Discrete Mathematics and Probability
Tutorial 5 – solutions

(1) Discuss your last tutorial/homework/class test with your peers.

(2) A telegraph sends out three symbols on the communication line. Represent the following events in a single Venn diagram:

\[ A_1 = \{ \text{only the first symbol is received} \} \]
\[ A_2 = \{ \text{at least one symbol is received} \} \]
\[ A_3 = \{ \text{exactly two symbols are received} \} \]
\[ A_4 = \{ \text{less than two symbols are received} \} \]
\[ A_5 = \{ \text{exactly one symbol is received} \} \]

Write 0 and 1 for whether a symbol is received. The sample space then consists of strings of three bits, or, interpreting them in binary notation, of the numbers \( \Omega = \{0, 1, 2, 3, 4, 5, 6, 7\} \).

\[
\begin{array}{cccccccc}
000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\]

The events become:

\[ A_1 = \{ 4 \} \]
\[ A_2 = \{ 1, 2, 3, 4, 5, 6, 7 \} \]
\[ A_3 = \{ 3, 5, 6 \} \]
\[ A_4 = \{ 0, 1, 2, 4 \} \]
\[ A_5 = \{ 1, 2, 4 \} \]

In a Venn diagram:

(3) Five cards are numbered as 1,2,3,4,5. Three cards are randomly selected from the set and are lined up next to each other to form 3 digit number \( x \). Find the probabilities of the following events:

(a) \( A = \{ x = 123 \} \)
(b) \( B = \{ x \text{ does not contain the digit 4} \} \)
(c) \( C = \{ x \text{ is even} \} \)
(d) \( D = \{ x \text{ contains at least one of the digits 1,2} \} \)

The number of possible permutations of 3 elements from a set of 5 is \( N = P_{3,5} = 5 \cdot 4 \cdot 3 = 60 \).
(a) Only the permutation 123 is selected, so the event \( A \) has cardinality 1, and thus probability \( P(A) = \frac{1}{60} \).

(b) To not contain the digit 4, we can choose only among digits 1,2,3,5, for which there are \( P_{3,4} = 4 \cdot 3 \cdot 2 = 24 \) possibilities, so \( P(B) = \frac{24}{60} = \frac{2}{5} \).

(c) For \( x \) to be even, its last digit must be 2 or 4. In each of those cases, the first two digits can have one of 4 values (1,3,4,5 in the first case, and 1,2,3,5 in the second case). Therefore event \( C \) has cardinality \( 2 \cdot P_{2,4} = 2 \cdot 4 \cdot 3 = 24 \), and \( P(C) = \frac{24}{60} = \frac{2}{5} \).

(d) Consider the complement \( D^c = \{x \text{ does not contain the digits 1,2}\} \).

Then every digit of \( x \) can be one of three values 3,4,5, and so \( D^c \) has cardinality \( P_{3,3} = 3! = 6 \), and so \( P(D^c) = \frac{6}{60} = \frac{1}{10} \), and therefore \( P(D) = 1 - P(D^c) = 1 - \frac{1}{10} = \frac{9}{10} \).

(4) In how many ways can you order the elements of the set \( \{1,2,\ldots,2n\} \) so that every even number is at an even position?

There are \( n \) even numbers in the set. There are also \( n \) even positions in a sequence of \( 2n \) numbers. The number of ways to order \( n \) numbers in \( n \) positions is \( P_{n,n} = n! \). Clearly, there are also \( n \) odd numbers in the set and they can also be ordered in \( n \) odd positions in \( n! \) possible ways. So for each ordering of even numbers we have \( n! \cdot n! = (n!)^2 \).

(5) A white ball is thrown into an urn containing \( n \) balls. Next, a ball is drawn at random from the urn. What is the probability that the selected ball is white? The urn may initially contain 0,1,2,\ldots or \( n \) white balls, and it is equally probable that the urn is in one of those \( n+1 \) initial states at the start of the experiment.

Let \( A = \{\text{white ball was drawn}\} \), and denote the following events expressing the initial state of the urn:

\[
B_1 = \{\text{the urn contains exactly 0 white balls}\}
\]

\[
B_2 = \{\text{the urn contains exactly 1 white balls}\}
\]

\[
B_3 = \{\text{the urn contains exactly 2 white balls}\}
\]

\[
\ldots
\]

\[
B_{n+1} = \{\text{the urn contains exactly } n \text{ white balls}\}
\]

Events \( B_i \) are pairwise mutually exclusive and exactly one of them must be true. Therefore \( \sum_{i=1}^{n+1} P(B_i) = 1 \). Since there are \( n+1 \) events, each of which is equally probable, we have

\[
P(B_i) = \frac{1}{n+1}.
\]

The conditional probability of event \( A \) given a particular initial state of the urn is

\[
P(A|B_i) = \frac{i}{n+1} : 1 \leq i \leq n+1.
\]
To see this, suppose that the initial state of the urn is $B_i$, that is, the urn contains $i - 1$ white balls out of $n$ balls. Then a white ball is thrown into the urn, after which it will contain $i$ white balls out of $n + 1$ balls. Now the probability to draw a white ball is $\frac{i}{n+1}$, which is exactly $P(A|B_i)$. Finally, using the law of total probability, we derive:

$$P(A) = \sum_{i=1}^{n+1} P(A|B_i)P(B_i) = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{i}{n+1}$$

$$= \frac{1 + 2 + \cdots + (n + 1)}{(n+1)^2} = \frac{(1 + (n + 1))(n + 1)}{2(n+1)^2} = \frac{n + 2}{2(n+1)}.$$