2\cdot 2^{k-1} \text{ and } 5^k = 5\cdot 5^{k-1} \\
&= (42\cdot 2^{k-1} + 70\cdot 5^{k-1}) - (30\cdot 2^{k-1} + 20\cdot 5^{k-1}) \\
&= (42 - 30)\cdot 2^{k-1} + (70 - 20)\cdot 5^{k-1} \\
&= 12\cdot 2^{k-1} + 50\cdot 5^{k-1} \\
&= 3\cdot 2^2\cdot 2^{k-1} + 2\cdot 5^2\cdot 5^{k-1} \\
&= 3\cdot 2^{k+1} + 2\cdot 5^{k+1} && \text{by algebra,}
\end{aligned}$$

[as was to be shown].

[Since both the basis and the inductive steps have been proved, we conclude that $P(n)$ is true for every integer $n \geq 0$.]

Task D

3. Proof (by strong mathematical induction): Let the property $P(n)$ be the sentence

$$c_n \text{ is even.}$$

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $P(0)$, $P(1)$, and $P(2)$ are true: By definition of $c_0, c_1, c_2, \ldots$, we have that $c_0 = 2$, $c_1 = 2$, and $c_2 = 6$ and 2, 2, and 6 are all even. So $P(0)$, $P(1)$, and $P(2)$ are all true.

Show that for every integer $k \geq 2$, if $P(i)$ is true for each integer i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 2$, and suppose

$$c_i \text{ is even for every integer } i \text{ with } 0 \leq i \leq k \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$c_{k+1} \text{ is even.}$$

Now by definition of $c_0, c_1, c_2, \ldots, c_{k+1} = 3c_{k-2}$. Since $k \geq 2$, we have that $0 \leq k-2 \leq k$, and so, by inductive hypothesis, c_{k-2} is even. Now the product of an even integer with any integer is even *[properties 1 and 4 of Example 4.2.3]*, and hence $3c_{k-2}$, which equals c_{k+1}, is also even *[as was to be shown]*.

[Since both the basis and the inductive steps have been proved, we conclude that $P(n)$ is true for every integer $n \geq 0$.]

Task E

18. $9^1 = 9$, $9^2 = 81$, $9^3 = 729$, $9^4 = 6561$, and $9^5 = 59049$.

Conjecture: For every integer $n \geq 0$, the units digit of 9^n is 1 if n is even and is 9 if n is odd.

Proof (by strong mathematical induction): Let the property $P(n)$ be the sentence

The units digit of 9^n is 1 if n is even and is 9 if n is odd.

We will prove that $P(n)$ is true for every integer $n \geq 0$.

Show that $P(0)$ and $P(1)$ are true: $P(0)$ is true because 0 is even and the units digit of $9^0 = 1$. $P(1)$ is true because 1 is odd and the units digit of $9^1 = 9$.

Show that for every integer $k \geq 1$, if $P(i)$ is true for each integer i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose:

For every integer i from 0 through k,

$$\text{the units digit of } 9^i = \begin{cases} 1 \text{ if } i \text{ is even} \\ 9 \text{ if } i \text{ is odd} \end{cases} \quad \leftarrow \quad \text{inductive hypothesis}$$

We must show that

$$\text{the units digit of } 9^{k+1} = \begin{cases} 1 \text{ if } (k+1) \text{ is even} \\ 9 \text{ if } (k+1) \text{ is odd} \end{cases} \quad \leftarrow \quad P(k+1)$$

Case 1($k+1$ iseven): In this case k is odd, and so, by inductive hypothesis, the units digit of 9^k is 9. This implies that there is an integer a so that $9^k = 10a + 9$, and hence

$$\begin{aligned} 9^{k+1} &= 9^1 \cdot 9^k && \text{by algebra (a law of exponents)} \\ &= 9(10a+9) && \text{by substitution} \\ &= 90a + 81 \\ &= 90a + 80 + 1 \\ &= 10(9a+8) + 1 && \text{by algebra.} \end{aligned}$$

Because $9a + 8$ is an integer, it follows that the units digit of 9^{k+1} is 1.

Case 2 ($k+1$ is odd): In this case k is even, and so, by inductive hypothesis, the units digit of 9^{k-1} is 1. This implies that there is an integer a so that $9^k = 10a + 1$, and hence

$$\begin{aligned} 9^{k+1} &= 9^1 \cdot 9^k && \text{by algebra (a law of exponents)} \\ &= 9(10a+1) && \text{by substitution} \\ &= 90a + 9 \\ &= 10(9a) + 9 && \text{by algebra.} \end{aligned}$$

Because $9a$ is an integer, it follows that the units digit of 9^{k+1} is 9.

Hence in both cases the units digit of 9^{k+1} is as specified in $P(k+1)$ *[as was to be shown]*.