Task 1. Retrieve your submissions from Homework 4 in Week 8 as well as the solution notes on the course website. Compare solutions around the group.

How did you count the number of permutations in Question 1(a)? What approach did you take to the calculation in Question 1(b)?

For Question 2 show your tables for part (a). How did you calculate the standard deviation for part (c)?

Task 2. I roll a fair six-sided die repeatedly until I get a result that is more than two. This takes me K rolls, giving a random variable with geometric distribution $K \sim \text{Geom}(p)$ where p is the chance of succeeding in one roll. Calculate the following.

(a) The value of p.

(b) The chance that it takes me exactly three rolls to get this result.

(c) $P(K > 3)$, the probability that it takes more than three rolls.

(d) The expectation $\text{E}(K)$. (You can use the formula for expectation of geometric distribution from last week's lecture slides or the textbook.)

Task 3. Discrete random variable X is known to have expected value $\text{E}(X) = 5.8$ and take three possible values: value $x_1 = 3$ with probability $p_1 = 0.2$, value $x_2 = 5$ with probability $p_2 = 0.4$, and value x_3 with probability p_3. Find p_3, x_3, standard deviation σ_X, and the cumulative distribution function $F(x)$ of X.

Task 4. A commuter travels by train twice a day: heading into town in the morning and coming back home in the evening. Trains run every 20 minutes, but the commuter arrives at the station randomly and always takes the next train. This means waiting at the station between 0 and 20 minutes two times each day. These waits are distributed uniformly $\text{Unif}(0, 20)$ and it turns out that the total waiting time each day in minutes is a random variable T with the following PDF.

$$f(t) = \begin{cases} \dfrac{t}{400} & 0 \le t < 20 \\ \dfrac{40-t}{400} & 20 \le t < 40 \\ 0 & t < 0 \text{ or } t \ge 40 \end{cases}$$

(a) Sketch a graph of the probability density function $f(t)$.

(b) Calculate the corresponding cumulative distribution function $F(t)$ for the random variable T. Sketch the graph of $F(t)$. (It is possible to do this by integration, but it's not essential and you may find it simpler to use geometry to calculate the area of certain triangles.)

(c) What is the probability that tomorrow the commuter has to wait no more than 10 minutes in total?

Solution Notes

Task 1. See the solution guide for Homework 4.

Task 2.

(a) Probability $p = 2/3$, the chance of rolling either 3, 4, 5, or 6 with a fair six-sided die.

(b) $P(K = 3) \;=\; \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27} \approx 0.074$

(c) $P(K > 3) \;=\; 1 - P(K = 1) - P(K = 2) - P(K = 3) \;=\; 1 - \frac{2}{3} - \frac{1}{3} \times \frac{2}{3} - \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{27-18-6-2}{27} = \frac{1}{27} \approx 0.037.$

Alternatively, we can observe that the chance of taking more than three rolls to succeed is the same as failing on each of the first three rolls, $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$.

(d) $\mathrm{E}(K) = 1/p = 1.5$. It's also possible to calculate this from the appropriate infinite series.

Task 3. The total probability mass must be 1, so we can calculate $p_3 = 1 - p_1 - p_2 = 1 - 0.2 - 0.4 = 0.4$. We can then use this in the formula for expected value as follows.

$$5.8 \;=\; \mathrm{E}(X) \;=\; p_1x_1 + p_2x_2 + p_3x_3 \;=\; 0.2 \times 3 + 0.4 \times 5 + 0.4 \times x_3 \;=\; 2.6 + 0.4x_3$$
$$x_3 \;=\; \frac{5.8 - 2.6}{0.4} \;=\; 8$$

From this the variance and standard deviation follow by standard calculation.

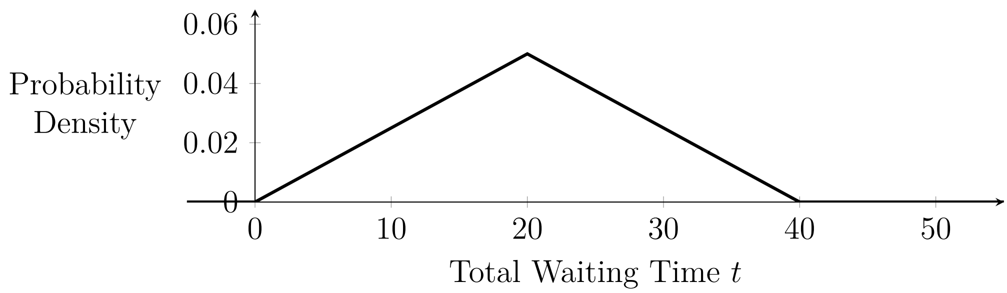
$$\begin{aligned}
\mathrm{E}(X^2) &= 0.2 \times 3^2 + 0.4 \times 5^2 + 0.4 \times 8^2 = 37.4 \\
\mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 37.4 - 5.8^2 = 3.76 \\
\sigma_X &= \sqrt{\mathrm{Var}(X)} = \sqrt{3.76} \approx 1.94
\end{aligned}$$

The CDF we obtain by adding up successive probabilities:

$$F(x) \;=\; \begin{cases} 0 & x < x_1 \\ p_1 & x_1 \le x < x_2 \\ p_1 + p_2 & x_2 \le x < x_3 \\ p_1 + p_2 + p_3 & x_3 \le x \end{cases} \qquad \text{which gives} \qquad F(x) \;=\; \begin{cases} 0 & x < 3 \\ 0.2 & 3 \le x < 5 \\ 0.6 & 5 \le x < 8 \\ 1 & 8 \le x \end{cases}.$$

Task 4.

(a) This is the shape of the probability density function.



Note that the graph reaches both to the left of $t = 0$ and to the right of $t = 40$ but has value zero there.

Here we just take the PDF as given, but the method for calculating it is in fact covered in Week 10: the PDF for a sum of independent random variables is a convolution of their individual PDFs.

$$f_{X+Y}(z) \;=\; \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\,dw$$

(b) The cumulative distribution function $F(t)$ measures the area under the probability density function $f(t)$. Rather than attempt the integration directly it's easier to divide up by cases and calculate the area under $f(x)$ geometrically.

- For $t < 0$ we have $f(t) = 0$ and hence $F(t) = 0$ too.
- For $0 \leq t < 20$ we need to measure the area of the right-angled triangle to the left of t. It has base t and height $f(t)$.

$$F(t) \;=\; \frac{1}{2} \times t \times f(t) \;=\; \frac{1}{2} \times t \times \frac{t}{400} \;=\; \frac{t^2}{800}$$

- For $20 \leq t \leq 40$ we can subtract the area of the right-angled triangle to the right of t from the total area, which we know to be 1.

$$F(t) \;=\; 1 \;-\; \frac{1}{2}(40-t)f(t) \;=\; 1 \;-\; \frac{1}{2}(40-t)\frac{(40-t)}{400} \;=\; 1 \;-\; \frac{(40-t)^2}{800}$$

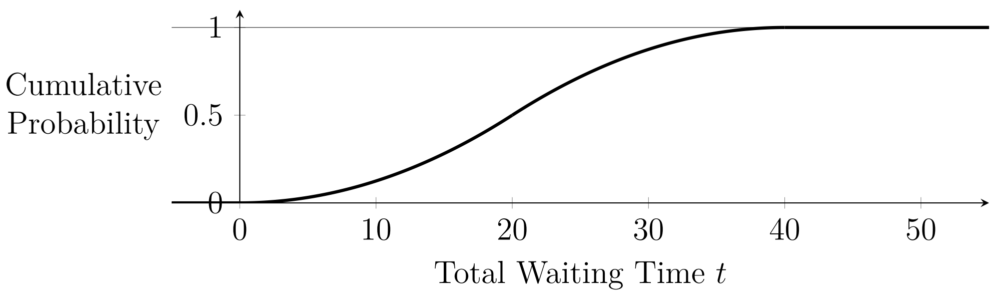
- For $t > 40$ we can see that all the probability mass lies to the left of t.

$$F(t) \;=\; 1$$

Summarizing:

$$F(t) \;=\; \begin{cases} 0 & t < 0 \\ \dfrac{t^2}{800} & 0 \leq t < 20 \\ 1 - \dfrac{(40-t)^2}{800} & 20 \leq t < 40 \\ 1 & 40 \leq t\,. \end{cases}$$

The graph for this function is below.



Here the values outside the main range of the curve are 0 for $t < 0$ and 1 for $t > 40$.

(c) The probability that tomorrow the commuter has to wait no more than 10 minutes in total is $P(T < 10)$ and can be calculated either with the CDF from the previous part or by calculating the relevant area under the PDF.

$$P(T < 10) \;=\; F(10) \;=\; \frac{10^2}{800} \;=\; \frac{1}{8}$$