Discrete Mathematics and Probability
Tutorial 6 – solutions

(1) *Discuss your last tutorial/homework/class test with your peers.*

(2) *A discrete random variable X has the following probability mass function:*

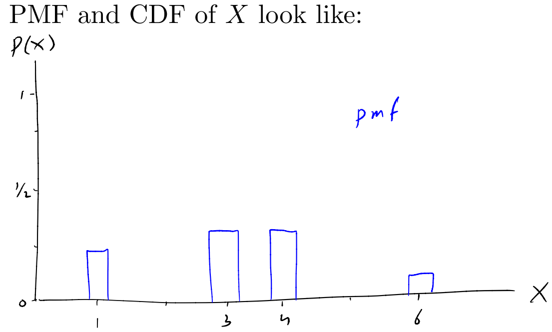
x_i	*1*	*3*	*4*	*6*
p_i	*0.25*	*0.3*	*0.3*	*0.15*

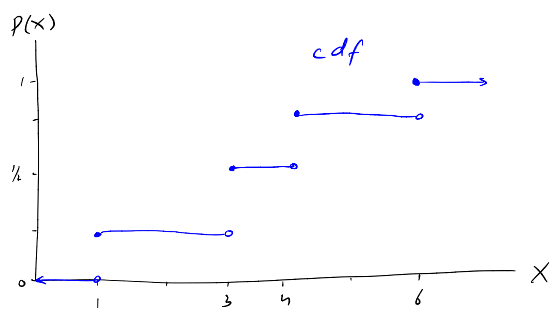
Compute the cumulative distribution function, and the probability $P(2 \leq X \leq 5)$. Plot the probability mass function and the cumulative distribution function.

The CDF of X is:

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.25 & \text{if } 1 \leq x < 3 \\ 0.55 & \text{if } 3 \leq x < 4 \\ 0.85 & \text{if } 4 \leq x < 6 \\ 1 & \text{if } 6 \leq x \end{cases}$$

Plots of the PMF and CDF of X look like:





The probability $P(2 \leq X \leq 5)$ is

$$P(2 \leq X \leq 5) = P((X = 3) \cup (X = 4)) = F(5) - F(2) = 0.85 - 0.25 = 0.6.$$

(3) *Alice and Bob play take turns throwing a six-sided die. The first one to throw a 5 or 6 wins. Alice starts. What are the probability of the events $A = \{Alice\ wins\}$ and $B = \{Bob\ wins\}$?*
(You may use the fact that $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots$.)

Write

$$A_k = \{\text{ on the } k\text{th throw Alice gets a 5 or 6}\}$$
$$B_k = \{\text{on the } k\text{th throw Bob gets a 5 or 6}\}$$

The probability that on any throw a player gets a 5 or 6 is $p = \frac{2}{6} = \frac{1}{3}$. Therefore $P(A_k) = P(B_k) = p = \frac{1}{3}$, and $P(A_k^c) = P(B_k^c) = q = 1 - p = \frac{2}{3}$. Alice wins in the event

$$A = A_1 \cup (A_1^c \cap B_1^c \cap A_2) \cup (A_1^c \cap B_1^c \cap A_2^c \cap B_2^c \cap A_3) \cup \cdots$$

Since the events A_k and B_k are independent and the events A_1 and $A_1^c \cap B_1^c \cap A_2$ and $A_1^c \cap B_1^c \cap A_2^c \cap B_2^c \cap A_3$ are pairwise mutually exclusive:

$$\begin{aligned} P(A) &= p + qqp + qqqqp + \cdots = p(1 + q^2 + q^4 + \cdots) \\ &= \frac{p}{1-q^2} = 0.6. \end{aligned}$$

Bob wins in the event

$$B = (A_1^c \cap B_1) \cup (A_1^c \cap B_1^c \cap A_2^c \cap B_2) \cup \cdots$$

Similarly

$$\begin{aligned} P(B) &= qp + qqqp + qqqqqp + \cdots = pq(1 + q^2 + q^4 + \cdots) \\ &= \frac{pq}{1-q^2} = 0.4. \end{aligned}$$

(4) *An exam has 4 questions. Each question has 4 answers, of which exactly 1 is correct. The exam is given to 256 students. Each student answers each question randomly. Describe the distribution of the number of correct answers, i.e. how many exams have 0 correct answers, how many exams have 1 correct answer, etc.*

The guessing of correct answers by a single student can be seen as a binomial experiment composed of $n = 4$ trials (4 questions). Since a student is randomly guessing each answer, the 4 trials are independent and the probability of success (i.e. guessing the correct answer) in each trial is $1/4$.

Let X be the number of correct answers guessed by a single student. This quantity can be modelled as a random variable whose values are distributed according to a binomial distribution $X \sim \text{Bin}(x; n = 4, p = 1/4)$. Therefore the probability that $X = x$ for $x = 0, 1, 2, 3, 4$ is

$$P(X = x) = \binom{4}{x} (1/4)^x (3/4)^{4-x}.$$

The expected number of correctly guessed answers is $256 \cdot P(X = x)$, and they are distributed over the values of x as follows:

$$256 \cdot P(X = x) = \begin{cases} 81 & x = 0 \\ 108 & x = 1 \\ 54 & x = 2 \\ 12 & x = 3 \\ 1 & x = 4 \end{cases}$$