Q1

Find the mistake in the following "proof" that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

"**Proof:** Suppose A, B, and C are any sets such that $A \subseteq B$ and $B \subseteq C$. Since $A \subseteq B$, there is an element x such that $x \in A$ and $x \in B$, and since $B \subseteq C$, there is an element x such that $x \in B$ and $x \in C$. Hence there is an element x such that $x \in A$ and $x \in C$ and so $A \subseteq C$."

. There is more than one error in the "proof." The most serious is the misuse of the definition of subset. To say that A is a subset of B means that for every x, **if** $x \in A$ **then** $x \in B$. It does not mean that there exists an element of A that is also an element of B. The second error in the proof occurs in the last sentence. Even if there is an element in A that is in B and an element in B that is in C, it does not follow that there is an element in A that is in C. For instance, suppose $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{3, 4\}$. Then there is an element in A that is in B (namely 2) and there is an element in B that is in C (namely, 3), but there is no element in A that is in C.

Q2

Find the mistake in the following "proof."

"**Theorem:**" For all sets A and B, $A^c \cup B^c \subseteq (A \cup B)^c$.

"**Proof:** Suppose A and B are any sets, and $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$ by definition of union. It follows that $x \notin A$ or $x \notin B$ by definition of complement, and so $x \notin A \cup B$ by definition of union. Thus $x \in (A \cup B)^c$ by definition of complement, and hence $A^c \cup B^c \subseteq (A \cup B)^c$."

The "proof" claims that because $x \notin A$ or $x \notin B$, it follows that $x \notin A \cup B$. But it is possible for "$x \notin A$ or $x \notin B$" to be true and "$x \notin A \cup B$" to be false. For example, let $A = \{1, 2\}$, $B = \{2, 3\}$, and $x = 3$. Then since $3 \notin \{1, 2\}$, the statement "$x \notin A$ or $x \notin B$" is true. But since $A \cup B = \{1, 2, 3\}$ and $3 \in \{1, 2, 3\}$, the statement "$x \notin A \cup B$" is false.

Q3

Recall that for sets A and B, $|A| = |B|$ if there is a bijection $f : A \to B$, a function f that is both injective (one-to-one) and surjective (onto). Let $E = \{0, 2, 4, \ldots\}$ be the set of non-negative even integers.

(a) Give an example of a function $g : E \to E$ that is injective but not surjective.

(b) Prove that $|\mathbb{Z}| = |E|$ by defining an explicit bijection.

(a) Any plausible g here such as $g(x) = 2x$, so $g(2) = 4$ and so on. This is clearly injective as $g(x) = g(y)$ implies $x = y$. However, it is not surjective as elements that are not divisible by 4, such as 2 and 6, are not mapped to. No marks if g is not a function from E to E.

(b) They need to produce a bijection from $g : \mathbb{Z} \to E$ such as $g(n) = 4n$ for $n \geq 0$ and $g(n) = -4n - 2$ for $n < 0$ and explain why it is a bijection. No marks if the function is not from $\mathbb{Z}$ to E. Reduce marks if the function is not given explicitly (such as, as an enumeration).

Q4

(a) Determine whether the function $f : (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z}$ is surjective if

i. $f(m, n) = m^2 + n^2$

ii. $f(m, n) = m$

iii. $f(m, n) = |n|$

iv. $f(m, n) = m - n$

Solution:

i. The function is not surjective because not every integer is the sum of two perfect squares. For example $-|n|$ and 3 are not the sum of two perfect squares (for any n).

ii. The function is surjective because for any $z \in \mathbb{Z}$ we can choose a pair $(z, x) \in \mathbb{Z} \times \mathbb{Z}$ and $f(z, x) = z$.

iii. The function is not surjective because $|n|$ is always positive, so there exists no (x, y) such that $f(x, y) = -|n|$.

iv. The function is surjective because for every z integer $f(z, 0) = z - 0 = z$.

$\sqcup$

(b) Assume functions $g : A \to B$ and $f : B \to C$. Prove or disprove the following statements.

i. If $f \circ g$ and g are injective then f is injective.

Solution:

This statement is not correct; let $A = \{a, b\} = C$ and $B = \{a, b, c\}$; let $g(a) = a$ and $g(b) = b$; and $f(a) = a$; $f(b) = b$ and $f(c) = a$. Now $f \circ g$ is injective since $(f \circ g)(a) \neq (f \circ g)(b)$; similarly g is injective; however, f is not injective because $f(a) = f(c)$. $\square$

ii. If $f \circ g$ and f are injective then g is injective.

Solution:

This statement is true. In fact, we prove the slightly stronger: if $f \circ g$ is injective then g is injective. By way of contradiction assume $f \circ g$ is injective and g is not. So, for some $a, a' \in A$, $a \neq a'$ and $g(a) = g(a')$; so, $f(g(a)) = f(g(a'))$, so $(f \circ g)(a) = (f \circ g)(a')$ which contradicts that $f \circ g$ is injective. $\square$

Q5

Let A and B be any sets. Then

$(A - (A \cap B)) \cap (B - (A \cap B))$

$$
\begin{aligned}
&= (A \cap (A \cap B)^c) \cap (B \cap (A \cap B)^c) && \text{by the set difference law (used twice)} \\
&= A \cap ((A \cap B)^c \cap (B \cap (A \cap B)^c)) && \text{by the associative law for } \cap \\
&= A \cap (((A \cap B)^c \cap B) \cap (A \cap B)^c) && \text{by the associative law for } \cap \\
&= A \cap ((B \cap (A \cap B)^c) \cap (A \cap B)^c) && \text{by the commutative law for } \cap \\
&= A \cap (B \cap ((A \cap B)^c \cap (A \cap B)^c)) && \text{by the associative law for } \cap \\
&= A \cap (B \cap (A \cap B)^c) && \text{by the idempotent law for } \cap \\
&= (A \cap B) \cap (A \cap B)^c && \text{by the associative law for } \cap \\
&= \emptyset && \text{by the complement law for } \cap.
\end{aligned}
$$