Discrete Mathematics and Probability

Tutorial 5 – solutions

(1) *Discuss your last tutorial/homework/class test with your peers.*

(2) *A telegraph sends out three symbols on the communication line. Represent the following events in a single Venn diagram:*

$$A_1 = \{\textit{only the first symbol is received}\}$$
$$A_2 = \{\textit{at least one symbol is received}\}$$
$$A_3 = \{\textit{exactly two symbols are received}\}$$
$$A_4 = \{\textit{less than two symbols are received}\}$$
$$A_5 = \{\textit{exactly one symbol is received}\}$$

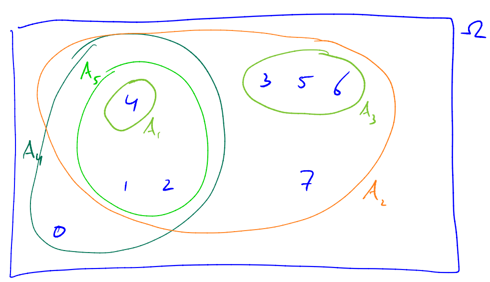
Write 0 and 1 for whether a symbol is received. The sample space then consists of strings of three bits, or, interpreting them in binary notation, of the numbers $\Omega = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

000	001	010	011	100	101	110	111
0	1	2	3	4	5	6	7

The events become:

$$A_1 = \{4\}$$
$$A_2 = \{1, 2, 3, 4, 5, 6, 7\}$$
$$A_3 = \{3, 5, 6\}$$
$$A_4 = \{0, 1, 2, 4\}$$
$$A_5 = \{1, 2, 4\}$$

In a Venn diagram:



(3) *Five cards are numbered as 1,2,3,4,5. Three cards are randomly selected from the set and are lined up next to each other to form 3 digit number x. Find the probabilities of the following events:*

(a) $A = \{x = 123\}$

(b) $B = \{x \textit{ does not contain the digit } 4\}$

(c) $C = \{x \textit{ is even}\}$

(d) $D = \{x \textit{ contains at least one of the digits } 1, 2\}$

The number of possible permutations of 3 elements from a set of 5 is $N = P_{3,5} = 5 \cdot 4 \cdot 3 = 60$.

(a) Only the permutation 123 is selected, so the event A has cardinality 1, and thus probability $P(A) = \frac{1}{60}$.
(b) To not contain the digit 4, we can choose only among digits 1,2,3,5, for which there are $P_{3,4} = 4 \cdot 3 \cdot 2 = 24$ possibilities, so $P(B) = \frac{24}{60} = \frac{2}{5}$.
(c) For x to be even, its last digit must be 2 or 4. In each of those cases, the first two digits can have one of 4 values (1,3,4,5 in the first case, and 1,2,3,5 in the second case). Therefore event C has cardinality $2 \cdot P_{2,4} = 2 \cdot 4 \cdot 3 = 24$, and $P(C) = \frac{24}{60} = \frac{2}{5}$.
(d) Consider the complement $D^c = \{x \text{ does not contain the digits } 1, 2\}$. Then every digit of x can be one of three values 3,4,5, and so D^c has cardinality $P_{3,3} = 3! = 6$, and so $P(D^c) = \frac{6}{60} = \frac{1}{10}$, and therefore $P(D) = 1 - P(D^c) = 1 - \frac{1}{10} = \frac{9}{10}$.

(4) *In how many ways can you order the elements of the set* $\{1, 2, \ldots, 2n\}$ *so that every even number is at an even position?*

There are n even numbers in the set. There are also n even positions in a sequence of $2n$ numbers. The number of ways to order n numbers in n positions is $P_{n,n} = n!$. Clearly, there are also n odd numbers in the set and they can also be ordered in n odd positions in $n!$ possible ways. So for each ordering of even numbers we have $n!$ orderings of odd numbers. Therefore the total number of orderings is $n! \cdot n! = (n!)^2$.

(5) *A white ball is thrown into an urn containing n balls. Next, a ball is drawn at random from the urn. What is the probability that the selected ball is white? The urn may initially contain* $0, 1, 2, \ldots$ *or n white balls, and it is equally probable that the urn is in one of those* $n + 1$ *initial states at the start of the experiment.*

Let $A = \{\text{white ball was drawn}\}$, and denote the following events expressing the initial state of the urn:

$$\begin{aligned}
B_1 &= \{\text{the urn contains exactly 0 white balls}\}\\
B_2 &= \{\text{the urn contains exactly 1 white balls}\}\\
B_3 &= \{\text{the urn contains exactly 2 white balls}\}\\
&\ldots\\
B_{n+1} &= \{\text{the urn contains exactly } n \text{ white balls}\}
\end{aligned}$$

Events B_i are pairwise mutually exclusive and exactly one of them must be true. Therefore $\sum_{i=1}^{n+1} P(B_i) = 1$. Since there are $n + 1$ events, each of which is equally probably, we have

$$P(B_i) = \frac{1}{n+1}.$$

The conditional probability of event A given a particular initial state of the urn is

$$P(A|B_i) = \frac{i}{n+1} : 1 \leq i \leq n + 1.$$

To see this, suppose that the initial state of the urn is B_i, that is, the urn contains $i-1$ white balls out of n balls. Then a white ball is thrown into the urn, after which it will contain i white balls out of $n+1$ balls. Now the probability to draw a white ball is $\frac{i}{n+1}$, which is exactly $P(A|B_i)$. Finally, using the law of total probability, we derive:

$$\begin{aligned} P(A) &= \sum_{i=1}^{n+1} P(A|B_i)P(B_i) = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{i}{n+1} \\ &= \frac{1+2+\cdots+(n+1)}{(n+1)^2} = \frac{(1+(n+1))(n+1)}{2(n+1)^2} = \frac{n+2}{2(n+1)}. \end{aligned}$$