DMP Class Test

Discrete Mathematics
23 October 2024

There are 30 marks to be earned, and you have 2 hours.

1. Prove that there exist no integer k such that k^2+k is odd. [3 marks]
2. Define the sequence $f_0, f_1, f_2, \ldots$ by $f_0 = 0$, $f_1 = 1$ and $f_i = f_{i-1} + f_{i-2}$ for all integers $i > 1$.
   Define the sequence $e_0, e_1, e_2, \ldots$ by $e_0 = 1$, $e_1 = 3$ and $e_i = e_{i-1} \cdot e_{i-2}$ for all integers $i > 1$.
   Prove that $e_n = 3^{f_n}$ for all integers $n \geq 0$. [4 marks]
3. Prove the identity $(A-B) \cup (B-C) \cup (C-A) = (B-A) \cup (C-B) \cup (A-C)$ by means of the element method.

   Note: What may look like a long solution to this can be simplified substantially using symmetry. Where a case distinction has many cases that are essentially the same, it is enough to work through just one case in detail and then indicate briefly how the others can be shown in the same way. [5 marks]

   Also give a Venn diagram showing either side of this equation. [2 marks]
4. Calculate the set of solutions of the Diophantine equation $63 \cdot x - 49 \cdot y = 56$. [4 marks]
5. Which of the following 4 functions are (i) injective, (ii) surjective, and (iii) bijective? (Note that these are 12 questions.) In each case, explain your answer. [8 marks]
   (a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = \begin{cases} 3x+4 & \text{if } x < 7 \\ 3x+5 & \text{if } x \geq 7 \end{cases}$.
   (b) $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(x) = \left\lfloor \frac{x}{3} \right\rfloor + 7$.
   Here $\lfloor r \rfloor$ means the real number r rounded down to the nearest integer.
   (c) $f \circ g$.
   (d) $g \circ f$.
6. Show that the set of all functions from the reals to the nonnegative integers is uncountable. [4 marks]